TRIGONOMETRIC POLYNOMIALS IN PRIME NUMBER THEORY[1]

BY
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1. Introduction

Let C_n be the set of n^{th} degree cosine polynomials g such that if

$$g(\phi) = a_0 + a_1 \cos \phi + \cdots + a_n \cos n\phi,$$

then $g(\phi) \geqq 0$ for all real ϕ, $a_1 > a_0 > 0$, and $a_k \geqq 0$ for $k = 2, 3, \cdots, n$. The estimates for the errors in approximate formulas obtained for various functions of prime numbers depend on the following two quotients formed from the coefficients of members of C_n :

$$R = R(g) = \frac{a_1 + a_2 + \cdots + a_n}{2(\sqrt{a_1} - \sqrt{a_0})^2},$$

$$S = S(g) = \frac{a_0 + a_1 + \cdots + a_n}{a_1 - a_0}.$$

Following standard notation, we denote by $\pi(x)$ the number of primes less than or equal to x. For $\pi(x)$, Landau [5, vol. 1, pp. 242–251] established the validity of the following approximation for all $\lambda > S + 2$:

$$\pi(x) = \int_2^x \frac{dy}{\log y} + O(xe^{-(\log x)^{1/\lambda}}).$$

By more sophisticated arguments Landau [5, vol. 1, pp. 321–333] also showed that for any $\rho > R$ we have

$$\pi(x) = \int_2^x \frac{dy}{\log y} + O(x\,(\log x)^{-1/2} e^{-\sqrt{(\log x)/\rho}}).$$

The last estimate for the error for large x depends on the following result concerning the zeros of the Riemann zeta-function $\zeta(s)$. If $a > R$, there exists a positive number γ_0 depending on a such that if $\beta + i\gamma$ is a zero of $\zeta(s)$ with $\gamma \geqq \gamma_0$, then $\beta < 1 - 1/(a \log \gamma)$. From these results it can be seen that the estimates for the error are decreased if S and R are made smaller.

The problem involving S has been treated by Landau, Tschakaloff and van der Waerden. Denoting the g.l.b. of $S(g)$ for $g \in C_n$ by P_n, the best results can be summarized as follows. Tschakaloff [8] proved that $P_2 = 7, P_3 = P_4 = P_5 = 6, P_6 = 5.92983 \cdots$, and $P_7 = P_8 = P_9 = 5.90529 \cdots$; he gave another

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proof for $6 \leqq n \leqq 9$ in [9]. Subsequently, Landau [6] proved these results for $2 \leqq n \leqq 5$ in a way different from that of Tschakaloff. Finally, van der Waerden [10] proved that for arbitrary n, $P_n \geqq 5.8642$.

Now we turn to the question of R, which is somewhat more difficult and has not been discussed as much in the literature. It has been established that $R \geqq 10.82$ for all $g \in C_n$; this is an unpublished result from a joint investigation by Schoenfeld and W. J. LeVeque. For $n = 3$, Landau [5, vol. 1, p. 324] gives the example

$$g(\phi) = (1 + \cos \phi)(1 + 2 \cos \phi)^2,$$

for which $R(g) < 18.53$. A slightly better example due to P. T. Bateman is

$$g(\phi) = (5 + 12 \cos \phi)^2(1 + \cos \phi),$$

for which $R(g) < 18.48$. Also, as was shown by Rosser-Schoenfeld [7, pp. 71, 78], choosing $\alpha = \frac{10}{8}$ in $g(\phi) = (1 + \cos \phi)^2(1 + \alpha \cos \phi)^2$ gives a $g \in C_4$ for which $R(g) < 17.51632$. Here we determine the g.l.b. of $R(g)$ for $g \in C_2$, we show that $R(g) > 16.2568$ for all $g \in C_n$ and we give stronger inequalities for R for smaller values of n. Our results are tabulated at the end of the paper.

For a complete discussion and proofs of results given herein the reader is referred to [3].

