A RELATION BETWEEN A CLASS OF LIMIT LAWS AND A RENEWAL THEOREM

BY

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Introduction

Let $\{X_k\}$ be a sequence of random variables and set

$$H(x) = \sum_{j=1}^{\infty} P(\sum_{k=1}^{j} X_k < x).$$

The asymptotic behavior of the function $H(x)$ in the case where the random variables, $\{X_k\}$, are independent but not necessarily identically distributed has been studied in some detail. Under additional assumptions [1] deals with $H(x)/x$, [3] with

$$\frac{1}{T}\int_0^T (H(x + h) - H(x))\, dx,$$

[5] and [7] with $H(x + h) - H(x)$, and [6] with a weighted renewal function,

$$\sum_{n=1}^{\infty} a_n\, P(\sum_{k=1}^{n} X_k < x).$$

A property common to the sequences of random variables considered in these four papers is that in each $(1/n)\sum_{k=1}^{n} X_k$ converges in probability to a constant as $n \to \infty$. It is the purpose of this paper to examine the case when the distribution function of $(1/n)\sum_{k=1}^{n} X_k$ converges to a limit law, proper or improper, and to discuss which limit laws can arise. Specifically:

THEOREM. *Let $\{X_k\}$ be a sequence of independent non-negative random variables. If the variables $\{X_k/n\}$ for $1 \leq k \leq n$ are infinitesimal and if there exists a probability distribution function F such that at every continuity point of F,*

$$\lim_{n\to\infty} P((1/n)\sum_{k=1}^{n} X_k < x) = F(x),$$

then for each h

$$(1)\quad \lim_{\varepsilon\to 0^+}\lim_{T\to\infty}\frac{1}{T}\int_0^T \sum_{j=1}^{\infty} P\left(x \leq \sum_{k=1}^{j}(X_k + \varepsilon) < x + h\right) dx = h\int_0^{\infty}\frac{dF(x)}{x}.$$

The proof of this theorem is found in Section 2 and is an adaptation of a Tauberian argument given in [3]. One would like to eliminate the limit on ε in (1) and to be able to conclude

$$(2)\qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T \sum_{k=1}^{\infty} P\left(x \leq \sum_{j=1}^{k} X_j < x + h\right) dx = h\int_0^{\infty}\frac{dF(x)}{x}.$$

Corollaries stated in Section 2 give conditions which permit this. If we write

$$H_\varepsilon(x) = \sum_{n=1}^{\infty} P(\sum_{k=1}^{n}(X_k + \varepsilon) < x)$$

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then, as is pointed out in [6 p. 677], (1) and (2) can be stated in the more compact forms:

$$(1') \qquad \lim_{\varepsilon\to 0^+} \lim_{x\to\infty} H_\varepsilon(x)/x = \int_0^\infty \frac{dF(x)}{x},$$

$$(2') \qquad \lim_{x\to\infty} H(x)/x = \int_0^\infty \frac{dF(x)}{x}.$$

Let $P = \{F : F$ is infinitely divisible, $F(x) = 0$ for each $x < 0\}$. We say that a probability distribution function, F, belongs to the class L if it is possible to find a sequence $\{X_k\}$ of independent random variables and constants A_n and $B_n > 0$ such that at every continuity point of F,

$$\lim_{n\to\infty} P((1/B_n)\textstyle\sum_{k=1}^n X_k - A_n < x) = F(x)$$

and such that the variables $\{X_k/B_n\}$ for $1 \leq k \leq n$ are infinitesimal. This is the class L as defined by Khintchine. In Section 1 the Laplace transform of functions in $P \cap L$ is discussed and it is pointed out that for any $F \in P \cap L$ there exists a sequence of random variables satisfying the conditions of the theorem with F as the limit law.

