GENERALIZED SCALAR OPERATORS WHOSE SPECTRA ARE CONTAINED IN A JORDAN CURVE

BY

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Introduction. In the previous paper [5], the author defined the *canonical* spectral representation for a C_c^m-scalar operator whose spectrum is contained in the real line. Such an operator will be called a C^m-real operator in this paper. H. G. Tillmann [9] and S. Kantorovitz [2] also treated similar kinds of operators on a Banach space and they gave characterizations for an operator to be C^m-real. One type of characterization was given in terms of the rate of growth of resolvents near the spectrum and another in terms of certain one-parameter group constructed from the original operator. The same types of characterization have also been discussed by F. Wolf [11] and S. Kantorovitz [2] for operators on a Banach space whose spectra are contained in the unit circle. Such operators may be called C^m-unitary. They generalize the notion of unitary operators on a Hilbert space, just as C^m-real operators generalize the notion of Hermitian operators (cf. [5], [9] and [2]).

Now, it would be natural to think that analogous discussions may hold if we replace the real line or the unit circle by a more general Jordan curve. In fact, for spectral operators on a Banach space, N. Dunford [1] gave some characterization theorems in terms of resolvents in the case the spectrum is contained in a Jordan curve.

Thus, in this paper, we consider a C_c^m-scalar operator S on a locally convex space such that its spectrum $\mathrm{Sp}(S)$ is contained in a C^m-Jordan curve and we shall be concerned with the following two problems: (a) Definition and existence of the spectral representations for S which are entitled to be called *canonical*; (b) Extension of characterization theorems for such an operator, especially for a C^m-unitary operator or a C^m-real operator on a locally convex space.

In Part I, we treat the case the curve is bounded. If the curve Δ is represented by a C^m-function, we can induce a natural C^m-structure on Δ. Then it is possible to talk about an operator of class $C^m(\Delta)$ (cf. [2]). It will be shown that the $C^m(\Delta)$-representation for such an operator is uniquely determined; this fact leads us to a definition of the canonical representation. We shall give two existence theorems for canonical representations, one of which is a corollary to a characterization theorem (§1.4). The characterization theorems, especially for C^m-unitary operators, will turn out to be similar to the results in F. Wolf [11] (and also in [2] and [9]), but our results extend and improve them. They may be summarized in the form of Theorem 3 in [6].

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In Part II, the case the curve is unbounded is treated. Definition of canonical representations can be discussed parallel to Part I. Essential difference of discussions in this part from those in Part I appear in the case $\mathrm{Sp}(S)$ is unbounded. In this case, however, we are unable to obtain general characterization theorems corresponding to those in Part I. Only some results on C^m-real operators will be given. We remark here that the same implication diagram as Theorem 3, (i) in [6] will hold for C^m-real operators having compact spectra, with certain modifications in the statements.

In the appendix, elementary proofs of approximation theorems, Lemma 1.1 and Lemma 2.1, will be given.[2]

Preliminaries. Throughout this paper, let E be a Hausdorff locally convex space over the complex field C such that $L(E)$, the space of all continuous linear operators on E into itself, is quasi-complete with respect to the bounded convergence topology τ_b.

Let m be a non-negative integer or $m = \infty$ and let C^m be the algebra of all complex-valued m-times continuously differentiable functions on $R^2(\cong C)$ and C_c^m be the subalgebra of C^m consisting of functions in C^m with compact support. We introduce the usual topologies in these spaces. (The topology of C^m is defined by uniform convergence of partial derivatives of order up to m on compact sets. For C_c^m, see e.g. [5, §1].)

A C_c^m-spectral (resp. C^m-spectral) representation is a continuous algebra homomorphism U of C_c^m (resp. C^m) into $L(E)$ with τ_b such that there exists a net $\{\varphi_\alpha\}$ in C_c^m with the property that $U(\varphi_\alpha)x \to x$ for each $x \in E$. (If U is C^m-spectral, then it follows that $U(1) = I$.) Such a net $\{\varphi_\alpha\}$ will be called *an identity net* for U.

Since C_c^m is dense in C^m, any C^m-spectral representation is C_c^m-spectral. Conversely, if U is a C_c^m-spectral representation and if the support of U ($\mathrm{Supp}\, U$) is compact, then U is C^m-spectral. Furthermore, in this case, we can define $U(\varphi)$ for a function φ defined and m-times continuously differentiable on a neighborhood σ of $\mathrm{Supp}\, U$ as the operator $U(\varphi\varphi_0)$, where $\varphi_0 \in C_c^m$ is equal to 1 on a neighborhood of $\mathrm{Supp}\, U$ and $\mathrm{Supp}\, \varphi_0 \subset \sigma$.

For the continuity of U, we remark: it is enough to assume that U is continuous with respect to the simple convergence topology in $L(E)$, since the continuity with respect to τ_b follows from the facts that C_c^m (resp. C^m) is bornological and that E is quasi-complete.

Let S be a closed linear transformation on E into itself and let D_S be its domain. S is called a C_c^m-scalar transformation (a C_c^m-scalar operator if $S \in L(E)$) if there exists a C_c^m-spectral representation U such that $U(\varphi)x \in D_S$ for each $\varphi \in C_c^m$, $x \in E$ and

$$(*) \qquad \lim_\alpha U(\lambda\varphi_\alpha)x = Sx \qquad \text{for all } x \in D_S,$$

[2] While preparing the manuscript, the author communicated with Professor E. Bishop who indicated the same proofs.

where $\{\varphi_\alpha\}$ is an identity net for U. (For $\varphi \in C_c^m$, $\lambda\varphi$ denotes the function $\lambda \to \lambda\varphi(\lambda)$, $\lambda \in C$. Similarly, the identity function $\varphi(\lambda) \equiv \lambda$ will be denoted by λ.) The condition $(*)$ can be replaced by

$$(*)' \qquad U(\varphi)Sx = U(\lambda\varphi)x \qquad \text{for all} \quad \varphi \in C_c^m \quad \text{and} \quad x \in D_S,$$

or by

$$(*)'' \qquad U(\varphi)S = SU(\varphi) \quad \text{on} \quad D_S \quad \text{and} \quad SU(\varphi) = U(\lambda\varphi).$$

Thus the definition of a C_c^m-scalar transformation does not depend on the choice of an identity net $\{\varphi_\alpha\}$. It follows that D_S is dense in E and is equal to

$$\{x \in E;\ \lim_\alpha U(\lambda\varphi_\alpha)x \quad \text{exists.}\}.$$

Also, we know that Supp $U \subseteq \mathrm{Sp}(S)$. ($\mathrm{Sp}(S)$ denotes the *spectrum* of S in Waelbroeck's sense. See [3] or [5, §1].)

$S \in L(E)$ is called a C^m-scalar operator if there exists a C^m-spectral representation U such that $S = U(\lambda)$. A C^m-scalar operator is a C_c^m-scalar operator. Conversely, if S is a C_c^m-scalar transformation and if Sp (S) is compact, then S is a C^m-scalar operator.

Most of these notions and properties are found in [3] and [5].

Part I. The case of bounded curves

1.1. C^m-curve and $C^m(\gamma)$-scalar operators. In Part I, we consider a Jordan curve, i.e., a closed curve in the complex plane C which is homeomorphic to the unit circle $\Gamma = \{\lambda \in C; |\lambda| = 1\}$. More precisely, we define:

DEFINITION 1.1. A compact set Δ in C is called a *C^m-curve* (m: integer ≥ 0 or $m = \infty$), if there exists a one-to-one continuous mapping γ of Γ into C such that

(i) $\gamma(\Gamma) = \Delta$;

(ii) γ can be extended to an open neighborhood σ of Γ (the extended map will also be denoted by γ) in such a way that γ is one-to-one on σ and γ and γ^{-1} are both m-times continuously differentiable on σ and $\gamma(\sigma)$ respectively as functions in two real variables.

The mapping γ is called a representation of Δ.

The unit circle Γ is a one-dimensional C^∞-manifold with the natural differential structure. Let $C^m(\Gamma)$ be the algebra of all complex-valued m-times continuously differentiable functions on Γ. Thus, a complex-valued function f on Γ belongs to $C^m(\Gamma)$ if and only if the periodic function $\tilde{f} : \tilde{f}(\theta) = f(e^{i\theta})$ of a real variable θ with period 2π is m-times continuously differentiable with respect to θ. We introduce the usual topology (defined by uniform convergence of derivatives of order up to m) in $C^m(\Gamma)$.

If γ is a representation of a C^m-curve Δ, then this mapping induces a differential structure on Δ, so that Δ is regarded as a one-dimensional C^m-manifold.

The space $C^m(\Delta)$ is isomorphic to $C^m(\Gamma)$ by the mapping $g \leftrightarrow g \circ \gamma$. This observation justifies the following definition (cf. [2]):

DEFINITION 1.2. Let γ be a representation of a C^m-curve. $S \in L(E)$ is called a *$C^m(\gamma)$-operator* if there exists a continuous algebra homomorphism W of $C^m(\Gamma)$ into $L(E)$ such that $W(1) = I$ and $W(\gamma) = S$. If γ is the identity map: $\gamma(e^{i\theta}) \equiv e^{i\theta}$ (so that $\gamma(\Gamma) = \Gamma$), then a $C^m(\gamma)$-operator is called a *C^m-unitary operator.*

Here, we remark again (see Preliminaries) that the topology of $L(E)$ may be either τ_b or the simple convergence topology for the continuity of W.

We shall show that the homomorphism W in the above definition is uniquely determined by S and γ. First, we state the following approximation theorem, a proof of which will be given in the appendix:

LEMMA 1.1 (Approximation Theorem). *Let γ be a representation of a C^m-curve and let λ_0 be a point lying inside the Jordan curve $\gamma(\Gamma)$. Then the set of functions of the form $Q \circ (\gamma - \lambda_0)$, where $Q(z) = P(z)/z^n$ with a polynomial P and an integer $n \geqq 0$, is dense in $C^m(\Gamma)$.*

THEOERM 1.1. *Let γ be a representation of a C^m-curve. If W_1 and W_2 are continuous homomorphisms of $C^m(\Gamma)$ into $L(E)$ such that $W_1(1) = W_2(1) = I$ and $W_1(\gamma) = W_2(\gamma)$, then $W_1 \equiv W_2$.*

Proof. Let λ_0 be a point inside $\gamma(\Gamma)$. Then

$$\frac{1}{\gamma - \lambda_0} \in C^m(\Gamma)$$

and

$$\begin{aligned} W_1\left(\frac{1}{\gamma - \lambda_0}\right) &= W_1\left(\frac{1}{\gamma - \lambda_0}\right)[W_2(\gamma) - \lambda_0 I]\, W_2\left(\frac{1}{\gamma - \lambda_0}\right) \\ &= W_1\left(\frac{1}{\gamma - \lambda_0}\right)[W_1(\gamma) - \lambda_0 I]\, W_2\left(\frac{1}{\gamma - \lambda_0}\right) \\ &= W_2\left(\frac{1}{\gamma - \lambda_0}\right). \end{aligned}$$

Hence, it follows that $W_1(Q \circ (\gamma - \lambda_0)) = W_2(Q \circ (\gamma - \lambda_0))$ for any Q given in Lemma 1.1. Hence this lemma and the continuity of W_1 and W_2 imply that $W_1(f) = W_2(f)$ for all $f \in C^m(\Gamma)$.

By this theorem, we see that W in Definition 1.2 is uniquely determined by S and γ. We shall call W the *$C^m(\gamma)$-representation* for S.

