ON CERTAIN PROBLEMS OF MINIMAL CAPACITY

BY

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1. A number of theorems in Function Theory can be characterized as expressing the fact that a certain compact plane set has minimal capacity within a given class of plane sets. By and large these can be traced back to the inspiration of the diameter theorem (see [2, p. 3]) in the theory of univalent functions. We single out in particular two results, one given some years ago by Pólya [5], the other quite recently by Pfluger [4]. Pólya's result is derived from a theorem of his concerning transfinite diameter, but in his explicit geometric result the extremal set displays certain rotational symmetry. On the other hand, Pfluger's extremal sets display reflectional symmetry. In each case the comparison sets are to comprise continua satisfying certain conditions. In this paper, by employing standard techniques of the method of the extremal metric, we will see that neither of these conditions is basic to the problem, but that they can be utilized to verify the conditions which we shall give. We will also point out relationships to a result of Rengel [6].

2. Let E be a set in the z-plane consisting of a finite number of Jordan arcs (specifically not Jordan curves) such that its complement D on the z-sphere is connected (thus a domain). Let $g(z)$ be the Green's function of D with logarithmic pole at the point at infinity. It is well known that an orthogonal trajectory of the level curves of g, apart from a finite number of exceptions, will be an open arc with limiting end points at the point at infinity and a point of E. Every point of E will be a limiting end point for two such orthogonal trajectories, with at most a finite number of exceptions. Let $\mathfrak{L}$ be the set of orthogonal trajectories which occur in such pairs and let T be the involutory transformation defined on $\mathfrak{L}$ by associating with an element of $\mathfrak{L}$ the other one with the same end point on E. There is a natural metric determined on $\mathfrak{L}$ by the variation of the conjugate of the Green's function. We will denote it by $d\mu$. In particular $\int_{\mathfrak{L}} d\mu = 2\pi$. We are now ready to state our principal result.

THEOREM. *Let E be a set consisting of a finite number of Jordan arcs in the z-plane such that its complement D on the z-sphere is connected.*

(a) *Let the involutory transformation T be measure preserving in the metric $d\mu$.*

(b) *Let S be a compact set in the z-plane such that if $l \in \mathfrak{L}$, S meets either $\bar{l}$ or Tl.*

Then

$$c(S) \geq c(E)$$

Received April 12, 1965.

[1] Research supported in part by the National Science Foundation.

where c denotes the capacity of the set in question. Equality can occur only if S differs from E at most by a set of capacity zero.

Let a compact plane set Q have positive capacitary density at each of its points and let $g(Q, z)$ be its Green's function with logarithmic pole at the point at infinity. The level curve $\lambda(Q, k) : g(Q, z) = k$ for k sufficiently large is a Jordan curve behaving asymptotically like the circle $| z | = c(Q)e^k$. Let $\Delta(Q, k)$ be the domain bounded by Q and $\lambda(Q, k)$ and $\Gamma(Q, k)$ be the class of locally rectifiable curves running in $\Delta(Q, k)$ from Q to $\lambda(Q, k)$. Let $m(Q, k)$ be the module [3, p. 13] of this class of curves. It is well known that $m(Q, k) = 2\pi k^{-1}$, the extremal metric being $k^{-1}| \operatorname{grad} g(Q, z) |$.

In the situation of our theorem let Q be the subset of S consisting of those points at which S has positive capacitary density. Evidently Q is compact and $c(Q) = c(S)$. It is readily verified that Q will likewise satisfy condition (b).

The transformation T induces a point transformation $\mathfrak{T}$ on a subset $\tilde{D}$ of D obtained by deleting a finite number of open analytic arcs and points (i.e., $\tilde{D}$ is the point set union of $l \in \mathfrak{L}$) by taking $\mathfrak{T}(P)$ for $P \in l$ to be the point on Tl with

$$g(E, \mathfrak{T}(P)) = g(E, P).$$

Under condition (a), $\mathfrak{T}$ is an anticonformal mapping on $\tilde{D}$ thus we can speak of its distortion $\tau(P)$.

