ON THE ZEROS OF AN ENTIRE FUNCTION AND ITS SECOND DERIVATIVE

BY

SIMON HELLERSTEIN[1]

INTRODUCTION

Recently, B. Ja. Levin and I. V. Ostrovskii have shown [4] that if f is a real entire function (i.e., real on the real axis) with only real zeros such that "most" of the zeros of f'' lie "near" the real axis, then f is at most of "small" infinite order. This is an important step in the direction of resolving an old question of Polya and Wiman. (For a statement of the problem, an account of partial results and bibliography we refer the reader to [4].)

In order to state the Theorem of Levin and Ostrovskii more precisely we recall the definition of an A-set of complex numbers.

DEFINITION. *An A-set is a sequence $\{a_k\}$ of complex numbers such that*

$$\sum_{a_k \neq 0} | I_m(1/a_k)| < +\infty .$$

The result referred to above takes the following form.

THEOREM A. *If all the zeros of the real entire function $f(z)$ are real and the zeros of $f''(z)$ form an A-set, then*

$$\log \log M(r, f) = O(r \log r).$$

Without the assumption of reality on f and its zeros Levin and Ostrovskii prove only the following somewhat sharpened version of an earlier theorem due to A. Edrei [2, Theorem 3].

THEOREM B. *If f is of the form*

(i) $f(z) = P(z)e^{Q(z)}$ *(P and Q entire)*

where $P(z)$ satisfies

(ii) $$\int_1^{\infty} \frac{\log^+ \log^+ M(t, P)}{t^2}\, dt < +\infty$$

and the zeros of $f(z)f''(z)$ form an A-set, then

(iii) $\log \log M(r, f) = O(r)$.

Moreover, $Q(z)$ is of exponential type and satisfies

(iv) $$\int_{-\infty}^{\infty} \frac{\log^+ | Q'(t)|}{1 + t^2}\, dt < +\infty .$$

Received May 5, 1965.

[1] This work was supported in part by a grant from the National Science Foundation.

Levin and Ostrovskii pose the question of whether the assumption that the zeros of $f(z)f''(z)$ form an A-set is already sufficient to yield the assertion that f is representable in the form (i) with P satisfying (ii), and consequently the conclusion of Theorem B holding for f and Q.

Although we are unable to settle this question, we succeed in showing that if the zeros of $f(z)f''(z)$ are to form an A-set, then f must have "almost as many" zeros as the growth of f will allow.

1. Statement of results.

We shall assume throughout that the reader is familiar with the standard notation and fundamental results of the Nevanlinna Theory.

Our main result is given by

THEOREM 1. *Let $f(re^{i\theta})$ be an entire function such that the zeros of ff'' form an A-set. Assume, moreover, that*

$$\limsup_{r\to\infty} \frac{\log_k^+ n\left(r, \frac{1}{f}\right)}{\log r} = \mathcal{K} \qquad (0 \leq \mathcal{K} \leq +\infty) \tag{1.1}$$

$$(\log_k^+ x = \log^+ (\log_{k-1}^+ x) \quad \textit{and} \quad \log_1^+ x = \log^+ x).$$

Then if $k \geq 3$,

$$\limsup_{r\to\infty} \frac{\log_k^+ T(r,f)}{\log r} = \mathcal{K}, \tag{1.2}$$

and if $k = 2$, we have only

$$\limsup_{r\to\infty} \frac{\log_2^+ T(r,f)}{\log r} = \tau \quad \textit{where} \quad \tau = \mathcal{K} \textit{ for } \mathcal{K} \geq 1, \quad \textit{and} \quad \tau \leq 1 \textit{ for } \mathcal{K} < 1. \tag{1.3}$$

Since the counting function $n(r, 1/f)$ of the zeros of f may grow so rapidly on some sequence of r-values that

$$\limsup_{r\to\infty} \{\log_k^+ n(r, 1/f)/\log r\} = \infty \qquad \text{for every} \quad k = 1, 2, \cdots,$$

we complement Theorem 1 with the following result in which we abandon the use of $\log r$ as a comparison function.