2. Determination of the g.l.b. of R for $n = 2$

We first note that C_1 is the null set since $g \in C_1$ implies $0 \leqq g(\pi) = a_0 - a_1$, which contradicts $a_1 > a_0$. Thus $n = 2$ is the simplest case, and we proceed to determine the g.l.b. of $R(g)$ for g in C_2. It is clear that R and the conditions imposed on $g(\phi)$, a_0, a_1, and a_2 remain unaltered if $g(\phi)$ is multiplied by a positive constant, so that it is no restriction to take $a_0 = 1$. Through the substitution $x = \cos \phi$, $g(\phi)$ is transformed into

$$h(x) = (1 - a_2) + a_1 x + 2a_2 x^2$$

which satisfies the condition $0 \leqq h(x)$ for all $x \in [-1, 1]$. Examination of the first and second order derivatives of $h(x)$ with respect to x shows that h assumes its absolute minimum at $x = x_0$ where $x_0 = -a_1/(4a_2)$. There are two possibilities.

First, suppose that $x_0 \in [-1, 1]$. For $x \in [-1, 1]$, $h(x) \geqq 0$ if and only if

$$\frac{a_1^2}{2} + \frac{(a_2 - \frac{1}{2})^2}{\frac{1}{2}^2} \leqq 1.$$

Second, suppose that $x_0 \notin [-1, 1]$. For $x \in [-1, 1]$, $h(x) \geqq 0$ if and only if

$$0 \leqq h(-1) = 1 - a_1 + a_2 .$$

Now since for fixed a_1, R decreases as a_2 decreases, it is clear that R has its minimum on the set $B_1 \cup B_2$, where

$$B_1 = \{(a_1, a_2) | a_2 = (1 - \sqrt{1 - a_1^2/2})/2, \tfrac{4}{3} < a_1 \leqq \sqrt{2}\}$$
$$B_2 = \{(a_1, a_2) | a_2 = a_1 - 1, 1 < a_1 \leqq \tfrac{4}{3}\}.$$

On B_2, inspection of the quotient R shows that R is minimal at $(a_1, a_2) = (\frac{4}{3}, \frac{1}{3})$, where it takes the value $V_2 = \frac{35}{2} + 10\sqrt{3} > 34.8$. On B_1, after differentiating the quotient R and simplifying, it is found that $dR/da_1 \leqq 0$ is equivalent to

$$81a_1^5 - 28a_1^4 - 220a_1^3 + 176a_1^2 + 128a_1 - 256 \equiv b(a_1) \leqq 0.$$

Sturm's theorem [1, p. 83] is applied to show that $b(a_1) = 0$ has exactly one root in $[1, \infty)$. This root, call it r, is located in [1.4126, 1.4127]. Since $b(a_1) > 0$ for $a_1 > r$ and $b(a_1) < 0$ for $1 \leqq a_1 < r$, on B_1, R is minimal at $a_1 = r$, where it takes the value $V_1 \in (26.5, 26.6)$.

It is clear that the g.l.b. of $R(g)$ for g in C_2 is V_1, where

$$V_1 = \frac{r + (1 - \sqrt{1 - r^2/2})/2}{2(\sqrt{r} - 1)^2}$$

and r is that zero of $b(a_1)$ satisfying $r \geqq 1$. This discussion also makes it clear that the g.l.b., V_1, is actually attained for the cosine polynomial having $a_0 = 1$, $a_1 = r$ and $a_2 = (1 - \sqrt{1 - r^2/2})/2$. In fact, a slight modification of the argument given by Tschakaloff [10] shows that for each n the

$$\operatorname*{g.l.b.}_{g \in C_n} R(g)$$

is attained.

3. Obtaining a lower bound for R by using results of Landau and van der Waerden

The complexity encountered in determining the g.l.b. of $R(g)$ for g in C_2 indicates that for $n \geqq 3$ the problem of determining the g.l.b. of $R(g)$ for g in C_n would be intractable. Thus we are led to search for lower bounds that will be near to the greatest lower bound.