The notation used is meant to be consistent with that used in [4].

$$\gamma(\nu) = \lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|\leqq\nu} x\, dP(X_{nk} < x) \quad \text{and} \quad \gamma = \lim_{\nu\to 0^+} \gamma(\nu).$$

$\{X_{nk}\}$ is said to be infinitesimal if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \sup_{1\leq k\leq k_n} P(|X_{nk}| > \varepsilon) = 0.$$

The function $n(x)$ is that found in the Lévy form for the Fourier transform of an infinitely divisible law. Strong use is made here of the fact that if $\{X_{nk}\}$ is a sequence of row independent, infinitesimal, random variables such that $\sum_{k=1}^{k_n} X_{nk}$ converges in distribution to a limit law, F, then at every continuity point of the function, $n(x)$, associated with F,

$$\lim_{n\to\infty} - \textstyle\sum_{k=1}^{k_n} P(X_{nk} \geq x) = n(x).$$

In each of the above k_n is finite for each n. In the main theorem of this paper, $X_k/n = X_{nk}$ with $k_n = n$.

Section 3 contains examples.

Section 1

LEMMA. *For any $F \in P$ the function $n(x)$ of the Lévy form for $\int_0^\infty e^{itx}\, dF(x)$ is integrable on every interval* $[0, b]$, $0 < b < \infty$, *and hence*

$$\int_0^\infty e^{-sx}\, dF(x) = \exp\left\{-\gamma s - \int_0^\infty (1 - e^{-sx})\, dn(x)\right\}.$$

Proof. By the corollary to Theorem 1, [2], XIII-29

$$\int_0^\infty e^{-sx}\, dF(x) = \exp\left\{-\int_0^\infty \frac{(1 - e^{-sx})}{x}\, dP(x)\right\}$$

where P is non-decreasing. It is therefore possible to construct a sequence, $\{X_{nk}\}$, of non-negative, row independent, infinitesimal, Poisson random variables such that at every continuity point of $F(x)$,

$$\lim_{n\to\infty} P(\textstyle\sum_{k=1}^{k_n} X_{nk} < x) = F(x).$$

We write

$$-\sum_{k=1}^{k_n} \int_0^b x\, dP(X_{nk} \geq x) = -\sum_{k=1}^{k_n} bP(X_{nk} \geq b) + \int_0^b \sum_{k=1}^{k_n} P(X_{nk} \geq x)\, dx$$

where b is a continuity point of $n(x)$. Applying Fatou's Lemma yields

$$\gamma(b) - bn(b) = \liminf_{n\to\infty} \int_0^b \sum_{k=1}^{k_n} P(X_{nk} \geq x)\, dx \geq \int_0^b - n(x)\, dx.$$

$n(x)$ is monotone and hence $\int_0^b - n(x)\, dx$ is finite for any $b > 0$. This proves the main assertion of the Lemma.

$n(x)$ is non-decreasing and $n(x) \in L_1(0, b)$, $0 < b < \infty$, so that $\lim_{x\to 0^+} xn(x) = 0$. An integration by parts shows that for fixed $s > 0$,

$$\lim_{b\to 0^+} \limsup_{n\to\infty} - \sum_{k=1}^{k_n} \int_0^b (1 - e^{-sx} - sx)\, dP(X_{nk} < x) = 0.$$

Also

$$\lim_{b\to 0^+} \lim_{n\to\infty} \sum_{k=1}^{k_n} \int_0^b x\, dP(X_{nk} < x) = \gamma,$$

and if $\{b_m\}$ is a sequence of continuity points of $n(x)$ converging to 0,

$$\lim_{b_m\to 0} \lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{b_m}^{\infty} (1 - e^{-sx})\, dP(X_{nk} < x) = \int_0^{\infty} (1 - e^{-sx})\, dn(x).$$

Consequently the Laplace transform of F has the stated form.