THEOREM 2. *Let $f(re^{i\theta})$ be entire and $h(r)$ a positive function of r. Assume that the zeros of ff'' form an A-set. If*

$$\lim_{r\to\infty} \frac{\log r}{h(r)} = 0, \tag{1.4}$$

and

$$\lim_{r\to\infty} \frac{\log_2^+ n\left(r, \frac{1}{f}\right)}{h(r)} = 0, \tag{1.5}$$

then

$$\lim_{r\to\infty} \frac{\log_2^+ T(r,f)}{h(r)} = 0, \tag{1.6}$$

provided that r avoids the values of an exceptional set of finite measure.

2. Preliminary lemmas

We remark that Theorems 1 and 2 already appear in a weaker form (involving an assumption on the zeros of f') in an unpublished chapter of the author's thesis [3, Chapter 3]. Also, the proofs we present here are considerably simpler since we shall avail ourselves of the following striking lemma due to Levin and Ostrovskii [4, Theorem 1].

LEMMA 1. *If the zeros of the entire function $f(z)$ and of its second derivative $f''(z)$ form an A-set, then*

$$\int_R^\infty \frac{m\left(r, \dfrac{f'}{f}\right)}{r^3}\, dr = O\left(\frac{\log R}{R}\right). \tag{2.1}$$

We shall also have need for Weierstrass products which do not grow unnecessarily fast and are associated with sequences of zeros whose exponent of convergence may be infinite. The following lemma provides us with such products whose growth is sufficiently restricted for our purposes.

LEMMA 2. *Let $\{a_\nu\}_{\nu=1}^\infty$ be a sequence of complex numbers (not necessarily distinct) with no finite point of accumulation, and assume that*

$$0 < |a_1| \le |a_2| \le |a_3| \le \cdots .$$

Then the function $\pi(z)$ defined by

$$\pi(z) = \prod_{\nu=1}^\infty \left(1 - \frac{z}{a_\nu}\right) \exp\left\{\frac{z}{a_\nu} + \frac{z^2}{2a_\nu^2} + \cdots + \frac{z^{p_\nu}}{p_\nu\, a_\nu^{p_\nu}}\right\} \tag{2.2}$$

where

$$p_\nu = [\log^{1+\varepsilon} \nu] \qquad (\varepsilon > 0,\ [\ \ ]\ \textit{denotes the largest integer function}) \tag{2.3}$$

is entire.

Moreover,

$$T(r, \pi) \le K\{r^{(\log^+ n(\sigma r, 1/\pi))^{1+\varepsilon}+1} + \exp\,(2/\log \sigma)^{1/\varepsilon}\} \qquad (\sigma > 1,\ K \ \ \textit{independent of} \ \ r.) \tag{2.4}$$

COROLLARY. *Let $f(z)$ be an entire function whose zeros have an infinite exponent of convergence. Then $f(z)$ may be represented in the form*

$$f(z) = z^l \pi(z) e^{Q(z)}, \tag{2.5}$$

where $l \geq 0$ is an integer, $Q(z)$ is entire and $\pi(z)$ is an entire function satisfying (2.4).

Proof. It is known [1, p. 131] that

$$\left|\left(1-\frac{z}{a_\nu}\right)\exp\left\{\frac{z}{a_\nu}+\frac{z^2}{2a_\nu^2}+\cdots+\frac{z^{p_\nu}}{p_\nu a_\nu^{p_\nu}}\right\}\right| \leq \exp\left\{\left|\frac{z}{a_\nu}\right|^{p_\nu+1}\right\}. \tag{2.6}$$

Hence in view of (2.2)

$$\log|\pi(z)| \leq \sum_{\nu=1}^{\infty}\left|\frac{z}{a_\nu}\right|^{p_\nu+1}. \tag{2.7}$$

From (2.3) we also have

$$p_\nu \leq \log^{1+\varepsilon}\nu < p_\nu + 1. \tag{2.8}$$

Now

$$\sum_{\nu=1}^{\infty}\left|\frac{z}{a_\nu}\right|^{p_\nu+1} = \sum_{|a_\nu|\leq\sigma r}\left|\frac{z}{a_\nu}\right|^{p_\nu+1} + \sum_{|a_\nu|>\sigma r}\left|\frac{z}{a_\nu}\right|^{p_\nu+1}, \qquad (\sigma > 1). \tag{2.9}$$

