A NOTE ON CERTAIN ARITHMETICAL CONSTANTS

BY
H. DAVENPORT AND A. SCHINZEL

In their famous memoir "Partitio Numerorum III" [5] Hardy and Littlewood formulated several conjectures about the asymptotic distribution of primes of various special forms. One of them was:

CONJECTURE K. *If c is any fixed integer other than a cube, then there are infinitely many primes of the form $m^3 + c$. The number $P(N)$ of such primes up to N is given asymptotically by*

$$P(N) \sim \frac{N^{1/3}}{\log N} \prod_p \left(1 - \frac{2}{p-1} (-c)_p\right), \tag{1}$$

where p runs through primes $\equiv 1 \pmod 3$ with $p \nmid c$, and $(-c)_p$ is 1 or $-\frac{1}{2}$ according as $-c$ is or is not a cubic residue $(\bmod\, p)$.

The problem of computing, for a particular c, the constant given by the product on the right of (1), and similar problems for more general conjectures, have engaged the attention of several mathematicians [1], [2], [3], [12]. A more general conjecture made by Bateman and Horn ([1]; see also [3]) was the following:

HYPOTHESIS H. *Let $f_1(x), \cdots, f_k(x)$ be distinct polynomials in one variable with integral coefficients and with highest coefficients positive, of degrees $h_1, \cdots, h_k$ respectively. Suppose that each of these polynomials is irreducible over the rational field and that there is no prime which divides $f_1(n) \cdots f_k(n)$ for all n. Let $Q(N)$ denote the number of positive integers n up to N for which $f_1(n), \cdots, f_k(n)$ are all primes. Then*

$$Q(N) \sim (h_1 \cdots h_k)^{-1} C(f_1, \cdots, f_k) \int_2^N (\log u)^{-k}\, du, \tag{2}$$

where

$$C(f_1, \cdots, f_k) = \prod\nolimits_p \{(1 - p^{-1})^{-k}(1 - p^{-1}\omega(p))\}. \tag{3}$$

Here the product is over all primes and $\omega(p)$ denotes the number of solutions of the congruence

$$f_1(x) \cdots f_k(x) \equiv 0 \pmod p.$$

This hypothesis implies Conjectures B, D, E, F, K, P of Hardy and Littlewood (cf. [11]).

Bateman and Horn showed that the convergence of the product (3) follows easily from the Prime Ideal Theorem. A similar deduction had been made

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earlier, for $k = 1$, by Nagell [8] and Rademacher [9]; see also Ricci [10]. However, since the product is not absolutely convergent, there is difficulty in estimating the error introduced when it is computed from a finite number of factors.

Bateman and Horn returned to the subject in [2]; they expressed $C(f_1, \cdots, f_k)$ in terms of an absolutely convergent product in the case when each of the polynomials f_i has the property that a zero of it generates a normal algebraic number field with an abelian Galois group. The question was raised of determining $C(f_1, \cdots, f_k)$ in more general cases, and in particular in the case $k = 1, f_1 = x^3 + c$ of Conjecture K.

In the present paper we express $C(f_1, \cdots, f_k)$ in terms of absolutely convergent products for arbitrary polynomials, and show how to evaluate the constant of Conjecture K when $c = 2$ or 3. No essentially new idea is required. We prove:

THEOREM. *Let $f_1(x), \cdots, f_k(x)$ be polynomials which satisfy the conditions of Hypothesis H. Let $f(x) = f_1(x) \cdots f_k(x)$ and let $f(x)$ have r_1 real zeros and r_2 pairs of conjugate complex zeros and have discriminant D. Let K_i be the field generated by a zero of f_i, let D_i be its discriminant, H_i its class-number, R_i its regulator, and let w_i be the number of roots of unity contained in K_i. Then*

$$(4)\quad \begin{aligned} C(f_1, \cdots, f_k) &= 2^{-r_1-r_2}\pi^{-r_2} \prod_{i=1}^{k} \frac{w_i \mid D_i \mid^{1/2}}{H_i R_i} \gamma(D) \\ &\times \textstyle\prod_{p \nmid D} (1 - p^{-1}\omega(p))(1 - p^{-1})^{-\omega(p)} \prod_{p \nmid D} \prod_{j \ge 2} (1 - p^{-j})^{-\omega_j(p)}. \end{aligned}$$

Here $\omega_j(p)$ denotes the number of irreducible factors of $f(x)$ mod p *that are of degree j, so that $\omega_1(p) = \omega(p)$, and*

$$(5)\quad \gamma(D) = \textstyle\prod_{p \mid D} (1 - p^{-1}\omega(p)) \prod_{i=1}^{k} \prod_{j \ge 1} (1 - p^{-j})^{-\omega_{i,j}(p)},$$

where $\omega_{i,j}(p)$ denotes the number of distinct prime ideal factors of p in K_i that are of degree j.