Now consider $\lambda(E, k)$ for k sufficiently large. There will be a level curve $\lambda(Q, k')$ lying inside $\lambda(E, k)$ and touching it with

$$k' = k + \log (c(E)/c(Q)) + o(1).$$

Let $\rho(z)$ be the extremal metric for $m(Q, k')$. Let

$$\begin{aligned} \rho'(z) &= \rho(z) \quad \text{for } z \text{ in } \Delta(E, k) \cap \Delta(Q, k') \\ &= 0 \qquad \text{elsewhere in } \Delta(E, k). \end{aligned}$$

Let

$$\begin{aligned} \tilde{\rho}(z) &= \tfrac{1}{2}(\rho'(z) + \tau(z)\rho'(\mathfrak{T}z)) \quad \text{for } z \text{ in } \Delta(E, k) \cap \tilde{D} \\ &= 0 \qquad \text{elsewhere in } \Delta(E, k). \end{aligned}$$

If $l(k)$ denotes the (open) arc on $l \in \mathfrak{L}$ for which $0 < g(E, z) < k$ we have

$$\int_{l(k)} \tilde{\rho}(z) | dz | \geq 1.$$

Since the $l(k)$ are precisely those curves in $\Gamma(E, k)$ which have length 1 in the extremal metric for the module problem defining $m(E, k)$ we have

$$\iint_{\Delta(E,k)} \tilde{\rho}^2(z) \, dA \geq m(E, k)$$

where dA denotes the element of area in the z-plane. Moreover

$$\iint_{\Delta(E,k)} \tilde{\rho}^2(z)\, dA = \tfrac{1}{4} \iint_{\Delta(E,k)} (\rho'(z) + \tau(z)\rho'(\mathfrak{T}z))^2\, dA$$

$$\leq \tfrac{1}{2} \iint_{\Delta(E,k)} (\rho'(z))^2\, dA + \tfrac{1}{2} \iint_{\Delta(E,k)} (\rho'(\mathfrak{T}z))^2(\tau(z))^2\, dA.$$

This last term is just $\iint_{\Delta(E,k)} (\rho'(z))^2\, dA \leq m(Q, k')$ (actually $=$ since E has zero area, see §5). Thus

$$m(Q, k') \geq m(E, k)$$

or

$$k \geq k' = k + \log (c(E)/c(Q)) + o(1).$$

So finally

$$c(Q) \geq c(E)$$

or

$$c(S) \geq c(E).$$

The standard equality argument in the method of the extremal metric [3, p. 20] shows that the inequality is strict unless Q coincides with E thus unless S differs from E only by a set of capacity zero.

3. We now wish to verify that the results mentioned in §1 actually are special cases of our theorem.

COROLLARY 1 (Pólya). *Let a and b be two distinct complex numbers, m a positive integer, such that*

$$0 \leq \arg (b/a) < \pi/m, \quad |a| > 0, \quad |b| > 0$$

and let $\omega = e^{2\pi i/m}$. Let S be a compact plane set which contains m (not necessarily disjoint) subcontinua, of which the j^{th} subcontinuum contains the two points $a\omega^{j-1}$ and $b\omega^{j-1}$, $j = 1, \cdots, m$. Then the capacity $c(S)$ satisfies

$$c(S) \geq |\tfrac{1}{4}(a^m - b^m)|^{1/m}.$$

Equality occurs for the set E consisting of m arcs which are the images of the segment joining a^m and b^m under the m^{th} root transformation and for a set S only if it differs from E at most by a set of capacity zero.

Let σ be the segment joining a^m and b^m. Then

$$g(E, z) = (1/m)g(\sigma, z^m).$$

From this it is clear that

$$c(E) = |\tfrac{1}{4}(a^m - b^m)|^{1/m}$$

and further that condition (a) of our theorem is satisfied. It is trivial that our condition (b) follows from the existence in S of subcontinua joining the end points of each of the arcs comprising E.

COROLLARY 2 (Pfluger). *Let E be a plane set consisting of a finite number of slits lying on rays emanating from the origin and reflectionally symmetric in each ray on which such slits occur. Let S be a compact plane set corresponding to which there exists a pairing of the end points of the segments in E such that each such pair can be joined by segments exterior to E on the ray adjacent to the end point in question plus a continuum in S. Then*

$$c(S) \geq c(E)$$

and equality occurs only if S differs from E at most by a set of capacity zero.