Given a representation γ of a C^m-curve, we consider the function

$$u_\gamma = \frac{\gamma^{-1}}{|\gamma^{-1}|} \equiv e^{i \arg \gamma^{-1}}$$

defined and m-times continuously differentiable on a neighborhood of $\gamma(\Gamma)$. u_γ coincides with γ^{-1} on $\gamma(\Gamma)$.

DEFINITION 1.3. Let U be a C_c^m-spectral representation. We say that U *satisfies the condition* (γ), if

$$\operatorname{Supp} U \subseteq \gamma(\Gamma) \quad \text{and} \quad U(\varphi \circ \gamma \circ u_\gamma) = U(\varphi) \quad \text{for all} \quad \varphi \in C_c^m.$$

The following theorem is a consequence of Theorem 1.1 in [3] and the above definition:

THEOREM 1.2. *Let S be a $C^m(\gamma)$-operator for a representation γ of a C^m-curve and let W be the $C^m(\gamma)$-representation for S. Then*

(i) *S is a C_c^m-scalar operator such that* $\operatorname{Sp}(S) \subseteq \gamma(\Gamma)$;

(ii) $U_0 : U_0(\varphi) = W(\varphi \circ \gamma)$ *for $\varphi \in C_c^m$ defines a C_c^m-spectral representation for S satisfying the condition (γ).*

A remark on the condition (γ). There are some alternate forms for the condition (γ); if U is a C_c^m-spectral representation and if $\operatorname{Supp} U \subseteq \gamma(\Gamma)$ for a representation γ of a C^m-curve, then the following conditions are mutually equivalent:

(i) $U(\varphi) = U(\varphi \circ \gamma \circ u_\gamma)$ for all $\varphi \in C_c^m$;

(ii) $U(\gamma^{-1}) = U(\overline{1/\gamma^{-1}})$;

(iii) $U(\varphi) = U(\varphi \circ \gamma \circ (\overline{1/\gamma^{-1}}))$ for all $\varphi \in C_c^m$;

(iv) $U(|\gamma^{-1}|) = I$.

Furthermore, if γ is analytic (i.e., if γ is a one-to-one holomorphic function on a neighborhood of Γ), then each one of (i)–(iv) is equivalent to

(v) $U(\lambda) = U(\gamma \circ (\overline{1/\gamma^{-1}}))$.

In particular, if γ is the identity map (so that $\operatorname{Supp} U \subseteq \Gamma$), then the following conditions are mutually equivalent:

(i)′ $U(\varphi_\Gamma) = U(\varphi)$ for all $\varphi \in C_c^m$, where $\varphi_\Gamma(\lambda) = \varphi(\lambda/|\lambda|)$;

(ii)′ $U(\lambda) = U(1/\bar\lambda)$;

(iii)′ $U(\hat\varphi) = U(\varphi)$ for all $\varphi \in C_c^m$, where $\hat\varphi(\lambda) = \varphi(1/\bar\lambda)$;

(iv)′ $U(|\lambda|) = I$.

To prove the equivalence, we use Proposition 2 in [5]. The equivalence of (i)–(iv) can be reduced to the equivalence of (i)′–(iv)′.

1.2. Canonical representation. Suppose $S \in L(E)$ is a C_c^m-scalar operator such that $\operatorname{Sp}(S)$ is contained in a C^m-curve Δ and suppose U is a C_c^m-spectral representation for S. If there exists a representation γ of Δ such that $U(\lambda) = U(\gamma \circ u_\gamma)$ (in particular, if U satisfies the condition (γ)), then $W(f) = U(f \circ u_\gamma)$ defines the $C^m(\gamma)$-representation for S, so that S is a $C^m(\gamma)$-operator. Thus, the theorems in the previous section indicate that C_c^m-spectral representation U satisfying the condition (γ), if it exists, is uniquely determined by S and Δ . The following theorem asserts that this is the case:

THEOREM 1.3. *Let U_1 and U_2 be two C_c^m-spectral representations with compact supports and suppose $U_1(\lambda) = U_2(\lambda)$, so that $\Sigma \equiv \operatorname{Supp} U_1 = \operatorname{Supp} U_2$. Let*

$\gamma_i (i = 1, 2)$ be a representation of a C^m-curve. If $\gamma_1(\Gamma) \cap \sigma = \gamma_2(\Gamma) \cap \sigma$ for some neighborhood σ of Σ and if each $U_i (i = 1, 2)$ satisfies the condition (γ_i), then $U_1 \equiv U_2$.

Proof. Let

$$W_1(f) = U_1(f \circ u_{\gamma_1})$$
$$W_2(f) = U_2(f \circ \gamma_1^{-1} \circ \gamma_2 \circ u_{\gamma_2})$$

for $f \in C^m(\Gamma)$. By our assumption that $\gamma_1(\Gamma) \cap \sigma = \gamma_2(\Gamma) \cap \sigma$, we see that W_2 is well defined. It is easy to see that W_1 and W_2 are continuous homomorphisms of $C^m(\Gamma)$ into $L(E)$ and $W_1(1) = W_2(1) = I$. Furthermore,

$$\begin{aligned} W_1(\gamma_1) &= U_1(\gamma_1 \circ u_{\gamma_1}) = U_1(\lambda) = U_2(\lambda) = U_2(\gamma_2 \circ u_{\gamma_2}) \\ &= U_2(\gamma_1 \circ \gamma_1^{-1} \circ \gamma_2 \circ u_{\gamma_2}) = W_2(\gamma_1). \end{aligned}$$

Hence, by Theorem 1.1, we have $W_1 \equiv W_2$. Therefore, for any $\varphi \in C_c^m$,

$$\begin{aligned} U_1(\varphi) &= U_1(\varphi \circ \gamma_1 \circ u_{\gamma_1}) = W_1(\varphi \circ \gamma_1) = W_2(\varphi \circ \gamma_1) \\ &= U_2(\varphi \circ \gamma_1 \circ \gamma_1^{-1} \circ \gamma_2 \circ u_{\gamma_2}) \\ &= U_2(\varphi \circ \gamma_2 \circ u_{\gamma_2}) = U_2(\varphi). \end{aligned}$$

DEFINITION. 1.4. Let S be a C_c^m-scalar operator such that $\mathrm{Sp}(S)$ is contained in a C^m-curve Δ. A C_c^m-spectral representation U for S is called *canonical with respect to* Δ if there exists a representation γ of Δ for which U satisfies the condition (γ).

By the above theorem, we see that the canonical representation is uniquely determined by S and Δ.

The following is a consequence of Theorem 1.2 and the observation at the beginning of this section:

COROLLARY. *Let $S \in L(E)$ and suppose $\mathrm{Sp}(S)$ is contained in a C^m-curve Δ. Then S is a C_c^m-scalar operator having the canonical representation with respect to Δ if and only if it is a $C^m(\gamma)$-operator for a representation γ of Δ.*

Example 1.1. Let Ψ be an algebra of complex-valued functions on a set and suppose Ψ contains constants. Let V be a homomorphism of Ψ into $L(E)$ such that $V(1) = I$. Suppose $f \in \Psi$ satisfies the following conditions:

(i) $\varphi \circ f \in \Psi$ for all $\varphi \in C_c^m$,
(ii) $\varphi \rightarrow V(\varphi \circ f)$ is a continuous mapping of C_c^m into $L(E)$,
(iii) The image of f is contained in a C^m-curve Δ.

Then, $V(f)$ is a C_c^m-scalar operator whose spectrum is contained in Δ and $U : U(\varphi) = V(\varphi \circ f)$ is the canonical representation for $V(f)$ with respect to Δ. In this case, the canonical representation is uniquely determined by $V(f)$ only.

Let S be a C_c^m-scalar operator such that $\mathrm{Sp}(S) = \Delta$ for some C^m-curve Δ. Then there is no other C^m-curve containing $\mathrm{Sp}(S)$. Hence, in this case, *the canonical representation for S* is uniquely determined by S. If $\mathrm{Sp}(S)$ does not coincide with a C^m-curve, then the example below (Example 1.2) shows that there can exist two different C^m-curves Δ_1 and Δ_2 such that $\mathrm{Sp}(S) \subseteq \Delta_1 \cap \Delta_2$ and two different C_c^m-spectral representations U_1 and U_2 for S which are canonical with respect to Δ_1 and Δ_2 respectively.

Remark 1.1. If there exists a neighborhood σ of $\mathrm{Sp}(S)$ such that $\Delta_1 \cap \sigma = \Delta_2 \cap \sigma$, then U_1 and U_2 in the above argument coincide by Theorem 1.3.

Example 1.2. Let $Q \in L(E)$ be nilpotent, i.e., $Q^2 = 0$. Then $\mathrm{Sp}(Q) = \{0\}$. Let

$$\Delta_1 = \Gamma + i \equiv \{\lambda \in C;\ |\lambda - i| = 1\},$$
$$\Delta_2 = \Gamma + 1 \equiv \{\lambda \in C;\ |\lambda - 1| = 1\},$$

$\gamma_1(\lambda) = \lambda + i$ and $\gamma_2(\lambda) = \lambda + 1$. γ_1 and γ_2 are representations of Δ_1 and Δ_2 respectively. Let

$$U_1(\varphi) = \varphi(0)I + \frac{\partial\varphi}{\partial\xi}(0)Q \quad \text{and} \quad U_2(\varphi) = \varphi(0)I + \frac{1}{i}\frac{\partial\varphi}{\partial\eta}(0)Q$$

for $\varphi \in C_c^1$ $(\lambda = \xi + i\eta)$. It is easy to see that U_1 and U_2 are C_c^1-spectral representations for Q and they satisfy the conditions (γ_1) and (γ_2) respectively. Obviously, $U_1 \neq U_2$.

THEOREM 1.4. *If S is a C_c^0-scalar operator whose spectrum is contained in a C^0-curve, then S has a unique C_c^0-spectral representation, which is canonical with respect to any C^0-curve containing* $\mathrm{Sp}(S)$.

Proof. It is enough to show that if U is any C_c^0-spectral representation for S and if γ is any representation of a C^0-curve containing $\mathrm{Sp}(S)$, then U satisfies the condition (γ). For any $x \in E$ and $x' \in E'$ (the dual of E), $\varphi \to \langle U(\varphi)x, x'\rangle$ is a continuous linear form on C_c^0. Hence, there exists a Radon measure $\mu_{x,x'}$ such that $\mu_{x,x'}(\varphi) = \langle U(\varphi)x, x'\rangle$ for all $\varphi \in C_c^0$. Obviously,

$$\mathrm{Supp}\, \mu_{x,x'} \subseteq \mathrm{Supp}\, U \subseteq \mathrm{Sp}(S) \subseteq \gamma(\Gamma).$$

Therefore, $\varphi \circ \gamma \circ u_\gamma = \varphi$ on $\mathrm{Supp}\, \mu_{x,x'}$; hence

$$\mu_{x,x'}(\varphi \circ \gamma \circ u_\gamma) = \mu_{x,x'}(\varphi).$$

Since this is true for any $x \in E$ and $x' \in E'$, U satisfies the condition (γ).

Canonical representations for C^m-unitary operators. If S is a C^m-unitary operator, then there exists a uniquely determined $C^m(e^{i\theta})$-representation W for S and $U(\varphi) = W(\varphi(e^{i\theta}))$ defines the canonical representation for S with respect to Γ. Conversely, if S is a C_c^m-scalar operator such that $\mathrm{Sp}(S) \subseteq \Gamma$ and if there exists the canonical representation U for S with respect to Γ, then it is a C^m-unitary operator (by corollary after Definition 1.4). Thus, for a C^m-unitary operator S, its canonical representation with respect to Γ is called *the canonical representation for* S. (Cf. Remark 2.4, Part II.)

