NORMAL, SIMPLE AND NEUTRAL CONGRUENCES ON LATTICES[1]

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It is known that the lattice of congruences of any lattice is a complete, Σ-distributive lattice and hence is pseudo-complemented. Thus given any congruence θ of a lattice L we can talk of its pseudo-complement (i.e., the pseudo-complement of θ in $\theta(L)$). In this paper we characterize the pseudo-complement of any congruence on a lattice L and establish sets of necessary and sufficient conditions for a congruence θ on L to be (i) normal and (ii) simple, in Theorems 1–3. This in turn enables us to give a characterization of weakly modular lattices in terms of its congruences (Theorem 4). Next we deal with neutral congruences and show in Theorem 5 that any congruence on a weakly complemented and dually weakly complemented lattice satisfying either chain condition is a neutral congruence. Further we show that a permutable congruence θ on a lattice L with 0, 1 is complemented if and only if θ is a principal neutral congruence on L (Theorem 6).

Another theorem of this paper is: If L is a lattice with zero satisfying the ascending chain condition then there is a (1-1) correspondence between neutral ideals and congruences on L if and only if L is a direct union of simple lattices (Theorem 7). Making use of the characterization of weakly modular lattices of Theorem 4, we arrive at a necessary condition for a (1-1) correspondence between neutral ideals and congruences for general lattices in Theorem 8.

We start with the definitions of some of the not too well known concepts used in the course of this paper.

An element n of a lattice L is said to be a neutral element of L if it satisfies the following equalities:

(i) $n(a + b) = na + nb$ for all a, b in L.
(ii) $n + ab = (n + a)(n + b)$ for all a, b in L.
(iii) $n + a = n + b; na = nb$ implies $a = b$ for all a, b in L.

An element n of a lattice L satisfying conditions (ii) and (iii) is called a standard element of L.

Any ideal I of L which is neutral (standard) considered as an element of $I(L)$, the lattice of ideals of L is called a neutral (standard) ideal of L.

An element a' of a lattice with zero is called the pseudocomplement of an element a in L if it satisfies the conditions: (i) $aa' = 0$ and (ii) $ax = 0$ implies $x \leq a'$ for all x in L.

An element of a lattice L is said to be normal if $a'' = a$, and is called simple if $a + a' = 1$.

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A congruence on a lattice L is said to be normal (simple) if it is normal (simple) when considered as an element of the lattice of congruences on L.

An interval (c, d) of a lattice L is said to be a lattice translate of an interval (a, b) of L if there exist elements $x_1, x_2, \cdots, x_n$ of L such that

$$c = f(a, x_1, x_2, \cdots, x_n) \quad \text{and} \quad d = f(b, x_1, x_2, \cdots, x_n);$$

where n is finite and f is a finite lattice polynomial.

An interval J of a lattice L is said to be effective if whenever J is a lattice translate of an interval I, there exists a nontrivial subinterval I_1 (other than a point) of I such that I_1 is a lattice translate of J.

A lattice L is called a weakly modular lattice if all its intervals are effective.

Next we give the characterization of the pseudo-complement of a congruence θ on L.

THEOREM 1. *Let θ be any congruence on a lattice L. The pseudocomplement θ' of θ in $\theta(L)$ can be got thus: $x \equiv y\ (\theta')$ if and only if $(x + y, xy)$ has no nontrivial lattice translate annulled by θ.*

Proof. It will suffice to prove θ' is the largest congruence which satisfies $\theta \wedge \theta' = 0$ (the null congruence).

First note that θ' is a congruence relation on L. For

(i) $x \equiv x\ (\theta')$

(ii) $x \equiv y\ (\theta')$ if and only if $xy \equiv x + y\ (\theta')$.

(iii) $x > y$ and $x \equiv y\ (\theta')$ implies $x + z \equiv y + z\ (\theta')$.

For if $(x + z, y + z)$ has a lattice translate annulled by θ then so does (x, y); as $(x + z, y + z)$ is a lattice translate of (x, y) and lattice translation is a transitive relation.