Two methods are available for obtaining inequalities of the form

$$b_0 a_0 + b_1 a_1 + \cdots b_n a_n \geqq 0. \tag{3.1}$$

One method uses the fact that if ϕ_0 is real, then $g(\phi_0) \geqq 0$; the other uses a function G such that $G(\phi) \geqq 0$ for $\phi \in [a, b]$ so that

$$\int_a^b G(\phi)g(\phi)\, d\phi \geqq 0. \tag{3.2}$$

When various inequalities of the type (3.1) or (3.2) are suitably combined, we obtain an inequality of the form

$$a_1 + a_2 + \cdots + a_n \geqq Aa_1 - Ba_0. \tag{3.3}$$

Following van der Waerden [10] we construct an inequality of the form (3.3). For $g \in C_n$ we have

$$g(\pi) = a_0 - a_1 + a_2 - a_3 + \cdots \geqq 0 \tag{3.4}$$

and

$$\int_{\pi/2}^{\pi} g(\phi)\, d\phi = \frac{\pi}{2} a_0 - a_1 + \sum_{\substack{k=2 \\ 2\nmid k}}^{n} (-1)^{(k+1)/2} \frac{a_k}{k} \geqq 0. \tag{3.5}$$

Multiplying (3.5) by $p \geqq 0$ and adding to (3.4) gives

$$\begin{aligned}(p\pi/2 + 1)a_0 - (p+1)a_1 + {\textstyle\sum_{k=2,2|k}^{n}} a_k - {\textstyle\sum_{k=2,2\nmid k}^{n}} a_k\{1 + (-1)^{(k-1)/2}p/k\} \\ = (p\pi/2+1)a_0 - (p+2)a_1 + {\textstyle\sum_{k=1}^{n}} a_k \\ - {\textstyle\sum_{k=3,2\nmid k}^{n}} a_k\{2 + (-1)^{(k-1)/2}p/k\} \geqq 0.\end{aligned}$$

If $p \leqq 6$, then $2 + (-1)^{(k-1)/2}p/k \geqq 0$, so that

$$a_1 + a_2 + \cdots + a_n \geqq (p+2)a_1 - (p\pi/2 + 1)a_0 \tag{3.6}$$

for $0 \leqq p \leqq 6$, and (3.6) is of the form (3.3). Therefore, if $g \in C_n$ and $0 \leqq p \leqq 6$, then

$$R(g) \geqq \frac{Aa_1 - Ba_0}{2(\sqrt{a_1} - \sqrt{a_0})^2} \tag{3.7}$$

where $A = p + 2$ and $B = p\pi/2 + 1$.

Selecting $[a, b] = [-\pi, \pi]$ and $G(\phi) = 1 - \cos\phi$ in (3.2) shows that $0 < a_1/a_0 < 2$. Letting $\beta = \sqrt{a_1/a_0} - 1$, we have $0 < \beta < \sqrt{2} - 1$; further, from (3.3) we have

$$R(g) \geqq [A(\beta+1)^2 - B]/(2\beta^2) \equiv F(\beta). \tag{3.8}$$

These results enable us to obtain a lower bound for $R(g)$, $g \in C_n$.

THEOREM. *For each $g \in C_n$ we have $R(g) > 16$.*

Proof. Suppose that there exists a $g \in C_n$ such that $R(g) \leqq 16$. Taking $p = 0$ in (3.6) we have

$$16 \geqq R \geqq \frac{2(\beta+1)^2 - 1}{2\beta^2}$$

so that $|\beta - \frac{1}{15}| \geqq \sqrt{34}/30 > \frac{11}{60}$. Since $\beta > 0$, this implies $|\beta - \frac{1}{15}| = \beta - \frac{1}{15}$; and hence, that $\beta > \frac{1}{4}$. Similarly, taking $p = 6$ in (3.6) we obtain $|\beta - \frac{1}{3}| > \frac{1}{10}$. The left side cannot be $\frac{1}{3} - \beta$, since this would imply $\beta < \frac{1}{3} - \frac{1}{10} < \frac{1}{4}$. Hence $\beta > \frac{1}{3} + \frac{1}{10} > \sqrt{2} - 1$, which contradicts $\beta < \sqrt{2} - 1$. This proves the theorem. It is clear that careful calculation in this proof will yield a value somewhat larger than 16; in fact, 16 could be replaced by 16.247.