Example 1.3. If E is a Hilbert space, then $S \in L(E)$ is a C^0-unitary operator if and only if it is similar to a unitary operator. In this sense, C^m-unitary operators on a locally convex space generalize the notion of unitary operators on a Hilbert space.

If E is a Banach space, we shall see (§1.5) that any one-to-one isometry on E is a C^2-unitary operator.

Example 1.4. Let $E = \mathcal{S}(R^n)$ be the Fréchet space of the rapidly decreasing function on R^n (see e.g. [3], Example 2.5). Let

$$\alpha = (\alpha_1, \cdots, \alpha_n) \in R^n.$$

If we define operators T_α, $U_\alpha(\varphi)$ ($\varphi \in C_c^\infty$) on E by

$$[T_\alpha f](x) = e^{i\langle\alpha,x\rangle} f(x)$$
$$[U_\alpha(\varphi) f](x) = \varphi(e^{i\langle\alpha,x\rangle}) f(x)$$

for $f \in \mathcal{S}(R^n)$, where $x = (x_1, \cdots, x_n)$ and $\langle\alpha, x\rangle = \alpha_1 x_1 + \cdots + \alpha_n x_n$, then T_α, $U_\alpha(\varphi) \in L(E)$ and we see that T_α is a C^∞-unitary operator and U_α is its canonical representation.

Let $\mathfrak{F}$ be the Fourier transform of $\mathcal{S}(R^n)$ onto itself. Since $\mathfrak{F}$ is a topological isomorphism, $\hat{T}_\alpha = \mathfrak{F} T_\alpha \mathfrak{F}^{-1}$ is again a C^∞-unitary operator and $\hat{U}_\alpha(\varphi) = \mathfrak{F} U_\alpha(\varphi) \mathfrak{F}^{-1}$ defines the canonical representation for $\hat{T}_\alpha$. We see that $[\hat{T}_\alpha f](x) = f(x + \alpha)$, i.e., $\hat{T}_\alpha$ is the translation of variable by α. Taking the dual, we also see that the above arguments hold on the space $E = \mathcal{S}(R^n)'$ of tempered distributions. Thus, we have seen that a translation is a C^∞-unitary operator on $\mathcal{S}(R^n)$ and on $\mathcal{S}(R^n)'$.

We shall see (§1.5) that the Fourier transform is C^2-unitary on $\mathcal{S}(R^n)$ and on $\mathcal{S}(R^n)'$.

Properties of the canonical representations

THEOREM 1.5. *Let S be a C_c^m-scalar operator such that* $\mathrm{Sp}(S)$ *is contained in a C^m-curve Δ and suppose it has the canonical representation U_Δ with respect to Δ.*

(i) *If $T \in L(E)$ commutes with S, then T commutes with each $U_\Delta(\varphi)$, $\varphi \in C_c^m$.*

(ii) *If F is a closed subspace of E and if S and $(\lambda_0 I - S)^{-1}$ leave F invariant for some λ_0 lying inside Δ, then each $U_\Delta(\varphi)$ leaves F invariant.*

Proof. (i) Let γ be a representation of Δ for which U_Δ satisfies the condition (γ). Let $W(f) = U_\Delta(f \circ u_\gamma)$ for $f \in C^m(\Gamma)$. Then W is the $C^m(\gamma)$-representation for S. It is enough to show that T commutes with each $W(f)$.

By assumption, $TW(\gamma) = W(\gamma)T$. Let λ_0 be a point inside Δ. Then

$$W\left(\frac{1}{\gamma - \lambda_0}\right)T = W\left(\frac{1}{\gamma - \lambda_0}\right)T(W(\gamma) - \lambda_0 I)W\left(\frac{1}{\gamma - \lambda_0}\right)$$
$$= W\left(\frac{1}{\gamma - \lambda_0}\right)(W(\gamma) - \lambda_0 I)TW\left(\frac{1}{\gamma - \lambda_0}\right)$$
$$= TW\left(\frac{1}{\gamma - \lambda_0}\right).$$

It follows that $W(Q \circ (\gamma - \lambda_0))T = TW(Q \circ (\gamma - \lambda_0))$ for any Q given in Lemma 1.1. Hence, by this lemma, we have $W(f)T = TW(f)$.

(ii) With the same notations in (i), our assumptions imply that $W(Q \circ (\gamma - \lambda_0))(F) \subseteq F$. For each $f \in C^m(\Gamma)$, there exists a sequence $\{Q_n\}$ such that $Q_n \circ (\gamma - \lambda_0) \to f$ in $C^m(\Gamma)$ by Lemma 1.1. If $x \in F$, then

$$W(Q_n \circ (\gamma - \lambda_0))x \in F \quad \text{and} \quad W(Q_n \circ (\gamma - \lambda_0))x \to W(f)x.$$

Hence $W(f)x \in F$, i.e., $W(f)$ leaves F invariant for each $f \in C^m(\Gamma)$; hence so does $U_\Delta(\varphi)$ for each $\varphi \in C_c^m$.

COROLLARY 1. *Let S be as in the theorem. Then any other C_c^m-spectral representation U for S commutes with U_Δ (i.e., $U(\varphi)U_\Delta(\psi) = U_\Delta(\psi)U(\varphi)$ for any $\varphi, \psi \in C_c^m$).*

COROLLARY 2. *Let S_i ($i = 1, 2$) be a $C_c^{m_i}$-scalar operator such that* $\mathrm{Sp}(S_i)$ *is contained in a $C_c^{m_i}$-curve Δ_i ($i = 1, 2$) and suppose S_i ($i = 1, 2$) has the canonical representation U_i with respect to Δ_i ($i = 1, 2$). If S_1 and S_2 commute, then U_1 and U_2 commute.*

THEOREM 1.6. *Let S be a C_c^m-scalar operator whose spectrum is contained in an analytic curve Δ (i.e., there exists a representation of Δ which is holomorphic on a neighborhood of Γ) and suppose S has the canonical representation U_Δ with respect to Δ. Let U be any other C_c^m-spectral representation for S. If m is finite, then there exists $Q \in L(E)$ such that $Q^{m+1} = 0$ and*

$$U(\varphi) = \sum_{k=0}^{m} \frac{Q^k}{k!} U_\Delta(D^k\varphi)$$

for all $\varphi \in C_c^{2m}$, where $D = \frac{1}{2}(\partial/\partial\xi + i\,\partial/\partial\eta)$ ($\lambda = \xi + i\eta$).

If $m = \infty$ and if E is a Banach space, then there exists $Q \in L(E)$ such that $Q^{m_0+1} = 0$ for some non-negative integer m_0 and

$$U(\varphi) = \sum_{k=0}^{m_0} \frac{Q^k}{k!} U_\Delta(D^k\varphi)$$

for all $\varphi \in C_c^\infty$.

Proof. By Corollary 1 above, U commutes with U_Δ. Let

$$Q = U(\bar{\lambda}) - U_\Delta(\bar{\lambda}).$$

By the remark at the end of §1.1, we see that

$$U_\Delta(\bar\lambda) = U_\Delta(\bar\gamma \circ (\overline{1/\gamma^{-1}})).$$

Since $\bar\gamma \circ (\overline{1/\gamma^{-1}})$ is holomorphic on a neighborhood of $\mathrm{Sp}(S)$, we have

$$U_\Delta(\bar\lambda) = U(\bar\gamma \circ (\overline{1/\gamma^{-1}})).$$

Thus, $Q = U(\bar\lambda - \bar\gamma \circ (\overline{1/\gamma^{-1}}))$. The function $\bar\lambda - \bar\gamma \circ (\overline{1/\gamma^{-1}})$ vanishes on $\mathrm{Supp}\, U \subseteq \gamma(\Gamma)$. Hence $Q^{m+1} = 0$ if m is finite. If we put

$$U_1(\varphi) = \sum_{k=0}^{m} \frac{Q^k}{k!} U_\Delta(D^k\varphi)$$

for $\varphi \in C_c^{2m}$, then we see that U_1 is a C_c^{2m}-spectral representation, $U_1(\lambda) = U(\lambda)$ and $U_1(\bar\lambda) = U(\bar\lambda)$. ' Hence, $U_1(\varphi) = U(\varphi)$ for all $\varphi \in C_c^{2m}$ by Proposition 2 in [5].

The last half of the theorem is proved in a similar way.

COROLLARY. *Let S, U_Δ and U be as in the theorem. If m is finite, then* $[U(\varphi) - U_\Delta(\varphi)]^{m+1} = 0$ *for any* $\varphi \in C_c^m$.

These results improve Theorem 2 and its Corollary in [5] in our special case where $\mathrm{Sp}(S)$ is contained in an analytic curve.

1.3. An existence theorem. Let S be a C_c^m-scalar operator such that $\mathrm{Sp}(S)$ is contained in a C^m-curve Δ. If $m \geq 1$, we do not know whether there always exists a canonical representation for S with respect to Δ. We can prove the following theorem which, in particular, asserts the existence for $m = \infty$.

THEOREM 1.7. *Let $m \geq 1$ and let S be a C_c^m-scalar operator such that* $\mathrm{Sp}(S)$ *is contained in a C^{2m}-curve Δ. (If $m = \infty$, we read $2\infty = \infty$.) Then there exists a canonical representation with respect to Δ for S as a C_c^{2m}-scalar operator.*

Proof. Let γ be a representation of Δ and let

$$g_k(\lambda) = [\lambda - \gamma(u_\gamma(\lambda))]^k \qquad (k = 0, 1, 2, \cdots).$$

g_k is defined and $2m$-times continuously differentiable on a neighborhood σ of Δ. Let U be a C_c^m-spectral representation for S. Since $\mathrm{Supp}\, U \subseteq \Delta$, $U(g_k)$ is well defined.

We consider a differential operator of the form

$$D = \omega(\xi, \eta)(\partial/\partial\xi + i\,\partial/\partial\eta) \qquad (\lambda = \xi + i\eta)$$

such that $\omega \in C^{2m-1}(\sigma)$ and $D(\gamma \circ u_\gamma) \equiv 1$ on σ. The existence of such a function ω can be easily seen by the fact that the Jacobian of the transformation γ^{-1} does not vanish on σ.

Case I. m is finite. If $k \geq m + 1$, then all the partial derivatives of g_k of order $\leq m$ vanish on Δ. Since $\varphi \to \langle U(\varphi)x, x'\rangle$ is a distribution of order

m whose support is contained in Δ for each $x \in E$ and $x' \in E'$, it follows $U(g_k) = 0$ for $k \geq m+1$ (see e.g., Théorème XXVIII of [7], Chapter III). Similarly, we see that

$$(\#) \qquad U(g_k \psi) = 0 \qquad \text{for } k = m+1, m+2, \cdots$$

for any $\psi \in C_c^m$. Now, we define U_Δ by

$$U_\Delta(\varphi) = \sum_{k=0}^{m} \frac{1}{k!} U[g_k D^k(\varphi \circ \gamma \circ u_\gamma)]$$

for $\varphi \in C_c^{2m}$. Then the right hand side is well defined and U_Δ is a continuous linear mapping of C_c^{2m} into $L(E)$. Furthermore, it is easy to see that $\operatorname{Supp} U_\Delta \subseteq \Delta$, $U_\Delta(1) = I$, $U_\Delta(\lambda) = U(\lambda) = S$ and $U_\Delta(\varphi \circ \gamma \circ u_\gamma) = U_\Delta(\varphi)$. From $(\#)$, it follows that U_Δ is multiplicative. Hence U_Δ is the canonical C_c^{2m}-respresentation for S with respect to Δ.