If $r < 1$,

$$\sum_{|a_\nu|\leq\sigma r}\left|\frac{z}{a_\nu}\right|^{p_\nu+1} \leq \sum_{|a_\nu|\leq\sigma r}\frac{1}{|a_\nu|^{p_\nu+1}}. \tag{2.10}$$

For $r \geq 1$, we have, in view of (2.8),

$$\sum_{|a_\nu|\leq\sigma r}\left|\frac{z}{a_\nu}\right|^{p_\nu+1} \leq r^{(\log^+ n(\sigma r,1/\pi))^{1+\varepsilon}+1}\sum_{|a_\nu|\leq\sigma r}\frac{1}{|a_\nu|^{p_\nu+1}}. \tag{2.11}$$

But

$$\begin{aligned}\sum_{|a_\nu|\leq\sigma r}\frac{1}{|a_\nu|^{p_\nu+1}} &= \sum_{|a_\nu|<e}\frac{1}{|a_\nu|^{p_\nu+1}} + \sum_{e\leq|a_\nu|\leq\sigma r}\frac{1}{|a_\nu|^{p_\nu+1}}\\ &\leq \sum_{|a_\nu|<e}\frac{1}{|a_\nu|^{p_\nu+1}} + \sum_{e\leq|a_\nu|\leq\sigma r}\frac{1}{e^{p_\nu+1}}.\end{aligned} \tag{2.12}$$

Moreover, from (2.8) it follows that

$$\sum_{e\leq|a_\nu|\leq\sigma r}\frac{1}{e^{p_\nu+1}} \leq \sum_{e\leq|a_\nu|\leq\sigma r}\frac{1}{e^{(\log\nu)^{1+\varepsilon}}} = \sum_{e\leq|a_\nu|\leq\sigma r}\frac{1}{\nu^{(\log\nu)^{\varepsilon}}} < +\infty. \tag{2.13}$$

Combining (2.10) or (2.11) with (2.12) and (2.13), we find that for all r,

$$\sum_{|a_\nu|\leq\sigma r}\left|\frac{z}{a_\nu}\right|^{p_\nu+1} \leq K\{r^{(\log^+ n(\sigma r,1/\pi))^{1+\varepsilon}+1} + 1\}. \tag{2.14}$$

In addition, by (2.8) putting $n(r) = n(r, 1/\pi)$, we find

$$\begin{aligned}\sum_{|a_\nu|>\sigma r}\left|\frac{z}{a_\nu}\right|^{p_\nu+1} &\leq \sum_{\nu\geq n(\sigma r)}\frac{1}{\sigma^{p_\nu+1}} \leq \sum_{\nu\geq n(\sigma r)}\frac{1}{\sigma^{(\log\nu)^{1+\varepsilon}}}\\ &= \sum_{\nu\geq n(\sigma r)}\frac{1}{\nu^{(\log\sigma)(\log\nu)^{\varepsilon}}}.\end{aligned} \tag{2.15}$$

Put $\nu_0 = [\exp\{(2/\log\sigma)^{1/\varepsilon}\} + 1]$ and observe that for $\nu > \nu_0$

$$\log\sigma(\log\nu)^{\varepsilon} > \log\sigma(\log\nu_0)^{\varepsilon} > 2.$$

Then,

$$\begin{aligned}(2.16)\qquad \sum_{\nu\geq n(\sigma r)}\frac{1}{\nu^{(\log\sigma)(\log\nu)^{\varepsilon}}} &\leq \sum_{\nu=n(\sigma r)}^{\nu_0}\frac{1}{\nu^{\log\sigma(\log\nu)^{\varepsilon}}} + \sum_{\nu_0+1}^{\infty}\frac{1}{\nu^2}\\ &\leq \nu_0 + \sum_{\nu=\nu_0+1}^{\infty}\frac{1}{\nu^2} \leq e^{(2/\log\sigma)^{1/\varepsilon}} + 1 + \frac{\pi^2}{6}.\end{aligned}$$

The conclusion of the lemma follows easily from (2.7), (2.9), (2.14) and (2.16).

We shall also find it convenient to apply the following modification of a well-known lemma of Borel [1, p. 12–16].