Corollary to a Theorem of Lévy's. *If $F \in P \cap L$ then there exists a sequence $\{Y_k\}$ of non-negative independent random variables such that the variables $\{Y_k/n\}$ for $1 \leq k \leq n$ are infinitesimal and such that at every continuity point of F,*

$$\lim_{n\to\infty} P((1/n) \textstyle\sum_{k=1}^{n} Y_k < x) = F(x).$$

Proof. Let $f(t) = \int_0^\infty e^{itx}\, dF(x)$. By Theorem 1, [4] p. 147, if $F \in L$ then for any α between 0 and 1, $f(t)$ can be written

$$f(t) = f(\alpha t) f_\alpha(t)$$

where $f_\alpha(t)$ is the Fourier transform of a probability distribution. As is done in the proof of this theorem of Lévy's, we set $\alpha = (k - 1)/k$ and look at the sequence $\{Y_k\}$ of independent random variables defined by

$$E(e^{itY_k}) = f_{(k-1)/k}(kt).$$

$$\prod_{k=1}^{n} E(e^{iY_k t/n}) = \prod_{k=1}^{n} \frac{f(kt/n)}{f(t(k-1)/n)}$$

and hence we have the desired convergence of distribution functions. It is also shown in the proof of this Lévy theorem that the variables $\{Y_k/n\}$ for $1 \leq k \leq n$ are infinitesimal. We add only the observation that if $F \in P \cap L$ and

$$F(x/k) = F(x/(k-1)) * P(Y_k < x) = \int_{-\infty}^{x} F\left(\frac{x-y}{k-1}\right) dP(Y_k < y)$$

then $P(Y_k < x) = 0$ for each k and each $x < 0$. If this were not the case then the following contradiction would arise. There would exist some k and some $\varepsilon > 0$ such that

$$P(Y_k \in [-2\varepsilon, -\varepsilon)) > 0.$$

Let $\gamma = \sup \{x/k : F(x/k) = 0\}$. $\gamma \geq 0$ and $F(\gamma) = 0$ because F is left continuous. Then

$$\begin{aligned} 0 = F(\gamma) &\geq \int_{-2\varepsilon}^{-\varepsilon} F\left(\frac{\gamma k - y}{k-1}\right) dP(Y_k < y) \\ &\geq F\left(\frac{\gamma k + \epsilon}{k-1}\right) P(Y_k \in [-2\varepsilon, -\varepsilon)) \\ &\geq F(\gamma + \varepsilon/k) P(Y_k \in [-2\varepsilon, -\varepsilon)) > 0. \end{aligned}$$

Hence the sequence $\{Y_k\}$ is non-negative and the proof is complete.

An alternative proof of this corollary is sketched in a remark appearing in Section 3 and following Example 1.

Section 2

Proof of the Theorem. Fix $h > 0$ and $\varepsilon > 0$. The function

$$G_\varepsilon(x) = \textstyle\sum_{k=1}^{\infty} P(x \leq \sum_{j=1}^{k} (X_j + \epsilon) < x + h)$$

is finite for each x. Set

$$\phi_k(s) = E(e^{-sX_k}) \quad \text{and} \quad \psi(s) = \int_0^{\infty} e^{-sx}\, dF(x).$$

Let $U_\varepsilon(T) = \int_0^T G_\varepsilon(x)\, dx$. We will show that

$$\begin{aligned} \lim_{s \to 0^+} s \int_0^{\infty} e^{-sx}\, dU_\varepsilon(x) &= \lim_{s \to 0^+} s \int_0^{\infty} e^{-sx} G_\varepsilon(x)\, dx \\ &= \lim_{s \to 0^+} \frac{(e^{sh} - 1)}{s}\, s \sum_{n=1}^{\infty} \prod_{k=1}^{n} e^{-n\varepsilon s} \phi_k(s) \\ &= h \int_0^{\infty} e^{-\varepsilon s} \psi(s)\, ds = h \int_0^{\infty} \frac{dF(x)}{x + \varepsilon}. \end{aligned}$$

An appeal to a standard Tauberian theorem gives the value of

$$\lim_{T \to \infty} (1/T) U_\varepsilon(T).$$

An application of the Monotone Convergence Theorem completes the proof. The interchange of integration and summation in the above is permissible because each of the summands in the definition of $G_\varepsilon(x)$ is non-negative.