Case II. $m = \infty$. If we apply the proof of Théorème XXVIII of [7], Chapter III to our $L(E)$-valued distribution U, we can conclude the following: For any continuous semi-norm q on $L(E)$, there exists a non-negative integer m_q such that if all the derivatives of $\varphi \in C_c^\infty$ of order $\leq m_q$ vanish on Δ, then $q[U(\varphi)] = 0$. Hence we see that

$$(\#_q) \qquad q[U(g_k \psi)] = 0 \quad \text{for } k = m_q+1, m_q+2, \cdots,$$

for any $\psi \in C_c^\infty$.

Now, we define T_p ($p = 1, 2, \cdots$) by

$$T_p(\varphi) = \sum_{k=0}^{p} \frac{1}{k!} U[g_k D^k(\varphi \circ \gamma \circ u_\gamma)]$$

for $\varphi \in C_c^\infty$. Then, for each p, T_p is a continuous linear mapping of C_c^∞ into $L(E)$ such that $\operatorname{Supp} T_p \subseteq \Delta$, $T_p(1) = I$, $T_p(\lambda) = S$ and $T_p(\varphi \circ \gamma \circ u_\gamma) = T_p(\varphi)$. The property $(\#_q)$ shows that $\{T_p(\varphi)\}$ is a Cauchy sequence in $L(E)$ for each $\varphi \in C_c^\infty$. Since $L(E)$ is quasi-complete, there exists $U_\Delta(\varphi) \in L(E)$ such that

$$T_p(\varphi) \to U_\Delta(\varphi) \qquad (p \to \infty)$$

for each $\varphi \in C_c^\infty$. In fact, $q(T_p(\varphi) - U_\Delta(\varphi)) = 0$ for $p \geq m_q$. It follows then that $\operatorname{Supp} U_\Delta \subseteq \Delta$, U_Δ is continuous linear on C_c^∞, $U_\Delta(1) = I$, $U_\Delta(\lambda) = S$ and $U_\Delta(\varphi \circ \gamma \circ u_\gamma) = U_\Delta(\varphi)$. Furthermore, $(\#_q)$ implies

$$q[T_{p_1}(\varphi\psi) - T_{p_2}(\varphi)T_{p_3}(\psi)] = 0$$

for any $p_1, p_2, p_3 \geq m_q$; $\varphi, \psi \in C_c^\infty$. It follows then that U_Δ is multiplicative. Hence, U_Δ is the canonical representation for S with respect to Δ.

Remark 1.2. If E is a Banach space, then so is $L(E)$. Hence there exists $m_0 \geq 0$ such that $U_\Delta = T_{m_0}$.

COROLLARY 1. *Let S_i ($i = 1, 2, \cdots, n$) be a C_c^∞-scalar operator such that $\operatorname{Sp}(S_i) \subseteq \Delta_i$ for a C^∞-curve Δ_i ($i = 1, \cdots, n$). If $S_1, \cdots, S_n$ commute* *with each other, then* $P(S_1, \cdots, S_n)$ *is a* C_c^∞*-scalar operator for any polynomial* P *in* n *variables.*

Proof. The previous theorem implies the existence of the canonical representation for each S_i with respect to Δ_i. By Corollary 2 to Theorem 1.5, these representations commute with each other. Hence, by (ii) of Corollary to Proposition 3.1 in [4], we see that $P(S_1, \cdots, S_n)$ is C_c^∞-scalar.

COROLLARY 2. *If* $S_1, \cdots, S_n$ *are* C^∞*-unitary operators commuting with each other, then* $S_1 \cdot \cdots \cdot S_n$ *is a* C^∞*-unitary operator.*

1.4. Characterization in terms of resolvents. Let $S \in L(E)$ be any C_c^m-scalar operator with compact spectrum. For $\lambda \notin \mathrm{Sp}(S)$, let $d_\lambda = \mathrm{dis}\,(\lambda, \mathrm{Sp}(S))$ and $R(\lambda) = (\lambda I - S)^{-1}$.

LEMMA 1.2. *For any continuous semi-norm* q *on* $L(E)$, *there exists a non-negative integer* m_q ($= m$, *if* m *is finite*) *such that*

$$q[R(\lambda)] \leq M_q(d_\lambda^{-m_q-1} + d_\lambda^{-1}) \qquad (M_q > 0)$$

for all $\lambda \notin \mathrm{Sp}(S)$.

Proof. For $d > 0$, we can choose $\varphi_d \in C_c^m$ in such a way that $\varphi_d \equiv 1$ on a neighborhood of $\mathrm{Sp}(S)$, $\varphi_d(z) \equiv 0$ if $d_z > \min\,(d/2, 1)$, $0 \leq \varphi_d \leq 1$ and $\| \varphi_d \|_{l,\Sigma} \leq K_l(d^{-l} + 1)$ ($l = 0, 1, 2, \cdots$; $K_l > 0$ is independent of d), where

$$\| \varphi \|_{l,\Sigma} = \sup \left\{ \left| \frac{\partial^{k_1+k_2}\varphi}{\partial \xi^{k_1} \partial \eta^{k_2}}(\lambda) \right| ; \quad 0 \leq k_1 + k_2 \leq l, \lambda \in \Sigma = \{z; d_z \leq 1\} \right\}.$$

(See, e.g., the proof of Théorème XXVIII of [7], Chapter III.) Let

$$\varphi_\lambda(z) = \frac{\varphi_{d_\lambda}(z)}{\lambda - z}.$$

Then
$$\| \varphi_\lambda \|_{l,\Sigma} \leqq K_l'(d_\lambda^{-l-1} + d_\lambda^{-1}),$$

$l = 0, 1, 2, \cdots$. Let U be a C_c^m-spectral representation for S. Since $R(\lambda) = U(\varphi_\lambda)$, the lemma follows from the continuity of U.

THEOREM 1.8 (Cf. Tillmann [9, Satz 1]). *Let* $m \geq 2$ *and let* $S \in L(E)$ *have a spectrum contained in a* C^m*-curve. If, for each continuous semi-norm* q *on* $L(E)$, *there exists an integer* m_q *with* $0 \leq m_q \leq m - 2$ *such that*

$$q[R(\lambda)] \leq M_q\, d_\lambda^{-m_q-1} \qquad (M_q > 0) \tag{1}$$

for all λ *with* $0 < d_\lambda \leq d_0$ ($d_0 > 0$), *then* S *is a* C_c^m*-scalar operator having a canonical representation with respect to any* C^m*-curve containing* $\mathrm{Sp}(S)$.

Proof. Let γ be a representation of a C^m-curve Δ containing $\mathrm{Sp}(S)$. First, we remark that (1) implies

$$q[R(\gamma(z))] \leq M_q' |\, 1 - |z|\, |^{-m_q-1} \qquad (M_q' > 0) \tag{1'}$$

for all z with $0 < |\, 1 - |z|\, | \leq \varepsilon_0$, ε_0 being taken sufficiently small.

By Theorem 1.2, it is enough to show that S is a $C^m(\gamma)$-operator. Let $\gamma_\delta(\theta) = \gamma\{(1+\delta)e^{i\theta}\}$ for $|\delta| \leq \varepsilon_0$. Let

$$\nu(\gamma) = \frac{1}{2\pi i}\int_0^{2\pi} \{[\gamma_\varepsilon(\theta) - \gamma(1)]^{-1}\gamma'_\varepsilon(\theta) - [\gamma_{-\varepsilon}(\theta) - \gamma(1)]^{-1}\gamma'_{-\varepsilon}(\theta)\}\, d\theta$$

for $0 < \varepsilon \leq \varepsilon_0$. Then $\nu(\gamma) = \pm 1$. For $f \in C^m(\Gamma)$ and ε $(0 < \varepsilon \leq \varepsilon_0)$, we define

$$W_\varepsilon(f) = \frac{\nu(\gamma)}{2\pi i}\int_0^{2\pi} f(e^{i\theta})\{R[\gamma_\varepsilon(\theta)]\,\gamma'_\varepsilon(\theta) - R\,[\gamma_{-\varepsilon}(\theta)]\,\gamma'_{-\varepsilon}(\theta)\}\, d\theta.$$

The integral exists and $W_\varepsilon(f) \in L(E)$, since all the functions in the integral are continuous in θ and $L(E)$ is quasi-complete. We shall show that $\lim_{\varepsilon\to 0} W_\varepsilon(f)$ exists for each $f \in C^m(\Gamma)$.

For a real δ with $|\delta| \leqq \varepsilon_0$ and for $f \in C^m(\Gamma)$, let

$$[\partial_\delta^{(1)} f](\theta) = -\frac{d}{d\theta} f(e^{i\theta})$$

$$[\partial_\delta^{(n)} f](\theta) = -\frac{d}{d\theta}\{[\partial_\delta^{(n-1)} f](\theta)[\gamma'_\delta(\theta)]^{-1}\}, \qquad n = 2, \cdots, m.$$

$\delta \to \partial_\delta^{(n)} f$ is continuous from $[-\varepsilon_0, \varepsilon_0]$ into $C^{m-n}(\Gamma)$ and $f \to \partial_\delta^{(n)} f$ is linear continuous from $C^m(\Gamma)$ into $C^{m-n}(\Gamma)$ for each δ.

Let σ_1 be the bounded component of $C - \Delta$. We may assume that $\gamma_{-\varepsilon}(\theta) \in \sigma_1$ for $0 < \varepsilon \leq \varepsilon_0$ and that $\nu(\gamma) = 1$. (We can similarly discuss the other cases.) Since σ_1 is simply connected, $R(\lambda)$ have n-fold indefinite integrals $R^{(-n)}(\lambda)$ in σ_1 for all $n = 1, 2, \cdots$. Then, integrating by parts, we have

$$\int_0^{2\pi} f(e^{i\theta})R[\gamma_{-\varepsilon}(\theta)]\,\gamma'_{-\varepsilon}(\theta)\, d\theta = \int_0^{2\pi} [\partial_{-\varepsilon}^{(n)} f](\theta) R^{(-n)}[\gamma_{-\varepsilon}(\theta)]\, d\theta \quad (n = 1, 2, \cdots, m).$$

On the unbounded component of $C - \Delta$, we define $R^{(-n)}(\lambda)$ for $\lambda = \gamma_\varepsilon(\theta)$, $0 < \varepsilon \leq \varepsilon_0$, $0 \leq \theta < 2\pi$ as follows:

$$R^{(0)}(\lambda) \equiv R(\lambda);$$

$$R^{(-n)}[\gamma_\varepsilon(\theta)] = \int_0^\theta R^{(-n+1)}[\gamma_{\varepsilon_0}(\alpha)]\,\gamma'_{\varepsilon_0}(\alpha)\, d\alpha + \int_{\varepsilon_0}^{\varepsilon} R^{(-n+1)}[\gamma_r(\theta)]\frac{d}{dr}\gamma_r(\theta)\, dr,$$

$n = 1, 2, \cdots$. Next, we define $S_n(f, \varepsilon) \in L(E)$, $n = 1, 2, \cdots, m$ by

$$S_1(f, \varepsilon) = 2\pi i\, f(1) I,$$

$$S_n(f, \varepsilon) = S_{n-1}(f, \varepsilon) + [\partial_\varepsilon^{(n-1)} f](0)[\gamma'_\varepsilon(0)]^{-1}\int_0^{2\pi} R^{(-n+1)}[\gamma_\varepsilon(\theta)]\gamma'_\varepsilon(\theta)\, d\theta.$$

$n = 2, \cdots, m$. Then, integrating by parts, we have

$$\int_0^{2\pi} f(e^{i\theta})R[\gamma_\varepsilon(\theta)]\gamma'_\varepsilon(\theta)\, d\theta = S_n(f, \varepsilon) + \int_0^{2\pi} [\partial_\varepsilon^{(n)} f](\theta) R^{(-n)}[\gamma_\varepsilon(\theta)]\, d\theta,$$

$n = 1, 2, \cdots, m$. We remark here that $\lim_{\varepsilon\to 0} S_n(f, \varepsilon)$ exists for each n, f and $f \to \lim_{\varepsilon\to 0} S_n(f, \varepsilon)$ is continuous on $C^m(\Gamma)$ for $n \leq m$.