The arguments that established (3.7) and proved the preceding theorem suggest that more careful selection of A and B in (3.3) and more information for β would produce a better lower bound for $R(g)$, $g \in C_n$. We now examine these possibilities with two formulas for A and B. The proof of the preceding theorem illustrates the general method of the following investigation.

In [5, vol. 2, pp. 891–893] Landau obtains the following formulas for A and B when $n \geqq 3$:

$$A = 2 + (x+2)(y+2)/4,$$
$$B = (1+y^2)/y + (y+2)(x^2+1)/2x$$

where $x > 0$ and $y > 0$.

In [10] van der Waerden obtains the following formulas for A and B:

$$A = p\cos\varepsilon + q + 1, \qquad B = p(\pi/2 - \varepsilon) + q$$

where $p \geqq 0$, $q \geqq 0$, $\varepsilon < \pi/2$, and p and q are such that for $k \geqq 2$, the coefficients b_k of (3.1) satisfy $b_k \leqq 1$. The b_k determined by van der Waerden's procedure are

$$b_k = b_k(\varepsilon) = m_k p + (-1)^k q,$$

where $m_k = -k^{-1}\sin k(\pi/2 + \varepsilon)$. For a given $\varepsilon < \pi/2$, let

$$T(\varepsilon) = \{(p, q) \mid p \geqq 0, q \geqq 0, b_k \leq 1 \quad \text{for} \quad k \geqq 2\}.$$

Then for $(p, q) \in T(\varepsilon)$ and with A and B as given above, (3.3) holds for all $g \in C_n$ if $\varepsilon < \pi/2$. Thus it will be necessary to examine the set $T(\varepsilon)$. The inequality $R \geqq F(\beta)$ will be used with both formulas for A and B to determine a domain for β and lower bounds for R.

4. The character of the set $T(\varepsilon)$

The discussion of §3 makes clear the necessity of determining the nature of the set $T(\varepsilon)$ if van der Waerden's formulas for A and B are to be used. For suitably defined m_k we can write $b_k = m_k p + (-1)^k q$ for $k \geqq 1$. For reasons that will be clear later we must determine

$$M_0 = \text{l.u.b.}_{2|k\geqq 2}\, m_k\,; \qquad M_1 = \text{l.u.b.}_{2\nmid k\geqq 3}\, m_k\,.$$

As we now show, these least upper bounds are non-negative and are assumed. If ε is a multiple of $\pi/2$, the statement is easily verified. Since $m_k \to 0$ as $k \to \infty$, it suffices to show that for ε not a multiple of $\pi/2$ there exist both an even and an odd $l \geqq 3$ such that $m_l > 0$.

If ε is a rational multiple of π, but not a multiple of $\pi/2$, we find l as follows. Let $\varepsilon = r\pi/s$ with r, s, relatively prime and $s \geqq 3$, $r \neq 0$. If s is odd there exists a positive integer n such that

$$2rn \equiv 1 - r \pmod{s};$$

and if s is even, r is odd so that there exists a positive integer n such that

$$rn \equiv (1-r)/2 \pmod{s/2}.$$

In either case, there is an integer m such that $(2n+1)r = ms + 1$. Now $l = 2(2n+1) \geqq 6$ is even and $m_l > 0$. Similarly, for suitable $a = 0$ or ± 1 and a suitable integer n, $(4n+3)r = 2ms + a$, and $l = 4n + 3 \geqq 3$ is odd and $m_l > 0$.

For ε not a rational multiple of π we use Kronecker's Theorem [4, p. 375], which we now state for our convenience.

KRONECKER'S THEOREM. *If θ is irrational, if α is arbitrary and if N and η are positive, then there exist integers n and m such that $n > N$ and $| n\theta - m - \alpha | < \eta$.*

We apply this with $\eta = \frac{1}{8} = N$ and $\theta = 2\varepsilon/\pi$ so that for suitable $\lambda \in (-1, 1)$, we have $4n\varepsilon = 2m\pi + 2\pi\alpha + (\pi/4)\lambda$. Now taking $\alpha = \frac{1}{4} - \varepsilon/\pi$ and defining $l = 2(2n + 1) \geqq 6$, we have l is even and $m_l > 0$; taking $\alpha = -3\varepsilon/(2\pi)$ and defining $l = 4n + 3 \geqq 7$, we have l is odd and $m_l > 0$.