Let $\delta > 0$ be given. Choose $R > 0$ so that

$$\int_R^\infty e^{-\varepsilon s}\psi(s)\,ds < \delta,$$

so that for fixed $s_0 > 0$,

$$\textstyle\sum_{n\geq R/s} se^{-n\varepsilon s}\prod_{k=1}^n \phi_k(s) \leq \delta$$

uniformly in s for $s \in [0, s_0]$, and so that $\delta/R < \frac{1}{2}$. Since

$$\lim_{m\to\infty} - \textstyle\sum_{k=1}^m P(X_k \geq mx) = n(x)$$

has the property that $\lim_{x\to 0^+} xn(x) = 0$ and since the sequence $\{X_k/n\}$ is infinitesimal, it is possible to find constants $b > 0$ and $N' > 0$, both depending on R and δ, such that for all n satisfying $N' \leq n \leq R/s$,

$$\begin{aligned}
&\sum_{k=1}^n \sum_{j=2}^\infty \frac{(1-\phi_k(s))^j}{j} \\
&\qquad \leq \sup_{1\leq k\leq n} (1-\phi_k(s)) \sum_{k=1}^n \frac{(1-\phi_k(s))}{\phi_k(s)} \\
&\qquad \leq \sup_{1\leq k\leq n} \left[\int_0^{nb} sx\,dP(X_k < x) + P(X_k \geq nb)\right] \sum_{k=1}^n \frac{(1-\phi_k(s))}{\phi_k(s)} \\
&\qquad \leq \frac{[Rb + \sup_{1\leq k\leq n} P(X_k \geq nb)]}{1 - [Rb + \sup_{1\leq k\leq n} P(X_k \geq nb)]} \left[\frac{1}{n}\sum_{k=1}^n \int_0^{nb} Rx\,dP(X_k < x) \right.\\
&\qquad\qquad \left. + \sum_{k=1}^n P(X_k \geq nb)\right] \\
&\qquad \leq -\log(1 - \delta/R).
\end{aligned}$$

There exists $s' < \min(s_0, R/N')$ such that if $0 \leq s \leq s'$, then

$$\textstyle\sum_{n=1}^{N'} se^{-n\varepsilon s}\prod_{k=1}^n \phi_k(s) < \delta.$$

Writing

$$\begin{aligned}
\textstyle\sum_{n=1}^\infty se^{-n\varepsilon s}\prod_{k=1}^n \phi_k(s) &= \textstyle\sum_{n=1}^{N'-1} se^{-n\varepsilon s}\prod_{k=1}^n \phi_k(s) \\
&\quad + \textstyle\sum_{n=N'}^{[R/s]} se^{-n\varepsilon s}\prod_{k=1}^n \phi_k(s) \\
&\quad + \textstyle\sum_{n>[R/s]} se^{-n\varepsilon s}\prod_{k=1}^n \phi_k(s)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=N'}^{[R/s]} se^{-n\varepsilon s} \exp\left(\sum_{k=1}^n \log \phi_k(s)\right) \\
&\qquad = \sum_{n=N'}^{[R/s]} se^{-n\varepsilon s} \exp\left\{\sum_{k=1}^n (1-\phi_k(s)) + \sum_{k=1}^n\sum_{j=2}^\infty \frac{(1-\phi_k(s))^j}{j}\right\}
\end{aligned}$$

and applying the above estimates gives for all $s \in [0, S']$,

$$\left| \sum_{n=1}^{\infty} se^{-n\varepsilon s} \prod_{k=1}^{n} \phi_k(s) - \sum_{n=N'}^{[R/s]} se^{-n\varepsilon s} - \sum_{k=1}^{n} (1 - \phi_k(s)) \right| \leq 3\delta.$$