From (1′), by a method of Tillmann [8] (also see [11]), we can conclude that $R^{(-m_q-2)}[\gamma_{\pm\varepsilon}(\theta)]$ has a q-limit as $\varepsilon \to 0$ and the convergence is uniform in θ. Since $m_q + 2 \leq m$ and $f \in C^m(\Gamma)$, it follows that $W_\varepsilon(f)$ has a q-limit as $\varepsilon \to 0$. Hence $W(f) = \lim_{\varepsilon\to 0} W_\varepsilon(f)$ exists and $W(f) \in L(E)$. Furthermore, the above arguments show that the mapping $f \to W(f)$ is continuous from $C^m(\Gamma)$ into $L(E)$. Obviously the mapping is linear.

If h is a function holomorphic on a neighborhood of Δ, then the operational calculus is written as

$$h(S) = \frac{1}{2\pi i}\int_0^{2\pi} \{h[\gamma_\varepsilon(\theta)]R[\gamma_\varepsilon(\theta)]\gamma'_\varepsilon(\theta) - h[\gamma_{-\varepsilon}(\theta)]R[\gamma_{-\varepsilon}(\theta)]\gamma'_{-\varepsilon}(\theta)\}\, d\theta$$

for sufficiently small $\epsilon > 0$. Hence

$$\begin{aligned} h(S) - W(h\circ\gamma) = \lim_{\varepsilon\to 0+} \frac{1}{2\pi i}\Big\{ &\int_0^{2\pi} \{h[\gamma_\varepsilon(\theta)] - h[\gamma_0(\theta)]\}R[\gamma_\varepsilon(\theta)]\gamma'_\varepsilon(\theta)\, d\theta \\ &- \int_0^{2\pi} \{h[\gamma_{-\varepsilon}(\theta)] - h[\gamma_0(\theta)]\}R[\gamma_{-\varepsilon}(\theta)]\gamma'_{-\varepsilon}(\theta)\, d\theta \Big\}. \end{aligned}$$

We can see that this limit is equal to zero, by repeating the arguments given above (also see [8] and [11]). Hence, $W(h\circ\gamma) = h(S)$; in particular, $W(1) = I$ and $W(\gamma) = S$. Furthermore, it follows that W is multiplicative on the set $\{h\circ\gamma;\ h$ is holomorphic on a neighborhood of $\Delta\}$, which is dense in $C^m(\Gamma)$ (Lemma 1.1). Since we have seen that W is continuous on $C^m(\Gamma)$, it follows that W is multiplicative on $C^m(\Gamma)$. Hence, W is a $C^m(\gamma)$-representation for S.

Corollary. *If S is a C_c^m-scalar operator $(m \geq 1)$ such that* Sp(S) *is contained in a C^{m+2}-curve, then there exists the canonical representation with respect to any C^{m+2}-curve containing* Sp(S) *for S as a C_c^{m+2}-scalar operator.*

This corollary improves Theorem 1.7 in the case m is finite ≥ 3. (It gives as good results in the cases $m = 2$ and $m = \infty$.) Thus, our best known result on the existence of the canonical representations is as follows:

If S is a C_c^m-scalar operator such that Sp(S) is contained in a $C^{m'}$-curve Δ, then there exists a canonical representation with respect to Δ for S as a $C_c^{m'}$-scalar operator, where $m' = 0$ if $m = 0$, $m' = 2$ if $m = 1$, $m' = m + 2$ if $2 \leq m < \infty$ and $m' = \infty$ if $m = \infty$.

Remark 1.3. In the above theorem, it is enough to assume that (1) is satisfied for each q belonging to a family of semi-norms on $L(E)$ which defines the topology τ_b. Hence, if E is a Banach space, we can say:

If there exists a non-negative integer m_0 such that Sp(S) is contained in a C^{m_0+2}-curve and

$$\| R(\lambda) \| \leqq M d_\lambda^{-m_0-1} \qquad (M > 0)$$

for all λ with $0 < d_\lambda \leq d_0$ $(d_0 > 0)$, then S is a C^{m_0+2}-scalar operator.

1.5. Another characterization of C^m-unitary operators.

THEOREM 1.9 (Cf. Kantorovitz [2, Lemma 2.9]).

(i) *If S is a C_c^m-scalar operator such that* $\mathrm{Sp}(S) \subseteq \Gamma$ (*in particular, if S is a C^m-unitary operator*), *then for each continuous semi-norm q on $L(E)$ there exists a non-negative integer $m_q(= m$, if m is finite) such that*

$$q(S^k) \leq M_q |\, k\,|^{m_q} \qquad (M_q > 0) \tag{2}$$

for all $k = \pm 1, \pm 2, \cdots$.

(ii) *Conversely, if $S \in L(E)$, S^{-1} exists and $\in L(E)$ and if* (2) *is satisfied for m_q with $0 \leq m_q \leq m - 2$ for each q, then S is a C^m-unitary operator.*

Proof. (i) For $0 < d < 1$, let $\varphi_d \in C_c^\infty$ be the function defined in the proof of Lemma 1.2 and let $\psi_k(\lambda) = \lambda^k \varphi_{1/|k|}(\lambda)$ for $k = \pm 1, \pm 2, \cdots$. Since

$$\mathrm{Supp}\, \psi_k \subseteq \{\lambda : 1 - 1/2|\, k\,| < |\,\lambda\,| < 1 + 1/2|\, k\,|\},$$

we have $\|\, \psi_k \,\|_{l,\Sigma} \leq K_l |\, k\,|^l$ for all $k = \pm 1, \pm 2, \cdots; l = 0, 1, 2, \cdots$. Since $S^k = U(\psi_k)$ for any C_c^m-spectral representation U for S, (2) follows from the continuity of U.

(ii) We may apply Kantorovitz' method [2, Lemma 2.9] to obtain the $C^m(e^{i\theta})$-representation for S. Here, we shall show that (2) implies

$$q(R(\lambda)) \leq M'_q |\, 1 - |\,\lambda\,|\,|^{-m_q - 1}$$

for $\lambda \notin \Gamma$, $|\,\lambda\,| < 2$. Then we conclude the existence of a $C^m(e^{i\theta})$-representation by Theorem 1.8. (In this way, we prove the implication III $(m) \Rightarrow$ IV (m) in Theorem 3, (i), of [6]. Cf. [11, 4.12].)

If $|\,\lambda\,| < 1$, then let $R_1(\lambda) = -\sum_{k=0}^\infty \lambda^k S^{-(k+1)}$ and if $|\,\lambda\,| > 1$, then let $R_1(\lambda) = \sum_{k=0}^\infty \lambda^{-(k+1)} S^k$. The condition (2) implies that these series converge in $L(E)$, so that $R_1(\lambda) \in L(E)$ for each $\lambda \notin \Gamma$. Furthermore, we see that $\lambda \to R_1(\lambda)$ is holomorphic on $\hat{C} - \Gamma$. Direct computations show that $(\lambda I - S)R_1(\lambda) = R_1(\lambda)(\lambda I - S) = I$ for each $\lambda \notin \Gamma$. Hence $\mathrm{Sp}(S) \subseteq \Gamma$ and $R(\lambda) = R_1(\lambda)$. Again by (2), we have

$$q(R(\lambda)) \leq M_q \sum_{k=0}^\infty |\,\lambda\,|^k |\, k + 1\,|^{m_q} \leq M'_q (1 - |\,\lambda\,|)^{-m_q - 1}$$

for $|\,\lambda\,| < 1$ and

$$q(R(\lambda)) \leq q(I)/|\,\lambda\,| + M_q \sum_{k=1}^\infty |\,\lambda\,|^{-(k+1)} k^{m_q} \leq M'_q (|\,\lambda\,| - 1)^{-m_q - 1}$$

for $1 < |\,\lambda\,| < 2$.

Remark 1.4. By this theorem, we see that a one-to-one isometry on a Banach space is C^2-unitary and the Fourier transform on $\mathcal{S}(R^n)$ or on $\mathcal{S}(R^n)'$ is C^2-unitary (see Examples 1.3 and 1.4). Also, we can directly show that any translation on $\mathcal{S}(R^n)$ or on $\mathcal{S}(R^n)'$ is a C^∞-unitary operator (cf. Example 1.4).

Remark 1.5. We showed that the condition (2) implies the condition (1). As a converse, we can prove the following: If $S \in L(E)$ has a spectrum contained in Γ and if (1) is satisfied for each q, then

$$q(S^k) \leq M'_q |k|^{m_q+1} \qquad (M'_q > 0)$$

for all $k = \pm 1, \pm 2, \cdots$. See, e.g., [11], 4.11.

COROLLARY 1. *Let S_i $(i = 1, 2)$ be a $C_c^{m_i}$-scalar operator such that* $\mathrm{Sp}(S_i) \subseteq \Gamma$ $(i = 1, 2)$. (*In particular, let S_i $(i = 1, 2)$* be a *C^{m_i}-unitary operator.*) If *S_1 and S_2 commute, then $S_1 \cdot S_2$ is a $C^{m_1+m_2+2}$-unitary operator.* (Cf. Corollary 2.10 in [2].)

Proof. If either $m_1 = \infty$ or $m_2 = \infty$, then this corollary reduces to Corollary 2 to Theorem 1.7. Suppose both m_1 and m_2 are finite. Then

$$B_1 = \{S_1^k / |k|^{m_1};\ k = \pm 1, \pm 2, \cdots\}$$

and

$$B_2 = \{S_2^k / |k|^{m_2};\ k = \pm 1, \pm 2, \cdots\}$$

are bounded sets in $L(E)$ by the above theorem. Since

$$B = \{(S_1 S_2)^k / |k|^{m_1+m_2};\ k = \pm 1, \pm 2, \cdots\} \subseteq B_1 \cdot B_2,$$

B is a bounded set, so that $q[(S_1 S_2)^k] \leq K_q |k|^{m_1+m_2}$ for all $k = \pm 1, \pm 2, \cdots$. Hence, $S_1 S_2$ is a $C^{m_1+m_2+2}$-unitary operator by (ii) of the above theorem.

COROLLARY 2. *Let S be a C_c^m-scalar operator such that* $\mathrm{Sp}(S)$ *is contained in a C^m-curve Δ, γ be a representation of Δ and U be a C_c^m-representation for S. For any continuous semi-norm q on $L(E)$, there exists a non-negative integer $m_q (= m$, if m is finite$)$ such that*

$$q[U(\gamma^{-1})^k] \leq M_q |k|^{m_q} \qquad (M_q > 0)$$

for all $k = \pm 1, \pm 2, \cdots$.