LEMMA 3. *Let $w(x)$ be a positive non-decreasing and unbounded function defined for $x \geq x_0$, and let α (>0) be given. If the points of discontinuity of w have no finite point of accumulation, then*

$$(2.17)\qquad w(x + 1/\log w(x)) < \{w(x)\}^{1+\alpha}$$

for all $x \geq x_0$, outside a set of values of x whose total measure is finite.

Proof. We remark first that if $w(x)$ is continuous this lemma coincides with that of Borel. Next, we show that the discontinuous case of our lemma follows easily from the continuous one.

In order to see this, we arrange the points of discontinuity of f in an increasing sequence $x_1 < x_2 < \cdots$. Given an arbitrary δ (>0) we choose a sequence of open intervals (x_n', x_n''), $(n = 1, 2, \cdots)$, such that

$$(2.18)\qquad x_n' < x_n < x_n'' < x_{n+1}',$$

and such that

$$(2.19)\qquad \sum\nolimits_n (x_n'' - x_n') \leq \delta.$$

We now define a continuous function $\tilde{w}(x)$ as follows. Put

$$\begin{aligned}\tilde{w}(x) &= w(x) &&\text{if}\quad x \notin (x_n', x_n'')\\ &= \frac{w(x_n'') - w(x_n')}{x_n'' - x_n'}(x - x_n') + w(x_n') &&\text{if}\quad x \in (x_n', x_n''),\\ & && (n = 1, 2, \cdots).\end{aligned}$$

Since $\tilde{w}(x)$ satisfies the assumptions of our lemma and is continuous, the lemma of Borel asserts that (2.17) holds for $x \geq x_0$, with $\tilde{w}$ in place of w, outside a set E of x values of finite measure. Since $w(x) = \tilde{w}(x)$ for $x \notin \bigcup_n (x_n', x_n'')$; the inequality (2.17) holds for $w(x)$ provided only that $x \notin E$ and $x \notin \bigcup_n (x_n', x_n'')$. In view of (2.19) and the fact that E is of finite measure, our conclusion follows.

3. Proofs of the theorems

Let λ denote the exponent of convergence of the zeros of f. If $\lambda < +\infty$, we first show that Theorems 1 and 2 are immediate consequences of Theorem B of Levin and Ostrovskii.

Proof of Theorems 1 and 2 for $\lambda < +\infty$. If $\lambda < +\infty$ then f may be represented in the form (i) of Theorem B where P is entire of finite order and therefore p satisfies condition (ii) of Theorem B.

Also for $\lambda < +\infty$, we have

$$\lim_{r\to\infty} \frac{\log_k^+ n\left(r, \frac{1}{f}\right)}{\log r} = 0 \qquad (k = 2, 3, \cdots). \tag{3.1}$$

Since

$$T(r, f) \le \log M(r, f), \tag{3.2}$$

it follows from (iii) of Theorem B and (3.2) that

$$\lim_{r\to\infty} \frac{\log_k^+ T(r, f)}{\log r} = 0 \qquad (k = 2, 3, \cdots), \tag{3.3}$$

which proves Theorem 1 in this case.

For $\lambda < +\infty$, in view of (1.4), Theorem 2 is an immediate consequence of Theorem 1 with $k = 2$.

Proof of Theorems 1 *and* 2 *for* $\lambda = +\infty$. Applying the corollary of Lemma 2 we represent $f(z)$ in the form (2.5) and taking the logarithmic derivative we have

$$f'(z)/f(z) = l/z + \pi'(z)/\pi(z) + Q'(z). \tag{3.4}$$

Since Q' is entire we have by definition, $T(r, Q') = m(r, Q')$ and from (3.4) and elementary properties of the Nevanlinna m-function, it follows that

$$T(r, Q') \le m(r, f'/f) + m(r, \pi'/\pi) + O(1). \tag{3.5}$$

Dividing in (3.5) by r^3 and integrating from R to σR, ($\sigma > 1$), we obtain

$$\int_R^{\sigma R} \frac{T(r, Q')}{r^3}\, dr \le \int_R^{\sigma R} \frac{m(r, f'/f)}{r^3}\, dr + \int_R^{\sigma R} \frac{m(r, \pi'/\pi)}{r^3}\, dr + O\left(\frac{1}{R^2}\right). \tag{3.6}$$

Since the zeros of ff'' form an A-set, we may apply Lemma 1 to the first integral on the right hand side of (3.6) and we find

$$\int_R^{\sigma R} \frac{T(r, Q')}{r^3}\, dr \le \int_R^{\sigma R} \frac{m(r, \pi'/\pi)}{r^3}\, dr + O\left(\frac{\log R}{R}\right). \tag{3.7}$$

(From this point on we shall find it convenient to use K to denote a constant independent of σ and r whose value may differ from one usage to the next.)