If the integral $\int_0^R e^{-\varepsilon s}\psi(s)\, ds$ is approximated by the Riemann sum and the lemma is used, we see that there exists $s'' > 0$ such that if $s \in [0, s'']$, then

$$\left| \int_0^R e^{-\varepsilon s}\psi(s)\, ds - \sum_{m=1}^{[R/s]} s \exp\left\{ -ms(\varepsilon + \gamma) - \int_0^{\infty} (1 - e^{-msx})\, dn(x) \right\} \right| \leq \delta.$$

Since

$$\begin{aligned} \sum_{k=1}^{m} \int_0^{\nu} x^2\, dP(X_k < mx) & \\ &\leq -\nu^2 \sum_{k=1}^{m} P(X_k \geq m\nu) + 2\nu \sum_{k=1}^{m} \int_0^{\nu} P(X_k \geq mx)\, dx \\ &\leq 2\nu \left[\frac{1}{m} \sum_{k=1}^{m} \int_0^{m\nu} x\, dP(X_k < x) + \nu \sum_{k=1}^{m} P(X_k \geq m\nu) \right], \end{aligned}$$

it is possible to find $N'' > 0$ and $\nu > 0$ so that for all m satisfying $N'' \leq m \leq R/s$,

$$\begin{aligned} \left| \sum_{k=1}^{m} \int_0^{\nu} (1 - e^{-msx} - msx)\, dP(X_k < mx) \right| &\leq m^2 s^2 \sum_{k=1}^{m} \int_0^{\nu} x^2\, dP(X_k < mx) \\ &\leq -\log (1 - \delta/R). \end{aligned}$$

It is also possible to choose $\nu > 0$ such that

$$\int_0^{\nu} (1 - e^{-msx})\, dn(x) \leq -\log (1 - \delta/R)$$

for all $m \leq R/s$, and such that $| \gamma(\nu) - \gamma | \leq -\log (1 - \delta/R)$. If $F \in L$ then as a consequence of Theorem 1, [4] p. 149, $n(x)$ is continuous on $(0, \infty)$. Fix ν. On the interval $[\nu, \infty)$, $n(x)$ is uniformly continuous and non-decreasing. Each of the functions, $-\sum_{k=1}^{m} P(X_k \geq mx)$, is also non-decreasing. Therefore on the interval $[\nu, \infty)$,

$$\lim_{m\to\infty} - \sum_{k=1}^{m} P(X_k \geq mx) = n(x)$$

uniformly in x. This fact combined with an integration by parts yields for all sufficiently large m,

$$\begin{aligned} \left| \int_{\nu}^{\infty} (1 - e^{-msx})\, dn(x) - \sum_{k=1}^{m} \int_{\nu}^{\infty} (1 - e^{-msx})\, dP(X_k < mx) \right| & \\ \leq -\log (1 - \delta/R).& \end{aligned}$$

Finally

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \int_0^{m\nu} x\, dP(X_k < x) = \gamma(\nu)$$

so that there exists N''' such that for all $s \in [0, s'']$,

$$\Bigg| \sum_{m=N'''}^{[R/s]} s \Bigg[\exp\Big\{ -ms(\gamma + \varepsilon) - \int_0^{\nu} (1 - e^{-msx})\, dn(x) - \int_{\nu}^{\infty} (1 - e^{-msx})\, dn(x) \Big\} - \exp\Big\{ -ms\Big(\frac{1}{m}\sum_{k=1}^{m} \int_0^{m\nu} x\, dP(X_k < x) + \varepsilon\Big) - \int_0^{\nu} (1 - e^{-msx} - msx) \sum_{k=1}^{m} dP(X_k < mx) - \int_{\nu}^{\infty} (1 - e^{-msx}) \sum_{k=1}^{m} dP(X_k < mx) \Big\} \Bigg] \Bigg| \le 9\delta.$$

Therefore there exists $r > 0$ such that for all $s \in [0, r]$,

$$\left| \int_0^{\infty} e^{-\varepsilon s}\psi(s)\, ds - \sum_{n=1}^{\infty} s e^{-n\varepsilon s} \prod_{k=1}^{n} \phi_k(s) \right| \le 3\delta + 9\delta + \delta + 3\delta = 16\delta.$$