Dually $x > y$ and $x \equiv y\ (\theta')$ implies $xz \equiv yz\ (\theta')$.

(iv) $x > y > z$ and $x \equiv y\ (\theta')$; $y \equiv z\ (\theta')$ implies $x \equiv z\ (\theta')$.

For if (x, y) has no nontrivial lattice translate annulled by θ and (y, z) has no nontrivial lattice translate annulled by θ, then (x, z) has no nontrivial lattice translate annulled by θ. For

$$\theta_{(x,z)} = \theta_{(x,y)} + \theta_{(y,z)} .$$

Therefore any lattice translate of (x, z) is a finite sum of lattice translates of (x, y) and (y, z) and hence if a nontrivial lattice translate of (x, z) is annulled by θ, then a nontrivial lattice translate of (x, y) or (y, z) is annulled by θ; which gives rise to a contradiction. Thus the restricted transitivity of θ' is established. Hence θ' is a congruence relation on L (cf. [3]).

Further $\theta \wedge \theta' = 0$ follows from definition of θ'. Next if $x \equiv y\ (\phi)$ with $\theta \wedge \phi = 0$ then $(x + y, xy)$ has no nontrivial lattice translate annulled by

θ; which implies $x \equiv y\ (\theta')$. That is $\phi \subset \theta'$ i.e., θ' is the largest congruence possessing the property $\theta \wedge \phi = 0$; i.e., θ' is the pseudocomplement of θ in $\theta(L)$.

As a corollary we get a theorem due to G. Gratzer and E. T. Schmidt (cf. [3]).

COROLLARY 1. *Let θ be any congruence on a weakly modular lattice L. The pseudo-complement θ' of θ in $\theta(L)$ can be got thus: $x \equiv y\ (\theta')$ if and only if $(x + y, xy)$ consists of single point congruence classes under θ.*

Proof. It will suffice to prove that in the case of weakly modular lattices an interval (a, b) has no lattice translates annulled by θ if and only if (a, b) consists of single point congruence classes under θ. Then the proof follows from Theorem 1.

The "only if" part follows as any subinterval of an interval is a lattice translate of itself.

The "if" part comes out as a consequence of the weak modularity; for if a lattice translate J of an interval I consisting of single point congruence classes under θ is annulled by θ then a nontrivial subinterval of I will be annulled by θ; as J is effective. This gives rise to a contradiction.

Next we have for the normality of any congruence on a lattice L, the theorem,

THEOREM 2. *Let θ be a congruence relation on L. θ is normal if and only if every interval I not annulled by θ has a nontrivial lattice translate J such that J has no nontrivial lattice translate annulled by θ.*

Proof. Now $\theta \subset \theta''$ is true for all θ. To prove the reverse inequality it will suffice to prove that if I is any interval not annulled by θ then I is not annulled by θ'' as well.

Consider $I = (a, b)$, any interval not annulled by θ. Then there exists a lattice translate J of I such that J has no lattice translate annulled by θ. This in turn implies J is annulled by θ' (by Theorem 1). Therefore J is not annulled by θ''. Hence I is not annulled by θ'' i.e., $\theta \supset \theta''$. Hence θ is normal.

For the other part, let θ be a normal congruence relation on L. Let if possible there be an interval I not annulled by θ such that every lattice translate of I has a lattice translate annulled by θ. This implies no lattice translate of I is annulled by θ'. Hence I is annulled by θ'', by Theorem 1. That is $\theta \subset_{\neq} \theta''$, a contradiction as θ is a normal congruence on L.

COROLLARY. *Let θ be a congruence on a weakly modular lattice L. θ is normal if and only if for every interval I not annulled by θ there exists a nontrivial subinterval J of I such that J consists of single point congruence classes under θ.*

THEOREM 3. *A congruence θ on a lattice L is simple if and only if between any pair of comparable elements a, b of L there exists a finite number of elements*

$$a = b_1 > b_2 > \cdots > b_n = b$$

such that either (b_{i-1}, b_i) is annulled by θ or (b_{i-1}, b_i) has no nontrivial lattice translate annulled by θ.