Now [5, pp. 62–63]

$$\int_R^{\sigma R} \frac{m(r, \pi'/\pi)}{r^3}\, dr \le K \int_R^{\sigma R} \frac{\log^+ T(r, \pi)}{r^3}\, dr. \tag{3.8}$$

From (3.7), (3.8) and the monotonicity of the characteristic function, we deduce that for $\sigma > 1$,

$$T(R, Q') \leq K\left(\log^+ T(\sigma R, \pi) + \frac{\sigma^2}{\sigma^2 - 1} R \log R\right). \tag{3.9}$$

Since Q is entire, we have by well-known inequalities

$$T(R, e^Q) \leq \log^+ M(R, e^Q) \leq M(R, Q) \leq RM(R, Q') + O(1) \tag{3.10}$$

and therefore, making use of a well-known inequality [5, p. 24], for $0 < R < \rho$,

$$\begin{aligned} \log^+ T(R, e^Q) &\leq \log^+ M(R, Q') + O(\log R) \\ &\leq \frac{\rho + R}{\rho - R} T(\rho, Q') + O(\log R). \end{aligned} \tag{3.11}$$

From the representation for f given by (2.5) together with (3.11) and (3.9), it follows that

$$\begin{aligned} \log^+ T(R, f) &\leq \log^+ T(R, \pi) + \log^+ T(R, e^Q) + l \log^+ R \\ &\leq K\left(\log^+ T(R, \pi) + \frac{\rho + R}{\rho - R}\left\{\log^+ T(\sigma\rho, \pi) \right.\right. \\ &\qquad \left.\left. + \frac{\sigma^2}{\sigma^2 - 1} \rho \log^+ \rho\right\}\right). \end{aligned} \tag{3.12}$$

Putting $\rho = \sigma R$, and restricting σ so that $1 < \sigma \leq e$, (3.12) implies that

$$\begin{aligned} &\log^+ T(R, f) \\ &\leq \frac{K}{(\sigma - 1)^2}\{\log^+ T(R, \pi) + \log^+ T(\sigma^2 R, \pi) + R \log^+ R\}. \end{aligned} \tag{3.13}$$

From (2.4) and (3.13) we find that

$$\begin{aligned} \log^+ T(R, f) \leq \frac{K}{(\sigma - 1)^2}&\left\{\left(\log^+ n\left(\sigma^3 R, \frac{1}{\pi}\right)\right)^{1+\varepsilon} \log^+ R \right. \\ &\left. + R \log^+ R + \left(\frac{2}{\log \sigma}\right)^{1/\varepsilon}\right\}. \end{aligned} \tag{3.14}$$

Taking $\log^+$ of the two sides of (3.14) and applying the elementary inequality

$$\begin{aligned} \log^+ (\alpha + \beta + \gamma) &\leq \log^+ \{3 \max (\alpha, \beta, \gamma)\} \\ &\leq \max \{\log^+ \alpha, \log^+ \beta, \log^+ \gamma\} + \log 3, \quad (\alpha, \beta, \gamma \quad \text{real}), \end{aligned}$$

we arrive at

$$\begin{aligned} \log_2^+ T(R, f) \leq \max \{&(1 + \varepsilon) \log_2^+ n(\sigma^3 R, 1/\pi) + \log_2^+ R, \\ &\log^+ R + \log_2^+ R, 1/\varepsilon \log^+ (2/\log \sigma)\} \\ &+ 2 \log^+ 1/(\sigma - 1) + \log^+ K + \log 3. \end{aligned} \tag{3.15}$$