That our condition (a) is satisfied is trivial from the symmetry property of E. To verify condition (b) we note first that each $l \in \mathfrak{L}$ runs from the point at infinity to E within one of the angles bounded by adjacent rays bearing slits of E. If then S did not meet $\bar{l}$ or Tl these would form together a Jordan curve on the z-sphere such that each of the sets complementary to it would contain an odd number of end points of slits in E. This contradicts the existence of a pairing of end points as in the statement of Corollary 2.

Not only does our theorem contain Pfluger's result, but the immediate conclusion drawn from it is sharper for numerous examples. Indeed consider Pfluger's example [4, p. 285] where E consists of segments joining the points $a\omega^k$, $b\omega^k$, $0 < a < b$, $\omega = e^{\pi i/l}$, $k = 1, 2, \cdots, 2l$. Then if S consists of the circular arcs

$$\{be^{i\varphi} : k\pi/l \leq \varphi \leq (k + 1)\pi/l\}$$

$k = 1, 3, \cdots, 2l - 1$, by the remarks of the preceding paragraph it is clear that from our theorem follows

$$c(S) \geq c(E).$$

On the other hand Pfluger's enunciation led him to include in S also the arcs

$$\{ae^{i\varphi} : (k - 1)\pi/l \leq \varphi \leq k\pi/l\}$$

$k = 1, 3, \cdots, 2l - 1$. Of course, he could have noted that by taking these latter arcs on the circle of radius $r < a$ and letting r tend to zero, he could have attained the same conclusion as above.

4. If the set E considered in §2 is connected so that its complementary domain is simply-connected, the finite number of orthogonal trajectories not in the set $\mathfrak{L}$ each has one limiting end point in E and its other limiting end point at the point at infinity. By an exceptional orthogonal trajectory we mean one of this number such that the limiting end point in E is the end point of one of the Jordan arcs comprising E and not on any other such Jordan arc.

COROLLARY 3. *Let E be a set consisting of a finite number of Jordan arcs in the z-plane such that its complement D on the z-sphere is connected and simply-*

connected and such that condition (a) *is satisfied. Then if S is a continuum in the z-plane meeting the closure of each exceptional orthogonal trajectory*

$$c(S) \geq c(E).$$

Equality can occur only if S coincides with E.

Indeed if $l \in \mathfrak{L}$ the Jordan curve formed from $\bar{l}$ and Tl will separate some two exceptional orthogonal trajectories. Thus S must meet $\bar{l}$ or Tl and condition (b) is satisfied.

Corollary 3 contains as a special case a familiar result of Rengel [6]. As above, the symmetry condition imposed by Rengel is effective only in verifying condition (a) and identifying the exceptional orthogonal trajectories. It should be pointed out that an even more direct proof of Rengel's result by the method of the extremal metric has long been known (at least to the author and probably to others) but not published.

We recall also that the continuum of minimal capacity containing a fixed finite set of points is a set E satisfying the conditions in Corollary 3 and that the limiting end points of the exceptional orthogonal trajectories in E are points of the fixed set. Thus such a continuum has minimal capacity also in a larger family of competing sets.

5. It is desirable to give a further characterization of the sets E and domains D which satisfy the conditions of our theorem. Let

$$\zeta = g(E, z) + ih(E, z)$$

where $h(E, z)$ denotes the (multiple-valued) conjugate of the Green's function. We see at once that $d\zeta^2$ is a quadratic differential on the z-sphere with a double pole at the point at infinity and a simple pole at the end point of any arc in E which is not on any other such arc. Moreover, each arc comprising E is made up of one or more trajectories together with their limiting end points. The domain D is thus an admissible domain for this quadratic differential [3, p. 49]. However, we do not obtain the most general quadratic differential with these characteristics, since in the present situation we cannot have orthogonal trajectories with a limiting end point on E at each end. We can use the standard construction methods for quadratic differentials [2] to obtain sets E which satisfy the conditions of our theorem, but which display no sort of reflectional or rotational symmetry.

We should remark that proofs along the line of that used for our theorem have long been known in simple special cases and indeed go back at least to Grötzsch [1]. Finally, there are numerous fairly obvious generalizations of the preceding methods, for example, to domains of infinite connectivity.

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