Proof follows from Theorem 1.

As a corollary we get a result due to G. Gratzer and E. T. Schmidt (cf. [3]).

COROLLARY. *A congruence θ on a weakly modular lattice L is simple if and only if it is separable.*

THEOREM 4. *Let L be an arbitrary lattice. Let θ be a congruence on L. Let θ' be the binary relation on L defined by $x \equiv y\ (\theta')$ if and only if the interval $(x + y, xy)$ consists of single point congruence classes under θ. The necessary and sufficient condition for L to be weakly modular is that for any congruence θ on L, the binary relation θ' (as defined above) is a congruence relation on L.*

Proof. Necessity follows from the corollary to Theorem 1.

Sufficiency. Let L be a non-weakly modular lattice. Let I be an ineffective interval of L. That is there exists an interval J in L such that I is a lattice translate of J but no non-trivial subinterval of J is a lattice translate of I.

Let θ be the congruence on L generated by I. We shall show that the binary relation θ' defined by $x \equiv y\ (\theta')$ if and only if $(x + y, xy)$ consists of single point congruence classes under θ is not a congruence relation on L. Now θ' annulls J but not I, which is impossible if θ' is a congruence relation. Thus θ' is not a congruence relation on L. This completes the proof, Q.E.D.

It is known that in a lattice L, if $aa' = 0$ and $a + a' = 1$ and if $x = xa + xa'$ and $x = (x + a)(x + a')$ for all x in L, then a and a' belong to the centre of L (cf. [9]).

Using this we have,

LEMMA 1. *In a lattice L with 0, 1; a complemented element a is neutral if and only if a is standard and a' is dually standard.*

Proof. It is known that if a is central, a' is also central. Hence a is standard and a' is dually standard.

Conversely let a be standard and a' dually standard in L. Then $x = x + aa' = (x + a)(x + a')$ for all x in L, as a' is dually standard in L. Also $x = x(a + a') = xa + xa'$ for all x in L, as a is standard in L. Therefore by the result stated above a is neutral.

LEMMA 2. *Any dually standard element of a weakly complemented lattice with* 1 is neutral.

Proof. Let s be a dually standard element of L. Let θ be the dually standard congruence on L generated by s (i.e., $x \equiv y\ (\theta)$ if and only if $xy = (x + y)t$ for some $t \geq s$). Let s' be the complement of s in L. Then the kernel under θ is the principal ideal generated by s'. For $s \equiv 1\ (\theta)$ implies $s' \equiv 0\ (\theta)$. Also if $p \equiv 0\ (\theta)$ and $p > s'$ then $ps = 0$ and $p + s = 1$. Hence p is also a complement of s in $(1, 0)$. But s being dually standard is uniquely complemented in L and so $s' = p$. Thus s' is a standard element of L, as in a weakly complemented lattice every congruence ideal is a standard ideal in L. Whence by Lemma 1, s, s' are neutral elements of L.

Dually we have,

LEMMA 3. *Any standard element of a dually weakly complemented lattice with zero is neutral.*

DEFINITION. The congruence on L generated by a neutral ideal of L is called a neutral congruence on L.

THEOREM 5. *Any congruence on a weakly complemented and dually weakly complemented lattice L satisfying either chain condition is a neutral congruence on L.*

Proof. Let L satisfy the ascending chain condition; then every ideal of L is a principal ideal and so is every congruence ideal of L. We also know that every congruence on a weakly complemented lattice is a standard congruence (cf. [4]). Thus any congruence θ on L is a principal standard congruence on L. Let s be the standard element corresponding to θ; then s is a neutral element of L, by Lemma 3 as L is dually weakly complemented. Thus θ is a neutral congruence on L. The case when L satisfies the descending chain condition can be proved dually.