For $p \geqq 0$, we now have that $b_k \leqq M_0 p + q$ for all even $k \geqq 2$ with equality for some k; likewise for all odd $k \geqq 3$ we have $b_k \leqq M_1 p - q$ with equality for some k. The nature of $T(\varepsilon)$ has now been determined. When ε is an odd multiple of $\pi/2$, $T(\varepsilon)$ is a strip; for other cases it is either a convex quadrilateral or a triangle. Also, it is clear that M_0 and M_1 must be determined for values of ε that we want to consider.

For $| \varepsilon | \leqq \arccos \sqrt{\frac{5}{6}}$, it is possible to prove that $M_0 = \max \{m_2, m_4\}$ and $M_1 = m_3$. To obtain this result it is necessary to prove various trigonometric inequalities and it can be verified by reference to [3].

5. Determining bounds for β

For general non-negative cosine polynomials Fejér [2] obtained the inequality

$$| a_1 | \leqq 2a_0 \cos [\pi/(n + 2)]$$

with equality holding for a certain g in C_n which is unique up to a multiplicative constant, so that

$$\beta \leqq \sqrt{2 \cos [\pi/(n + 2)]} - 1. \tag{5.1}$$

This result will be used to determine lower bounds of R for certain n; however, it is apparent that for large n, this bound only slightly improves $\beta \leqq \sqrt{2} - 1$. The procedure to determine domains for β and corresponding lower bounds for R is as follows. Assuming that $g \in C_n$ and $R(g) < c$ we obtain, for certain numbers c, values $\xi(c)$ and $\nu(c)$ such that $\beta \in [\xi(c), \nu(c)]$. We use $\xi(c)$ and $\nu(c)$ to obtain a number $\rho(c)$ that is a lower bound for $F(\beta)$ on this interval. Hence we will have that $R \geqq \min \{c, \rho(c)\}$.

If $R < c$, then from (3.8) we obtain

$$(2c - A)\beta^2 - 2A\beta + B - A > 0. \tag{5.2}$$

Let the discriminant of the quadratic in (5.2) be denoted by 4Δ. Then (5.2) and $A < 2c$ imply

$$\left(\beta - \frac{A}{2c - A}\right)^2 > \frac{\Delta}{(2c - A)^2}. \tag{5.3}$$

If $B = (\lambda + 1)^2 A - 2c\lambda^2$, then $\Delta \geqq 0$. Now if $\lambda \leqq \beta$ for all $g \in C_n$ such that $R(g) < c$, then

$$\beta > 4c/(2c - A) - \lambda - 2. \tag{5.4}$$

If $2c \leqq A$, then from (5.2) it can be deduced that, whenever $R(g) < c$ and $g \in C_n$,

$$\beta < (B - A)/(\sqrt{\Delta} + A). \tag{5.5}$$

These results are now used with the van der Waerden and Landau formulas for A and B to obtain bounds on β in the sense described above.

The quantity $4c/(2c - A)$ appearing on the right side of (5.4) increases as A increases, and with van der Waerden's formulas, A is maximal for $\varepsilon = 0$ if $(p, q) = (6, 1)$. This choice in conjunction with $B = (\lambda + 1)^2 A - 2c\lambda^2$ fixes λ and maximizes $4c/(2c - A)$, but does not necessarily maximize the right side of (5.4). With the choice $\varepsilon = 0$, in [3] it is shown by the aid of linear programming that it is best to take $(p, q) = (0, 1), (6, 1)$ as we do to obtain the lower bounds given below. Further, the corresponding upper bound procedure using van der Waerden's formulas with $\varepsilon = 0$ yields no information.