Hence

$$\lim_{s \to 0^+} s \int_0^{\infty} e^{-sx} G_\varepsilon(x)\, dx = h \int_0^{\infty} e^{-s\varepsilon}\psi(s)\, ds$$

and consequently

$$\lim_{T\to\infty} \frac{1}{T} \int_0^{T} G_\varepsilon(x)\, dx = h \int_0^{\infty} e^{-\varepsilon s}\psi(s)\, ds = h \int_0^{\infty} \frac{dF(x)}{x + \varepsilon}.$$

Letting $\varepsilon \to 0$ gives the desired result.

In the corollaries that follow are found conditions which make it possible to replace (1) by the more desirable (2) in the conclusion of the theorem. In the proof of the theorem, the introduction of ε served two purposes. It made it possible to choose $R > 0$ so that

$$\int_R^{\infty} e^{-\varepsilon s}\psi(s)\, ds \quad \text{and} \quad \sum_{n \ge R/s} s e^{-n\varepsilon s} \prod_{k=1}^{n} \phi_k(s)$$

could both be made small, in the latter, uniformly in s for s in some $[0, s_0]$. The remaining estimates in the proof were made independent of ε.

COROLLARY 1. *If in addition to the assumptions of the theorem, it is assumed that there exists some $M > 0$ such that*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_0^{M} x\, dP(X_k < x) = C > 0$$

then (2) *holds.*

Proof. For each $\nu > 0$, $\gamma(\nu) \ge C$ and hence $\lim_{\nu\to 0^+} \gamma(\nu) = \gamma > 0$. By the lemma $\psi(s) \in L_1(0, \infty)$.

$$\begin{aligned}\sum_{n\geq R/s} s\prod_{k=1}^{n}\phi_k(s) &\leq \sum_{n\geq R/s} s\exp\left(-\sum_{k=1}^{n}(1-\phi_k(s))\right)\\ &\leq \sum_{n\geq R/s} s\exp\left(-\sum_{k=1}^{n}\int_0^M (1-e^{-sx})\,dP(X_k<x)\right)\\ &\leq \sum_{n\geq R/s} s\exp\left(\frac{-(1-e^{-sM})}{M}\sum_{k=1}^{n}\int_0^M x\,dP(X_k<x)\right).\end{aligned}$$

Therefore there exists $s_0 > 0$ such that for all $s \in (0, s_0]$,

$$\begin{aligned}\textstyle\sum_{n\geq R/s} s\prod_{k=1}^{n}\phi_k(s) &\leq \textstyle\sum_{n\geq R/s} se^{-nC(1-e^{-sM})/2M}\\ &\leq \frac{se^{-RC(1-e^{-sM})/2sM}}{1-e^{-C(1-e^{-sM})/2M}}.\end{aligned}$$

This inequality together with the integrability of $\psi(s)$ and the remark in the paragraph preceding Corollary 1 completes the proof.

COROLLARY 2. *If in addition to the assumptions of the theorem it is also assumed that* $\psi(s) \in L_1(0, \infty)$ *and that*

$$\lim_{n\to\infty}\left(\textstyle\prod_{k=1}^{n}\phi_k(s/n)\right)/\psi(s) = 1$$

uniformly in s *for* $s \geq 0$, *then* (2) *holds.*

Proof. For R/s large,

$$\sum_{n\geq R/s} s\prod_{k=1}^{n}\phi_k(s) \leq 2\sum_{n\geq R/s} s\psi(sn) \leq 2\int_{R-s}^{\infty}\psi(s)\,ds.$$

For the same reasons as stated at the conclusion of the proof of Corollary 1, Corollary 2 is proven.