We observe that the assumptions about the f_i imply that these polynomials are algebraically coprime, so that $D \neq 0$. We observe further that since

$$(1 - p^{-1}\omega(p))(1 - p^{-1})^{-\omega(p)} = 1 - \tfrac{1}{2}\omega(p)(\omega(p) - 1)p^{-2} + O(p^{-3}),$$

the infinite products occurring in (4) are absolutely convergent.

Proof. By the infinite product for the Dedekind ζ-function, we have

$$(6)\quad \textstyle\prod_{i=1}^{k} \zeta(s)/\zeta_{K_i}(s) = \prod_p (1 - p^{-s})^{-k} \prod_{i=1}^{k} \prod_{j \ge 1} (1 - p^{-js})^{\omega_{i,j}(p)}.$$

If $p \nmid D$, it follows from Dedekind's theorem on the connection between prime ideals and the factorization of a polynomial mod p, together with the fact that the $f_i(x)$ are coprime mod p, that

$$\textstyle\sum_{i=1}^{k} \omega_{i,j}(p) = \omega_j(p),$$

and in particular that

$$\sum_{i=1}^{k} \omega_{i,1}(p) = \omega_1(p) = \omega(p).$$

Hence

$$\prod_{i=1}^{k} \zeta(s)/\zeta_{K_i}(s) = A_D(s) \prod_{p \nmid D} (1 - p^{-s})^{-k+\omega(p)} \prod_{j \geq 2} (1 - p^{-js})^{\omega_j(p)},$$

where $A_D(s)$ denotes the product on the right of (6) extended over primes which divide D. The last expression is

$$A_D(s) \prod_{p \nmid D} (1 - p^{-s})^{-k}(1 - \omega(p)p^{-s}) \frac{(1 - p^{-s})^{\omega(p)}}{(1 - \omega(p)p^{-s})} \prod_{j \geq 2} (1 - p^{-js})^{\omega_j(p)}.$$

It is known [7, Theorem 123] that

$$\lim_{s \to 1} \prod_{i=1}^{k} \frac{\zeta(s)}{\zeta_{K_i}(s)} = 2^{-r_1 - r_2}\pi^{-r_2} \cdot \prod_{i=1}^{k} \frac{w_i \mid D_i \mid^{1/2}}{H_i R_i}.$$

Since

$$\prod_p (1 - p^{-1})^{-k}(1 - \omega(p)p^{-1})$$

is known to converge, we can make $s \to 1$ and apply Abel's theorem. This gives

$$A_D(1) \prod_{p \nmid D} (1 - p^{-1})^{-k}(1 - \omega(p)p^{-1}) \frac{(1 - p^{-1})^{\omega(p)}}{(1 - \omega(p)p^{-1})} \prod_{j \geq 2} (1 - p^{-j})^{\omega_j(p)}$$

$$= 2^{-r_1 - r_2}\pi^{-r_2} \prod_{i=1}^{k} \frac{w_i \mid D_i \mid^{1/2}}{H_i R_i} = B, \quad \text{say.}$$

Hence

$$C(f_1, \cdots, f_k) = \prod_{p \mid D} (1 - p^{-1})^{-k}(1 - \omega(p)p^{-1}) B A_D(1)^{-1}$$

$$\times \prod_{p \nmid D} \frac{(1 - \omega(p)p^{-1})}{(1 - p^{-1})^{\omega(p)}} \prod_{j \geq 2} (1 - p^{-j})^{-\omega_j(p)},$$

and on inserting the value of $A_D(1)$ we obtain

$$C(f_1, \cdots, f_k) = B\gamma(D) \prod_{p \nmid D} \frac{(1 - \omega(p)p^{-1})}{(1 - p^{-1})^{\omega(p)}} \prod_{j \geq 2} (1 - p^{-j})^{-\omega_j(p)},$$

where $\gamma(D)$ is as defined in (5). This proves (4).

COROLLARY 1. *We have*

(7)
$$C(x^3 \pm 2) = \frac{3\sqrt{3}}{\pi \mid \log (\sqrt[3]{2} - 1) \mid} \prod\nolimits_1 \left(1 - \frac{3p - 1}{(p - 1)^3}\right) \prod\nolimits_2 \left(1 - \frac{1}{p^2}\right)^{-1}$$

$$\times \prod\nolimits_3 \left(1 - \frac{1}{p^3}\right)^{-1} = 1.29 \cdots,$$

where $\prod_1$ *is over primes* p *representable as* $a^2 + 27b^2$, *and* $\prod_2$ *is over primes* $p > 2$ *satisfying* $p \equiv 2 \pmod 3$, *and* $\prod_3$ *is over primes* $p \equiv 1 \pmod 3$ *not representable as* $a^2 + 27b^2$.