Using van der Waerden's formulas with $\varepsilon = 0$, $p = 0$ and $q = 1$ so that $A = 2$ and $B = 1$, $B = (\lambda + 1)^2 A - 2c\lambda^2$ is equivalent to

$$2c\lambda^2 - 2(\lambda + 1)^2 + 1 = 0 \tag{5.6}$$

whose roots we denote by s_1 and s_2 with $s_1 < s_2$. Now it is clear that for $c > 1$ we have $s_1 < 0 < \beta$ and (5.4) shows that

$$\beta > s_2 . \tag{5.7}$$

With $\varepsilon = 0$, $p = 6$ and $q = 1$ so that $A = 8$ and $B = 3\pi + 1$,

$$B = (\lambda + 1)^2 A - 2c\lambda^2$$

is equivalent to

$$2c\lambda^2 - 8(\lambda + 1)^2 + 3\pi + 1 = 0 \tag{5.8}$$

whose roots we denote by λ_1 and λ_2. It is clear that for certain values of c we have $\lambda_1 < \beta$ so that by (5.4), $\beta > \lambda_2$. If $u_2 \in [16.686, 16.687]$ is the largest root of

$$(\pi - 2)^2 c^2 - 2(\pi^2 + \pi - 2)c + (\pi - 1)^2 = 0, \tag{5.9}$$

then it can be shown that $c \leqq u_2$ implies $\lambda_1 \leqq s_2$. Therefore, by (5.7)

$$\beta > \lambda_2 \tag{5.10}$$

for such c.

Now $c = 16.30$, $c = 16.66$, $c = u_2$ and $c = 16.69$ imply, respectively, $\beta > .4094$, $\beta > .3796$, $\beta > .3772$ and $\beta > .2532$. The last bound, although not used, shows the behavior of the bound when the estimate (5.7) is used.

Using Landau's formulas for A and B and choosing $A = 63$, $B = 120.3046$ and $c = 16.2568$, we have $2c < A$. Therefore, (5.5) implies that $\beta < .413440261$ when $R(g) < c$, $g \in C_n$ and $n \geqq 3$. Subject to $A \geqq 2c$, these choices for Landau's parameters are optimal as is shown in [3].

6. Lower bounds for R

Under the assumption $R(g) < 16.2568$, it has been shown that $\beta \in [.4094, .413440261]$. In §3 it was indicated that for $A > 0$ and $\beta \in [\xi, \nu]$, $F(\beta) \geqq \min\{F(\xi), F(\nu)\}$. With van der Waerden's formulas for A and B, it is shown by using the nature of $T(\varepsilon)$ and linear programming and taking $|\varepsilon| \leqq \text{arc cos}\sqrt{\frac{5}{6}}$ that the choice $\varepsilon = 0, (p, q) = (6, 1)$ provides the best bound for R. In [3] it is shown that the bounds obtained for R using Landau's formulas are not so good as those obtained with van der Waerden's.

Using van der Waerden's formulas for A and B and taking $\varepsilon = 0, (p, q) = (6, 1)$, we have $\min\{F(.4094), F(.413440261)\} = F(.413440261) > 16.2570$. Therefore, for all $g \in C_n$ we have

$$R(g) > 16.2568. \tag{6.1}$$

For smaller values of n we can give stronger inequalities for R than that given by (6.1). For $3 \leqq n \leqq 7$ we take $c = u_2$ which implies $\beta > \lambda_2 > .3772$. Also, for $3 \leqq n \leqq 7$ Fejér's result (5.1) implies $\beta < .371$. The obvious contradiction shows that for these n we have $R > u_2$. For $n \geqq 8$, we take $c = 16.66$ so that $\beta > .3796$; on letting ν_n be the right side of (5.1) and taking $\varepsilon = 0, (p, q) = (6, 1)$ we find that

$$R(g) \geqq 4 + 8/\nu_n - (3\pi - 7)/(2\nu_n^2).$$

This, together with (6.1) and the result of §2 enables us to construct the following table that gives the best results we have obtained in this study.

n	2	$3, \cdots, 7$	8	9	10	all n
$R >$	26.5	16.6865	16.6565	16.5883	16.5418	16.2568

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