Proof. $U(\gamma^{-1})$ is a C_c^m-scalar operator and $\mathrm{Sp}\,(U(\gamma^{-1})) \subseteq \Gamma$.

COROLLARY 3. *Let $S \in L(E)$ and suppose* $\mathrm{Sp}(S)$ *is compact. If there exists a one-to-one holomorphic function h on a neighborhood of* $\mathrm{Sp}(S)$ *such that $h(S)^{-1} \in L(E)$ and if for each continuous semi-norm q on $L(E)$ there exists an integer m_q with $0 \leq m_q \leq m - 2$ such that*

$$q\{[h(S)]^k\} \leq M_q |k|^{m_q} \qquad (M_q > 0)$$

for all $k = \pm 1, \pm 2, \cdots$, then S is a C_c^m-scalar operator having a canonical representation with respect to a C^∞-curve.

Proof. By Theorem 1.9, (ii), $h(S)$ is a C^m-unitary operator. Let V be the canonical representation for $h(S)$. If we define $U(\varphi) = V(\varphi \circ h^{-1})$ for $\varphi \in C_c^m$, then U is a C_c^m-spectral representation for $U(\lambda) = V(h^{-1}) = h^{-1}(h(S)) = S$. Since we can find a representation γ of a C^∞-curve such that $\gamma \equiv h^{-1}$ on a neighborhood of $\mathrm{Sp}(h(S))$ and since U satisfies the condition (γ) for such γ, we have the corollary.

Part II. The case of unbounded curves

2.1. Unbounded C^m-curve and $C^m(\gamma)$-transformation. In Part II, we consider a Jordan curve in $\hat{C} = C \cup \{\infty\}$ passing through the point ∞, or its restriction to C. Such a curve is homeomorphic to the extended real line $\hat{R} = R \cup \{\infty\}$.

Definition 2.1. A closed set Δ in $\hat{C}$ is called a *C^m_∞-curve* (m : integer ≥ 0 or $m = \infty$), if there exists a one-to-one continuous mapping γ of $\hat{R}$ into $\hat{C}$ such that

(i) $\gamma(\hat{R}) = \Delta$, $\gamma(\infty) = \infty$; (Hence, $\gamma(R) = \Delta - \{\infty\}$.)

(ii) γ can be extended to an open neighborhood σ of R in C (the extended map will also be denoted by γ) in such a way that γ is one-to-one on σ and γ and γ^{-1} are both m-times continuously differentiable on σ and $\gamma(\sigma)$ respectively as functions in two real variables.

The mapping γ is called a representation of Δ.

Let $C^m_c(R)$ be the space of all complex-valued m-times continuously differentiable functions with compact support in R. We introduce the usual topology in $C^m_c(R)$ similar to that in C^m_c.

If γ is a representation of a C^m_∞-curve Δ, then this mapping induces a differential structure on $\Delta - \{\infty\}$, so that $\Delta - \{\infty\}$ is regarded as a one-dimensional C^m-manifold. The space $C^m_c(\Delta - \{\infty\})$ is defined to be isomorphic to $C^m_c(R)$ by the mapping $g \leftrightarrow g \circ \gamma$. Thus, corresponding to Definition 1.2, we define:

Definition 2.2. Let γ be a representation of a C^m_∞-curve. A closed transformation S with domain D_S is called a *$C^m(\gamma)$-transformation* (*$C^m(\gamma)$-operator*, if $S \in L(E)$), if there exist a continuous algebra homomorphism W of $C^m_c(R)$ into $L(E)$ and a net $\{f_\alpha\}$ in $C^m_c(R)$ such that $W(f_\alpha)x \to x$ for each $x \in E$, $W(f)x \in D_S$ for any $f \in C^m_c(R)$ and

$$(**) \qquad \lim_\alpha W(\gamma f_\alpha)x = Sx \qquad \text{for all } x \in D_S .$$

The net $\{f_\alpha\}$ is called an *identity net* for W. If γ is the identity map $\gamma(t) \equiv t$ (so that $\gamma(R) = R$), then a $C^m(t)$-transformation (resp. a $C^m(t)$-operator) is called a *C^m-real transformation* (resp. a *C^m-real operator*). (Cf. [5], [2], and [9]).

Remark 2.1. The condition $(**)$ can be replaced by

$$(**)' \qquad W(f)Sx = W(\gamma f)x \quad \text{for all } f \in C^m_c(R) \text{ and } x \in D_S ,$$

or by

$$(**)'' \qquad W(f)S = SW(f) \text{ on } D_S \quad \text{and} \quad SW(f) = W(\gamma f).$$

Hence, the definition of a $C^m(\gamma)$-transformation does not depend on the choice of $\{f_\alpha\}$ (cf. Preliminaries). Furthermore,

$$D_S = \{x \in E;\ \lim_\alpha W(\gamma f_\alpha) \text{ exists}\}.$$

Also, we can make the same remark as in Preliminaries on the topology of $L(E)$ and the continuity of W.

An approximation theorem corresponding to Lemma 1.1 may be formulated as follows (see appendix):

LEMMA 2.1 (Approximation Theorem). *Let γ be a representation of a C_{∞}^{m}-curve. Given a compact set Σ on R and an m-times continuously differentiable function f on R, there exists a sequence $\{P_n\}$ of polynomials (in one complex variable) such that*

$$(P_n \circ \gamma)g \to fg \qquad (n \to \infty)$$

in $C_c^m(R)$ for all $g \in C_c^m(R)$ with Supp $g \subseteq \Sigma$.

THEOREM 2.1. *Let γ be a representation of a C_{∞}^{m}-curve. If W_1 and W_2 are continuous homomorphisms of $C_c^m(R)$ into $L(E)$ with identity nets $\{f_\alpha^{(1)}\}$ and $\{f_\beta^{(2)}\}$ respectively such that*

$$\lim_\alpha W_1(f_\alpha^{(1)}\gamma)x = \lim_\beta W_2(f_\beta^{(2)}\gamma)x \tag{1}$$

then $W_1 \equiv W_2$. (Here, (1) means that if the limit of one side exists, then so does the limit of the other side and they are equal.)

Proof. First, we observe that $\lim_\alpha W_1(f_\alpha^{(1)}\gamma)W_2(g)x$ exists and is equal to $W_2(g\gamma)x$ for any $g \in C_c^m(R)$ and $x \in E$. We shall show that

$$W_1(g_1\gamma^k)W_2(g_2) = W_1(g_1)W_2(g_2\gamma^k), \qquad k = 0, 1, 2, \cdots \tag{2}$$

for any $g_1, g_2 \in C_c^m(R)$. If $k = 0$, then (2) is trivial. Suppose (2) is true for a k. Then, for any $x \in E$,

$$\begin{aligned} W_1(g_1\gamma^{k+1})W_2(g_2)x &= W_1(g_1\gamma^{k+1}) \lim_\alpha W_1(f_\alpha^{(1)})W_2(g_2)x \\ &= \lim_\alpha W_1(g_1\gamma^k)W_1(f_\alpha^{(1)}\gamma)W_2(g_2)x \\ &= W_1(g_1\gamma^k)W_2(g_2\gamma)x \\ &= W_1(g_1)W_2(g_2\gamma^{k+1})x. \end{aligned}$$

Hence, by induction, we have (2). It follows then that

$$W_1[g_1(P \circ \gamma)]W_2(g_2) = W_1(g_1)W_2[g_2(P \circ \gamma)] \tag{3}$$

for any polynomial P. Let $f \in C_c^m$ be given. By Lemma 2.1, there exists a sequence $\{P_n\}$ of polynomials such that

$$g_1(P_n \circ \gamma) \to g_1 f \quad \text{and} \quad g_2(P_n \circ \gamma) \to g_2 f \qquad (n \to \infty)$$

in C_c^m. Hence, (3) implies

$$W_1(g_1)W_1(f)W_2(g_2) = W_1(g_1)W_2(f)W_2(g_2).$$

Now, taking $g_1 = f_\alpha^{(1)}$, $g_2 = f_\beta^{(2)}$ and taking limits, we see that $W_1(f) = W_2(f)$.

By this theorem and Remark 2.1, we see that W in Definition 2.2 is uniquely determined by S and γ. We shall call W the *$C^m(\gamma)$-representation* for S.

DEFINITION 2.3. Let γ be a representation of a C_∞^m-curve and let U be a C_c^m-spectral representation. We say that U *satisfies the condition* (γ), if

$$\operatorname{Supp} U \subseteq \gamma(R) \quad \text{and} \quad U(\varphi \circ \gamma \circ \operatorname{Re} \gamma^{-1}) = U(\varphi)$$

for all $\varphi \in C_c^m$.

THEOREM 2.2. *Let S be a $C^m(\gamma)$-transformation for a representation γ of a C_∞^m-curve Δ and let W be the $C^m(\gamma)$-representation for S. Then*

(i) *S is a C_c^m-scalar transformation such that* $\operatorname{Sp}(S) \subseteq \Delta$;

(ii) $U_0 : U_0(\varphi) = W(\varphi \circ \gamma)$ *for $\varphi \in C_c^m$ defines a C_c^m-spectral representation for S satisfying the condition* (γ).

Proof. It is easy to see that U_0 is a C_c^m-spectral representation with an identity net $\{f_\alpha \circ \gamma^{-1}\}$, where $\{f_\alpha\}$ is an identity net for W and that U_0 satisfies the condition (γ). If $x \in D_S$, then

$$U_0[\lambda(f_\alpha \circ \gamma^{-1})]x = W(\gamma f_\alpha)x \rightarrow Sx .$$

Hence S is a C_c^m-scalar transformation and U_0 is for S.

A remark on the condition (γ). Corresponding to the similar remark in Part I, we can state some equivalent forms for the condition (γ) in the case γ is a representation of a C_∞^m-curve; if U is a C_c^m-spectral representation and if $\operatorname{Supp} U \subseteq \gamma(R)$, then the following conditions are mutually equivalent:

(i) $U(\varphi) = U(\varphi \circ \gamma \circ \operatorname{Re} \gamma^{-1})$ for all $\varphi \in C_c^m$;

(ii) $U(\gamma^{-1}\varphi) = U(\overline{\gamma^{-1}\varphi})$ for all $\varphi \in C_c^m$;

(iii) $U(\varphi) = U(\varphi \circ \gamma \circ \overline{\gamma^{-1}})$ for all $\varphi \in C_c^m$;

(iv) $U[(\operatorname{Im} \gamma^{-1})\varphi] = 0$ for all $\varphi \in C_c^m$.

Furthermore, if γ is analytic, then each one of (i)–(iv) is equivalent to

(v) $U(\lambda\varphi) = U[(\overline{\gamma \circ \overline{\gamma^{-1}}})\varphi]$ for all $\varphi \in C_c^m$.

In particular, if γ is the identity map (so that $\operatorname{Supp} U \subseteq R$), then the following are mutually equivalent:

(i)′ $U(\varphi_R) = U(\varphi)$ for all $\varphi \in C_c^m$, where $\varphi_R(\lambda) = \varphi(\operatorname{Re} \lambda)$;

(ii)′ $U(\lambda\varphi) = (\bar{\lambda}\varphi)$ for all $\varphi \in C_c^m$;

(iii)′ $U(\varphi^*) = U(\varphi)$ for all $\varphi \in C_c^m$, where $\varphi^*(\lambda) = \varphi(\bar{\lambda})$;

(iv)′ $U[(\operatorname{Im} \lambda)\varphi] = 0$ for all $\varphi \in C_c^m$.