To prove Theorem 1, we first note that fixing $\sigma = e$ in (3.15) and recalling that the zeros of $f(z)$ and $z^l\pi(z)$ are the same, it follows that for $k \geq 3$,

$$\begin{aligned} \log_k^+ T(R,f) &\leq \log_k^+ n(e^3R, 1/\pi) + O(\log_{k-1} R) \\ &\leq \log_k^+ n(e^3R, 1/f) + O(\log_{k-1} R). \end{aligned} \tag{3.16}$$

Further, by the First Fundamental Theorem and the monotonicity of the n-function we also have

$$\begin{aligned} T(R,f) \geq N\left(R, \frac{1}{f}\right) + O(1) &\geq \int_{R/e}^{R} \frac{n\left(r, \frac{1}{f}\right)}{r}\,dr + O(1) \\ &\geq n\left(\frac{R}{e}, \frac{1}{f}\right) + O(1). \end{aligned} \tag{3.17}$$

From (3.17) it follows easily that

$$\limsup_{R\to\infty} \frac{\log_k^+ n\left(R, \frac{1}{f}\right)}{\log R} \leq \limsup_{R\to\infty} \frac{\log_k^+ T(R,f)}{\log R} \qquad (k = 2, 3, \cdots). \tag{3.18}$$

If $k \geq 3$, Theorem 1 now follows from (3.18) and (3.16).

For $k = 2$, we make use of (3.15) with $\sigma = e$ to obtain

$$\limsup_{R\to\infty} \frac{\log_2^+ T(R,f)}{\log R} \leq \max\left\{(1+\varepsilon) \limsup_{R\to\infty} \frac{\log_2^+ n\left(R, \frac{1}{f}\right)}{\log R}, 1\right\}. \tag{3.19}$$

Since ε (>0) is arbitrary, (3.18) together with (3.19) imply (1.3) which completes the proof of Theorem 1.

In order to prove Theorem 2 for $\lambda = +\infty$, we choose an $R_0 > 2$ such that $n(R_0, 1/\pi) \geq 3$ and define

$$\sigma = \sigma(R) = \left\{1 + \frac{1}{R \log n\left(R, \frac{1}{\pi}\right)}\right\}^{1/3}, \qquad (R \geq R_0). \tag{3.20}$$

Using the elementary inequalities: $\log(1 + x) > x/2$ for $0 < x < 1$ and $(1 + x)^{1/3} - 1 > (x/3)2^{-2/3} > x/6$ for $0 < x < 1$, it follows from (3.20) that for $R > R_0$,

$$\log^+ (2/\log \sigma) \leq \log (12R \log n(R, 1/\pi)), \tag{3.21}$$

and

$$\log^+ 1/(\sigma - 1) \leq \log (6R \log n(R, 1/\pi)). \tag{3.22}$$

In addition, applying Lemma 3 to the function $n(R, 1/\pi)$ with $\alpha = 1$, we have for $R \geq R_0$,

$$\begin{aligned} &\log_2^+ n\left(\sigma^3 R, \frac{1}{\pi}\right) \\ (3.23)\qquad &= \log\left\{\log n\left(R + \frac{1}{\log n\left(R, \frac{1}{\pi}\right)}\right)\right\} \leq \log_2^+ n\left(R, \frac{1}{\pi}\right) + \log 2 \end{aligned}$$

outside a set E of R values whose total measure is finite.

In view of (3.21), (3.22), and (3.23) we find from (3.15) that

$$\begin{aligned} \log_2^+ T(R, f) \leq \max\{&(1 + \varepsilon) \log_2^+ n(R, 1/\pi) + \log_2 R + \log 2, \\ (3.24)\qquad &\log^+ R + \log_2^+ R, (1/\varepsilon) \log (12R \log n(R, 1/\pi)) \\ &+ 2 \log (6R \log n(R, 1/\pi)) + \log^+ K\} \\ &(\varepsilon > 0, R \notin E). \end{aligned}$$

Dividing in (3.24) by $h(R)$ and making use of the conditions (1.4) and (1.5), we easily deduce

$$(3.25)\qquad \lim_{R\to\infty} \frac{\log_2 T(R, f)}{h(R)} = 0.$$

This completes our proof.

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University of Wisconsin
Madison, Wisconsin