Proof. Here we have $k = 1, r_1 = r_2 = 1, K_1 = Q(\sqrt[3]{2}), D_1 = D = -108$, $H_1 = 1, R_1 = |\log(\sqrt[3]{2} - 1)|, w_1 = 2$ (see [4, p. 141]). In order to calculate $\gamma(D)$ we factorize the principal ideals (2) and (3) in $Q(\sqrt[3]{2})$ and get

$$(2) = (\sqrt[3]{2})^3, \qquad (3) = (\sqrt[3]{2} + 1)^3,$$

whence

$$\begin{aligned} \omega_{i,j}(2) = \omega_{i,j}(3) &= 1 \quad \text{when } j = 1, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Since $\omega(2) = \omega(3) = 1$, this gives $\gamma(D) = 1$. For $p > 3$, by the cubic reciprocity law ([6, p. 67])

$$\begin{aligned} \omega(p) &= 3 \quad \text{if } p = a^2 + 27b^2, \\ &= 1 \quad \text{if } p \equiv 2 \pmod 3, \\ &= 0 \quad \text{otherwise;} \end{aligned}$$

$$\begin{aligned} \omega_2(p) &= 1 \quad \text{if } p \equiv 2 \pmod 3, \\ &= 0 \quad \text{otherwise;} \end{aligned}$$

$$\begin{aligned} \omega_3(p) &= 1 \quad \text{if } p \equiv 1 \pmod 3 \text{ and } p \neq a^2 + 27b^2, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Substituting in (4), we obtain (7).

The three infinite products on the right of (7) have the approximate values 0.993, 1.06, 1.004, whence the numerical value of C.

COROLLARY 2. *We have*

$$C(x^3 \pm 3) = \frac{9\sqrt{3}}{2\pi\,|\log(\sqrt[3]{9} - 2)|} \prod\nolimits_4 \left(1 - \frac{3p - 1}{(p - 1)^3}\right) \prod\nolimits_5 \left(1 - \frac{1}{p^2}\right)^{-1} \times \prod\nolimits_6 \left(1 - \frac{1}{p^3}\right)^{-1} = 1.38 \cdots, \tag{8}$$

where $\prod_4$ *is over primes* p *such that* $4p = a^2 + 243b^2$, *and* $\prod_5$ *is over primes* $p \equiv 2 \pmod 3$, *and* $\prod_6$ *is over primes* $p \equiv 1 \pmod 3$ *with* $4p$ *not representable as* $a^2 + 243b^2$.

Proof. Here we have $k = 1, r_1 = r_2 = 1, K_1 = Q(\sqrt[3]{3}), D_1 = D = -243$, $H_1 = 1, R_1 = |\log(\sqrt[3]{9} - 2)|, w_1 = 2$ (see [4, p. 141]). Since $(3) = (\sqrt[3]{3})^3$ and $\omega(3) = 1$, we have $\gamma(D) = 1$. For $p \neq 3$, by the cubic reciprocity law [6, p. 67],

$$\begin{aligned} \omega(p) &= 3 \quad \text{if } 4p = a^2 + 243b^2, \\ &= 1 \quad \text{if } p \equiv 2 \pmod 3, \\ &= 0 \quad \text{otherwise;} \end{aligned}$$

$$\begin{aligned} \omega_2(p) &= 1 \quad \text{if} \quad p \equiv 2 \pmod 3, \\ &= 0 \quad \text{otherwise;} \\ \omega_3(p) &= 1 \quad \text{if} \quad p \equiv 1 \pmod 3 \quad \text{and} \quad 4p \neq a^2 + 243b^2, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

This gives (8). The three infinite products on the right are approximately 0.997, 1.41, 1.004, whence the numerical value of C.

The numerical values found in Corollaries 1 and 2 agree with those found empirically by Bateman and Horn from a finite product, and confirmed by their count of the numbers of primes up to 14000. These values were 1.29 for $x^3 \pm 2$ and 1.38 for $x^3 \pm 3$. They quote also an empirical value 2.88 for $C(x, (x^3 - x + 18)/6)$. This constant could also be calculated from (5); but since the cubic field generated by a root of $x^3 - x + 18 = 0$ is not tabulated in [4], the necessary computations would be rather long.

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Trinity College
Cambridge, England
Mathematical Institute of the Polish Academy of Sciences
Warsaw, Poland