2.2. Canonical representation. Corresponding to Theorem 1.3, we have the following theorem, which leads to the definition of a uniquely determined canonical representation:

THEOREM 2.3. *Let U_1 and U_2 be two C_c^m-spectral representations with identity nets $\{\varphi_\alpha^{(1)}\}$ and $\{\varphi_\beta^{(2)}\}$ respectively and suppose*

$$\lim_\alpha U_1(\lambda\varphi_\alpha^{(1)})x = \lim_\beta U_2(\lambda\varphi_\beta^{(2)})x,$$

so that Supp U_1 = Supp $U_2(\equiv\Sigma)$. *Let* $\gamma_i(i = 1, 2)$ *be a representation of a* C_∞^m*-curve. If* $\gamma_1(R) \cap \sigma = \gamma_2(R) \cap \sigma$ *for some neighborhood* σ *of* Σ *and if each* $U_i(i = 1, 2)$ *satisfies the condition* (γ_i), *then* $U_1 \equiv U_2$.

Proof. Let

$$W_1(f) = U_1(f \circ \operatorname{Re} \gamma_1^{-1})$$
$$W_2(f) = U_2(f \circ \gamma_1^{-1} \circ \gamma_2 \circ \operatorname{Re} \gamma_2^{-1})$$

for $f \in C_c^m(R)$. Then, W_1 and W_2 are well defined and they are continuous homomorphisms of $C_c^m(R)$ into $L(E)$. Let

$$f_\alpha^{(1)} = \varphi_\alpha^{(1)} \circ \gamma_1 \quad \text{and} \quad f_\beta^{(2)} = \varphi_\beta^{(2)} \circ \gamma_1 .$$

Then $\{f_\alpha^{(1)}\}$ and $\{f_\beta^{(2)}\}$ are identity nets for W_1 and W_2 respectively. Furthermore,

$$\begin{aligned} W_1(f_\alpha^{(1)}\gamma_1) &= U_1[\{(\varphi_\alpha^{(1)} \circ \gamma_1)\gamma_1\} \circ \operatorname{Re} \gamma_1^{-1}] \\ &= U_1[(\lambda\varphi_\alpha^{(1)}) \circ \gamma_1 \circ \operatorname{Re} \gamma_1^{-1}] = U_1(\lambda\varphi_\alpha^{(1)}). \end{aligned}$$

Similarly, we see that $W_2(f_\beta^{(2)}\gamma_1) = U_2(\lambda\varphi_\beta^{(2)})$. Hence

$$\lim_\alpha W_1(f_\alpha^{(1)}\gamma_1)x = \lim_\beta W_2(f_\beta^{(2)}\gamma_1)x.$$

Therefore, we have $W_1 \equiv W_2$ by Theorem 2.1. Hence, for any $\varphi \in C_c^m$,

$$\begin{aligned} U_1(\varphi) &= U_1(\varphi \circ \gamma_1 \circ \operatorname{Re} \gamma_1^{-1}) = W_1(\varphi \circ \gamma_1) = W_2(\varphi \circ \gamma_1) \\ &= U_2(\varphi \circ \gamma_1 \circ \gamma_1^{-1} \circ \gamma_2 \circ \operatorname{Re} \gamma_2^{-1}) \\ &= U_2(\varphi \circ \gamma_2 \circ \operatorname{Re} \gamma_2^{-1}) = U_2(\varphi). \end{aligned}$$

DEFINITION 2.4. Let S be a C_c^m-scalar transformation such that $\operatorname{Sp}(S)$ is contained in a C_∞^m-curve Δ. A C_c^m-spectral representation U for S is called *canonical with respect to* Δ, if there exists a representation γ of Δ for which U satisfies the condition (γ).

By the above theorem, we see that the canonical representation is uniquely determined by S and Δ.

COROLLARY. *Let S be a closed transformation such that* $\operatorname{Sp}(S)$ *is contained in a* C_∞^m*-curve* Δ. *Then S is a* C_c^m*-scalar transformation having a canonical representation with respect to* Δ *if and only if it is a* $C^m(\gamma)$*-transformation for a representation* γ *of* Δ.

Proof. Theorem 2.2 is the "if" part. If S is a C_c^m-scalar transformation having the canonical representation U with respect to Δ, then U satisfies the condition (γ) for some representation γ of Δ. If we define $W(f) = U(f \circ \operatorname{Re} \gamma^{-1})$, then we can see, by an argument similar to the proof of the previous theorem, that S is a $C^m(\gamma)$-transformation with the $C^m(\gamma)$-representation W.

Remark 2.2. We can formulate an example corresponding to Example 1.1 with an extra condition on the existence of an identity net.

If S is a C_c^m-scalar transformation such that $\mathrm{Sp}(S) = \Delta$ for some C_∞^m-curve Δ, then there is no other C_∞^m-curve containing $\mathrm{Sp}(S)$. Hence, in this case, the *canonical representation for* S is uniquely determined by S.

If $\mathrm{Sp}(S)$ does not coincide with a whole C_∞^m-curve, then we can make remarks similar to Remark 1.1.

Remark 2.3. In the arguments in Part I, if $\mathrm{Sp}(S)$ does not coincide with a whole C^m-curve, then it is enough to assume that the representation γ of the curve can be extended to be one-to-one and C^m only on a neighborhood of $\gamma^{-1}(\mathrm{Sp}(S))$. If $\mathrm{Sp}(S)$ is contained in a C_∞^m-curve and if it is compact, so that $\gamma^{-1}(\mathrm{Sp}(S)) \subseteq [-\tau, \tau]$ for some $\tau > 0$, then

$$\tilde{\gamma} = (1 + \mathrm{Im}\,\gamma^{-1}) \exp\,(i\pi\, \mathrm{Re}\,\gamma^{-1}/2\tau)$$

gives a one-to-one C^m-mapping of a neighborhood of $\mathrm{Sp}(S)$ onto a neighborhood of $\tilde{\gamma}(\mathrm{Sp}(S)) \subseteq \Gamma$ and $\gamma_1 \circ u_{\gamma_1} = \gamma \circ \mathrm{Re}\,\gamma^{-1}$, where $\gamma_1 = \tilde{\gamma}^{-1}$. Hence, we can regard $\mathrm{Sp}(S)$ to be contained in a C^m-curve in the sense remarked above. Thus, if $\mathrm{Sp}(S)$ is compact, then we can reduce our arguments to those in Part I. This means that essential difference of Part II from Part I, in general discussions, appears only when $\mathrm{Sp}(S)$ is *not* compact. In this respect, we shall state in this part, only those theorems which deal with general transformations having non-compact spectrum or with C^m-real operators.

THEOREM 2.4. *If S is a C_c^0-scalar transformation whose spectrum is contained in a C_∞^0-curve, then S has a unique C_c^0-spectral representation, which is canonical with respect to any C_∞^0-curve containing* $\mathrm{Sp}(S)$.

The proof of this theorem is similar to that of Theorem 1.4.

Canonical representations for C^m-real operators (cf. [5]). If S is a closed transformation whose spectrum is contained in $\hat{R}$, then S is C_c^m-scalar with a canonical representation with respect to $\hat{R}$ if and only if it is a C^m-real transformation. Thus, for a C^m-real transformation S, its canonical representation with respect to $\hat{R}$ is called *the canonical representation for* S. (This definition is equivalent to the definition in [5,§7], where a C^m-real transformation was called a real C_c^m-scalar operator.)

Remark 2.4. If $\mathrm{Sp}(S) \subseteq \{1, -1\} (= \Gamma \cap R)$, then S can be C^m-unitary and C^m-real at the same time. In this case, the canonical representation for S as a C^m-unitary operator may be different from that for S as a C^m-real operator. Example: $S = I + Q$ with a nilpotent operator Q.

Examples of C^m-real transformations were given in [5].

Properties of the canonical representations.

THEOREM 2.5. *Let S be a C_c^m-scalar transformation such that* $\mathrm{Sp}(S)$ *is contained in a C_∞^m-curve Δ and suppose it has the canonical representation U_Δ with respect to Δ.*

(i) *If $T \in L(E)$ commutes with S on D_S, then T commutes with each $U_\Delta(\varphi)$, $\varphi \in C_c^m$.*

(ii) *If F is a closed subspace of $E_S = \{U_\Delta(\varphi)x;\ \varphi \in C_c^m,\ x \in E\}$ and if S leaves F invariant, then each $U_\Delta(\varphi)$ leaves F invariant.*

Proof. (i) Let γ be a representation of Δ for which U_Δ satisfies the condition (γ). Then there exists the $C^m(\gamma)$-representation W for S. It is enough to show that T commutes with each $W(f)$, $f \in C_c^m(R)$. For any $g_1, g_2 \in C_c^m(R)$ and $x \in E$, we have

$$W(g_1\gamma)TW(g_2)x = W(g_1)STW(g_2)x = W(g_1)TSW(g_2)x = W(g_1)TW(g_2\gamma)x.$$

Similarly, we have

$$W(g_1\gamma^k)TW(g_2)x = W(g_1)TW(g_2\gamma^k)x, \quad k = 0, 1, 2, \cdots.$$

Hence,

$$W[g_1(P \circ \gamma)]TW(g_2) = W(g_1)TW[g_2(P \circ \gamma)]$$

for any polynomial P. Given $f \in C_c^m(R)$, we approximate it by $P \circ \gamma$ on $\mathrm{Supp}\, g_1 \cup \mathrm{Supp}\, g_2$ (Lemma 2.1) and we obtain

$$W(g_1)W(f)TW(g_2) = W(g_1)TW(f)W(g_2).$$

Letting g_1, g_2 be members of an identity net for W and taking limits, we finally have $W(f)T = TW(f)$.

(ii) With the same notations as in (i), we shall show that each $W(f)$ leaves F invariant. If $x \in F \subseteq E_S$, then there exists $f_0 \in C_c^m(R)$ such that $x = W(f_0)x$. Then

$$W(f_0\gamma)x = Sx \in F$$

by assumption. By induction, we see that $W(f_0\gamma^k)x \in F$ for all $x \in F$. It follows that

$$W[f_0(P \circ \gamma)]x \in F$$

for any polynomial P. Hence, given $f \in C_c^m(R)$, we conclude that

$$W(f)x = W(f_0 f)x \in F$$

by Lemma 2.1.

COROLLARY 1. *Let S be as in the theorem. Then any other C_c^m-spectral representation for S commutes with U_Δ.*

COROLLARY 2. *Let $S_i \in L(E)\,(i = 1, 2)$ be a $C_c^{m_i}$-scalar operator such that $\mathrm{Sp}(S_i)$ is contained in a $C_\infty^{m_i}$-curve $\Delta_i (i = 1, 2)$ and suppose $S_i (i = 1, 2)$ has the canonical representation U_i with respect to $\Delta_i (i = 1, 2)$. If S_1 and S_2 commute, then U_1 and U_2 commute.*

2.3. An existence theorem. Let S be a C_c^m-scalar transformation such that $\mathrm{Sp}(S)$ is contained in a C_∞^{2m}-curve Δ. If $\mathrm{Sp}(S)$ is compact, then we can reduce the arguments to Part I (see Remark 2.3), so that we have Theorem 1.7 for an existence of the canonical representation. In this connection, we correct an error in [5], i.e., Proposition 6 in [5] should read:

Proposition 6. If S is a C_c^m-scalar operator whose spectrum is compact and contained in the real line, then there exists a unique C_c^{2m}-spectral representation U such that $S = S_U = S_U^*$.