Section 3

Example 1. Let $F(x) = (1/\Gamma(\alpha))\int_0^x t^{\alpha-1}e^{-t}\,dt$ and

$$\psi(s) = \int_0^\infty e^{-sx}\,dF(x) = (1+s)^{-\alpha} = \exp\left\{-\int_0^\infty ((1-e^{-sx})\alpha e^{-x}/x)\,dx\right\},$$

[2, xiii, p. 29]. Here $n(x) = -\int_x^\infty (\alpha e^{-y}/y)\,dy$ and $\gamma = 0$. For $\alpha > 1$ let $\{X_k\}$ be a sequence of independent random variables with

$$\begin{aligned}E(e^{-sX_k}) &= \exp\left\{-\int_0^\infty ((1-e^{-ksx})\alpha e^{-x}/k)\,dx\right\}\\ &= \exp\{-\alpha s/(1+ks)\}.\end{aligned}$$

The sequence satisfies the conditions of the theorem with

$$\lim_{n\to\infty} P((1/n)\textstyle\sum_{k=1}^{n} X_k < x) = F(x).$$

Furthermore $(1+s)^{-\alpha}$ is integrable on $[0, \infty)$ and

$$\textstyle\sum_{n\geq R/s} s\prod_{k=1}^{n} E(e^{-sX_k})$$

can be made small uniformly for all s in some $[0, s_0]$ by taking R large so that it is possible to conclude the stronger (2). The value of the limit in (2) is

$$h \int_0^\infty \frac{dF(x)}{x} = h\Gamma(\alpha - 1)/\Gamma(\alpha) = h/(\alpha - 1).$$

For each k, $E(X_k) = \alpha$ and we see that the reciprocal of the common mean does not appear in the limit.

Remark. Given any $F \epsilon P \cap L$ the method of construction used in the above example provides an alternative proof of the corollary of Section 1. If

$$\int_0^\infty e^{-sx}\, dF(x) = \exp\left\{ - \gamma s - \int_0^\infty (1 - e^{-sx})\, dn(x) \right\}$$

then

$$\exp\left\{ - \gamma s - \int_0^\infty (1 - e^{-sx})\, d(- xn'(x^+)) \right\}$$

is the Laplace transform of an infinitely divisible law, [4, p. 149]. Defining X_k by

$$E(e^{-sX_k}) = \exp\left\{ - \gamma s - (1/k) \int_0^\infty (1 - e^{-ksx})\, d(- xn'(x^+)) \right\}$$

yields a sequence of random variables satisfying the conditions of the theorem with F as limit law.

Example 2. Let $\{X_k\}$ be a sequence of independent random variables defined by

$$P(X_k = 1) = 1 - 1/k \quad \text{and} \quad P(X_k = k + 1) = 1/k.$$

Let $u_n = \sum_{k=1}^\infty P(\sum_{j=1}^k X_j = n)$. From Corollary 1 we can conclude (2) which here takes the form

$$\lim_{n\to\infty} (1/n) \sum_{k=1}^n u_k = \int_0^\infty \exp\left\{ - s - \int_0^1 \frac{(1 - e^{-sx})}{x}\, dx \right\} ds.$$

Here $n(x) = \log x$ for $x \leq 1$ and $n(x) = 0$ for $x > 1$. $\gamma = 1$. For all k, $E(X_k) = 2$ and yet the average of the probabilities of a renewal at time n approaches a value between $\frac{1}{2}$ and 1.

Example 3. Let $\{X_k\}$ be a sequence of independent random variables defined by

$$E(e^{-sX_k}) = \exp\left\{ - \int_0^\infty ((1 - e^{-ksx})/k)\, d(- x^{-\alpha}) \right\} = \exp\{- k^{\alpha-1} s^\alpha C_\alpha\}$$

where $0 < \alpha < 1$ and $C_\alpha = \alpha \int_0^\infty (1 - e^{-v})v^{-\alpha-1}\, dv$. For the same reasons as stated in Example 1, it is possible to conclude (2). The value of the limit is $h \int_0^\infty e^{-s^\alpha C_\alpha/\alpha}\, ds$. A non-zero limit is obtained despite the fact that for each k, $E(X_k) = \infty$.

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