We can easily prove

LEMMA 4. *Let $L = N + M$ (direct union); then N and M are both neutral ideals of L.*

COROLLARY. *Any decomposition congruence on L is a neutral congruence on L.*

LEMMA 5. *Let L be any lattice. Let θ be a neutral congruence on L determined by a principal ideal $N = \mu(n)$. Then θ is complemented and the complement θ' is determined by the principal dual ideal $\alpha(n)$.*

Proof. As N is a neutral ideal of L, n is a neutral element of L. Let θ' be the binary relation on L defined by $x \equiv y\ (\theta')$ if and only if $xn = yn$. As n is a neutral element of L, n distributes all sums in L, i.e., $n(x + y) = nx + ny$ for all x, y in L and hence θ' is a congruence relation on L.

Now $0 \equiv n\ (\theta)$ and $n = 1\ (\theta')$. Thus $\theta + \theta' = 1$. Also $\theta \wedge \theta' = 0$. For if $a \equiv b\ (\theta \wedge \theta')$ then $a \equiv b\ (\theta)$ and $a \equiv b\ (\theta')$ implies $n + a = n + b$ and $na = nb$; which in turn implies $a = b$, as n is neutral. Thus θ is complemented and θ' is its complement. Further θ' is determined by the dual ideal $\alpha(n)$.

LEMMA 6. *Let L be any lattice with* 0 *and* 1. *Let θ be a decomposition congruence on L. Then θ is a principal neutral congruence on L.*

Proof. Let θ be a decomposition congruence on L; then θ is a neutral congruence on L by the corollary to Lemma 4. Let N be the neutral ideal of L corresponding to θ and $M = L/\theta$. Then we have $L = N + M$. Also $N \cdot M = (0)$, and L is a lattice with 0 and 1. Therefore the sum and the intersection of the neutral ideals N and M are principal. Hence N and M are principal (cf. [4]). Thus θ is a principal neutral congruence on L.

Combining Lemmas 5 and 6, we get

THEOREM 6. *Let L be any lattice with* 0 *and* 1 *and θ a permutable congruence on L. Then θ is complemented if and only if θ is a principal neutral congruence on L.*

COROLLARY. *Let L be a weakly complemented lattice with* 0 *and* 1. *Any congruence θ on L is complemented if and only if the kernel of θ is a principal neutral ideal of L.*

Proof follows as any congruence of a weakly complemented lattice is permutable.

Next we get s slight generalization of a result due to J. Hashimoto (cf. [5]).

THEOREM 7. *If L is a lattice with zero satisfying the ascending chain condition, then there is a* (1-1) *correspondence between neutral ideals of L and congruences on L if and only if L is a direct union of simple lattices.*

Proof. Let L be a lattice satisfying the ascending chain condition then any ideal of L is principal. Let there be a (1-1) correspondence between neutral ideals of L and congruences on L. Then as every congruence on L is a principal neutral congruence on L, by Lemma 5 every congruence on L is complemented. Hence $\theta(L)$ is a Boolean algebra. Further $\theta(L)$ is isomorphic to the centre of L and L satisfies the ascending chain condition; whence $\theta(L)$ is a finite Boolean algebra. Let $\theta_1, \theta_2, \cdots, \theta_n$ be the maximal elements of $\theta(L)$. Then $\theta_1 \theta_2 \cdots \theta_n = 0$ and L/θ_i is simple, for each 1. Thus L is a direct union of $L/\theta_1, L/\theta_2, \cdots, L/\theta_n$ as every congruence on L is permutable. Thus L is a direct union of a finite number of simple lattices.

Conversely let L be a direct union of a finite number of simple lattices. Then every congruence on L is a decomposition congruence relation and hence a neutral congruence on L. Hence there is a (1-1) correspondence between neutral ideals of L and congruences on L.

COROLLARY. *A weakly complemented and dually weakly complemented lattice satisfying either chain condition is a direct union of a finite number of simple lattices.*

Proof follows from Theorems 5 and 7.

Next we state the following lemma (Lemma 5 of [4]) without proof as it is made use of for the theorem which follows.

LEMMA A (G. Gratzer and E. T. Schmidt). *Let S be a standard ideal of L. Then the congruence generated by S in $I(L)$ (the lattice of ideals of L) is the extension to $I(L)$ of the congruence generated by S in L, and the congruence generated by S in L is the restriction to L of the congruence generated by S in $I(L)$.*

THEOREM 8. *A necessary condition for a (1-1) correspondence between neutral ideals and congruences on any lattice L is that L is weakly modular.*

Proof. Let there be a (1-1) correspondence between congruences and neutral ideals of a lattice L. Let θ be any congruence on L. Let N be the neutral ideal of L corresponding to θ. Consider the congruence θ^I on $I(L)$ generated by N. Now as θ^I is a principal neutral congruence on $I(L)$ it is complemented (by Lemma 5). Let θ'^I be the complement of θ^I on $I(L)$. Then θ'^I is generated by the principal dual ideal corresponding to N in $I(L)$, (by Lemma 5); i.e.,

$$\theta^I + \theta'^I = 1 \quad \text{and} \quad \theta^I \wedge \theta'^I = 0 \text{ on } I(L).$$

Let θ' be the restriction of θ'^I on L. Then we shall prove that θ' is a congruence on L defined by the binary relation $a \equiv b\ (\theta')$ if and only if $(a + b, ab)$ consists of single point congruence classes under θ. Then the weak modularity of the lattice will follow by Theorem 4, as the choice of θ is arbitrary.

Let $a \equiv b\ (\theta')$ in L. Then $A \equiv B\ (\theta'^I)$ in $I(L)$, where A, B are the principal ideals generated by a, b of L respectively. Hence $(A + B, AB)$ consists of single point congruence classes under θ^I on $I(L)$. Thus $(a + b, ab)$ consists of single point congruence classes under θ on L, as θ is the restriction of θ^I on L, by Lemma A.

Next the interval $(a + b, ab)$ consists of one element congruence classes under θ on L implies the interval $(A + B, AB)$ consists of single point congruence classes under θ^I on $I(L)$. For otherwise let $AB \leq I <_{\neq} J \leq A + B$ in $I(L)$ and let $I \equiv J\ (\theta^I)$. As θ^I is the extension of θ on L, we have for any x in I, a y in J such that $x \equiv y\ (\theta)$ and conversely. Now $I <_{\neq} J$. Therefore there exists a p in J such that $p \notin I$. Let q be the element of I such that $p \equiv q\ (\theta)$. This implies $pq \equiv p + q\ (\theta)$ on L. Also $pq \not\equiv p + q$ as $p + q \notin I$ and $pq \in I$. Put $pq = c$ and $p + q = d$, then $c < d$ and $c \equiv d\ (\theta)$ on L. This implies $c + ab \equiv d + ab\ (\theta)$ on L and $c + ab \not\equiv d + ab$, for $c + ab$ is in I and $d \notin I$. Also as $c + ab, d + ab \in J$ and $J \subseteq A + B$; $c + ab, d + ab \leq a + b$. Thus $c + ab$, $d + ab$ are elements in the interval $(a + b, ab)$ such that they are congruent under θ. This is a contradiction to our assumption. Hence the conclusion.

Now $(A + B, AB)$ consists of single point congruence classes under θ^I on $I(L)$ and θ^I is simple on $I(L)$. Hence

$$AB \equiv A + B\ (\theta'^I),$$

i.e., $ab \equiv a + b\ (\theta')$ as θ' is the restriction of θ'^I on L. Thus given any congruence θ on L the binary relation θ' defined by $a \equiv b\ (\theta')$ if and only if

$(a + b, ab)$ consists of single point congruence classes under θ is a congruence relation on L.

Combining this with Theorem 5 we get,

COROLLARY. *Any weakly complemented and dually weakly complemented lattice satisfying either chain condition is weakly modular.*

Remark. Weak modularity of a lattice L is not a sufficient condition for a (1-1) correspondence between neutral ideals and congruences on L.

Proof follows from the fact that if we take a distributive lattice L other than a relatively complemented lattice then there is no (1-1) correspondence between neutral ideals of L and congruences on L; even though any distributive lattice is weakly modular.

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