The three lines after the definition in p. 148 of [5] should also be changed in accordance with the above correction.

If $\mathrm{Sp}(S)$ is not compact, we do not know if the corresponding theorem holds in general. When we try to apply the method of Theorem 1.7 to this case, a difficulty appears in showing the existence of an identity net for U_Δ. However, we can prove:

THEOREM 2.6. *Let $m \geq 1$ and let S be a C^m-scalar operator such that $\mathrm{Sp}(S)$ is contained in a C_∞^{2m}-curve Δ. Then there exists the canonical representation with respect to Δ for S as a C_c^{2m}-scalar operator. Furthermore this representation is C^{2m}-spectral.*

The proof is similar to that of Theorem 1.7. Here we use the function $\mathrm{Re}\,\gamma^{-1}$ instead of u_γ. We take a C^m-spectral representation U for S and construct U_Δ.

COROLLARY 1. *Let $S_i (i = 1, \cdots, n)$ be a C^∞-scalar operator such that $\mathrm{Sp}(S_i) \subseteq \Delta_i$ for a C_∞^∞-curve $\Delta_i (i = 1, \cdots, n)$. If $S_1, \cdots, S_n$ commute with each other, then $P(S_1, \cdots, S_n)$ is a C^∞-scalar operator for any polynomial P in n variables.*

Proof. By the above theorem and Corollary to Theorem 2.5, we see that $S_1, \cdots, S_n$ have C^∞-spectral representations $U_1, \cdots, U_n$ respectively which commute with each other. Then, we see that $P(S_1, \cdots, S_n)$ is C^∞-scalar by (i) of Corollary to Proposition 3.1 in [4].

COROLLARY 2. *If $S_1, \cdots, S_n$ are C^∞-scalar operators such that $\mathrm{Sp}(S_i) \subseteq \hat{R}$ for all $i = 1, \cdots, n$ and if they commute with each other, then $P(S_1, \cdots, S_n)$ is a C^∞-real operator for any polynomial P in n variables with real coefficients.*

2.4. Rate of growth of resolvents.

LEMMA 2.2. *Let S be a C_c^m-scalar transformation. For $\lambda \notin \mathrm{Sp}(S)$, let $d_\lambda = \mathrm{dis}\,(\lambda, \mathrm{Sp}(S))$ and $R(\lambda) = (\lambda I - S)^{-1}$.*

For any continuous semi-norm p on E and $x \in E_S$, there exists a non-negative integer $m_1 = m(p, x)$ ($=m$, if m is finite) such that

$$p(R(\lambda)x) \leq M_{p,x}(d_\lambda^{-m_1-1} + d_\lambda^{-1}) \qquad (M_{p,x} > 0) \tag{4}$$

for all $\lambda \notin \mathrm{Sp}(S)$.

If S is C^m-scalar, then for any continuous semi-norm q on $L(E)$, there exists a non-negative integer m_q ($=m$, if m is finite) such that

$$q(R(\lambda)) \leq M_q(d_\lambda^{-m_q-1} + d_\lambda^{-1}) \qquad (M_q > 0) \tag{5}$$

for all $\lambda \notin \mathrm{Sp}(S)$.

Proof. (i) First, let S be a C_c^m-scalar transformation and let U be a C_c^m-spectral representation for S. Given $x \in E_S$, there exists $\varphi_x \in C_c^m$ such that $U(\varphi_x)x = x$. Then,

$$\Sigma_x = \mathrm{Sp}(S) \cap \mathrm{Supp}\, \varphi_x$$

is a compact set. For any $d > 0$, we choose (cf. the proof of Lemma 1.2) $\varphi_d \in C_c^m$ in such a way that $\varphi_d \equiv 1$ on a neighborhood of Σ_x, $0 \leqq \varphi_d \leqq 1$, $\varphi_d(z) = 0$ if dis $(z, \Sigma_x) > \min(d/2, 1)$ and $\| \varphi_d \|_{l,\Sigma} \leqq K_l(d^{-l} + 1)$ for all $l = 0, 1, 2, \cdots$, where $\Sigma = \{z; \text{dis}\,(z, \Sigma_x) \leq 1\}$. Let

$$\psi_{\lambda,x}(z) = \frac{\varphi_x(z)\varphi_{d_\lambda}(z)}{\lambda - z}$$

for $\lambda \notin \mathrm{Sp}(S)$. Then, $\| \psi_{\lambda,x} \|_{l,\Sigma} \leq K_l'(d_\lambda^{-l-1} + d_\lambda^{-1})$ and $R(\lambda)x = U(\psi_{\lambda,x})x$. Hence, by the continuity of the mapping $\varphi \to U(\varphi)x$, we have (4).

(ii) If S is C^m-scalar, then there is a C^m-spectral representation U for S. For any continuous semi-norm q on $L(E)$, there exists a compact set Σ' and an integer m_q ($=m$, if m is finite) such that

$$q[U(\varphi)] \leqq M_q' \| \varphi \|_{m_q,\Sigma'} \qquad (M_q' > 0)$$

for all $\varphi \in C^m$. Then, we easily obtain (5) by expressing $R(\lambda) = U(\psi_\lambda)$ with a suitably chosen $\psi_\lambda \in C^m$.

It is an open problem to formulate a general theorem corresponding to Theorem 1.8 in the case $\mathrm{Sp}(S)$ is not compact. Here, we discuss only the case $\mathrm{Sp}(S) \subseteq \hat{R}$.

LEMMA 2.3. *Let S be a closed transformation. If $\mathrm{Sp}(S) \subseteq \hat{R}$ and if, for each continuous semi-norm q on $L(E)$, there exists a non-negative integer $m_q \leq m - 2 (m \geq 2)$ such that*

$$q[R(\xi + i\eta)] \leq M_q(|\eta|^{-m_q-1} + |\eta|^{-1}) \tag{6}$$

for all ξ, η ($\eta \neq 0$), then

(i) *$\tilde{S} = (S - i)(S + i)^{-1} \equiv I + 2iR(-i)$ is a C^m-unitary operator;*

(ii) *There exists a continuous homomorphism W of $C_c^m(R)$ into $L(E)$ such that $W(f)x \in D_S$ for all $f \in C_c^m(R)$, $x \in E$, $W(f)Sx = SW(f)x$ for all $f \in C_c^m(R)$ and $x \in D_S$ and $SW(f) = W(tf)$.*

Proof. (i) First, we remark that $\tilde{S} \in L(E)$. For $z \notin \Gamma$, let

$$\tilde{R}(z) = \frac{1}{1-z}(S + i)R\left(i\frac{1+z}{1-z}\right) \equiv \frac{I}{z-1} + \frac{2i}{(z-1)^2} R\left(i\frac{1+z}{1-z}\right).$$

It is easy to see that $\tilde{R}(z) = (zI - \tilde{S})^{-1}$ for all $z \notin \Gamma$. It follows that $\mathrm{Sp}(\tilde{S}) \subseteq \Gamma$.

Since

$$\mathrm{Im}\left(i\frac{1+z}{1-z}\right) = \frac{1 - |z|^2}{|1 - z|^2},$$

(6) implies

$$q(R(z)) \leq M'_q \,|\, 1 - |z| \,|^{-m_q-1}$$

for all z with $|z| < 2$, $|z| \neq 1$. Hence, by Theorem 1.8, $\tilde{S}$ is a C^m-unitary operator.

(ii) Let $\tilde{W}$ be the $C^m(e^{i\theta})$-representation for $\tilde{S}$. For $f \in C_c^m(R)$, let

$$\tilde{f}(e^{i\theta}) = f\left(i\frac{1+e^{i\theta}}{1-e^{i\theta}}\right).$$

Then $\tilde{f} \in C^m(\Gamma)$ and $1 \notin \operatorname{Supp} \tilde{f}$. Let $W(f) = \tilde{W}(\tilde{f})$ for $f \in C_c^m(R)$. Then, we easily see that W is a continuous homomorphism of $C_c^m(R)$ into $L(E)$. If $x \in E$ and $f \in C_c^m(R)$, then

$$W(f)x = \tilde{W}(\tilde{f})x = (\tilde{S} - I)\tilde{W}\left(\frac{\tilde{f}}{e^{i\theta}-1}\right)x = 2iR(-i)\tilde{W}\left(\frac{\tilde{f}}{e^{i\theta}-1}\right)x \in D_S.$$

Since $\tilde{S}$ commutes with $\tilde{W}(\tilde{f})$, it follows that $SW(f)x = W(f)Sx$ for $x \in D_S$. Now,

$$W(tf) = \tilde{W}\left(i\frac{1+e^{i\theta}}{1-e^{i\theta}}\tilde{f}\right) = i(I+\tilde{S})\tilde{W}\left(\frac{\tilde{f}}{1-e^{i\theta}}\right).$$

Since

$$W(f) = 2iR(-i)\tilde{W}\left(\frac{\tilde{f}}{1-e^{i\theta}}\right)$$

as we have seen above, we have

$$\begin{aligned} W(tf) &= \tfrac{1}{2}(I+\tilde{S})(S+i)W(f) \\ &= \tfrac{1}{2}\{I + (S-i)(S+i)^{-1}\}(S+i)W(f) = SW(f). \end{aligned}$$

THEOREM 2.7. *Any C^m-scalar operator S such that* $\operatorname{Sp}(S) \subseteq \hat{R}$ *is a* C^{m+2}-*real operator.*

Proof. In view of Theorem 2.6, it is enough to prove the case $m \geq 2$. By the previous two lemmas, we see that $\tilde{S} = (S-i)(S+i)^{-1}$ is a C^{m+2}-unitary operator. Let $\tilde{W}$ and W be as in the proof of the previous lemma. (Here, $\tilde{W}$, W are continuous homomorphisms of $C^{m+2}(\Gamma)$ and $C_c^{m+2}(R)$, respectively, into $L(E)$.)

By Theorem 2.6, there exists a continuous homomorphism W_1 of $C^{2m}(R)$ into $L(E)$ such that $W_1(1) = I$ and $W_1(t) = S$ (i.e., $W_1(f) = U_\Delta(f(\operatorname{Re} z))$ for $f \in C^{2m}(R)$). We shall show that $W_1(f) = W(f)$ for $f \in C_c^{2m}(R)$. Since $2m \geq m+2$, it follows that W is a $C^{m+2}(t)$-representation for S and the theorem will be proved.

Let $u(t) = (t-i)/(t+i)$ $(t \in R)$. u is a one-to-one C^∞-mapping of R onto $\Gamma - \{1\}$. For $g \in C^{2m}(\Gamma)$, $g \circ u \in C^{2m}(R)$ and $g \to g \circ u$ is continuous from $C^{2m}(\Gamma)$ into $C^{2m}(R)$. Hence, $\tilde{W}_1(g) = W_1(g \circ u)$ defines a continuous homomorphism $\tilde{W}_1$ of $C^{2m}(\Gamma)$ into $L(E)$. Furthermore,

$$\tilde{W}_1(1) = I \quad \text{and} \quad \tilde{W}_1(e^{i\theta}) = W_1\left(\frac{t-i}{t+i}\right) = \tilde{S}.$$

Hence, by Theorem 1.1, $\tilde{W}(g) = \tilde{W}_1(g)$ for all $g \in C^{2m}(\Gamma)$. Since $W(f) = \tilde{W}(f \circ u^{-1})$, it follows that $W(f) = W_1(f)$ for all $f \in C_c^{2m}(R)$.

Remark 2.5. If the condition (6) of Lemma 2.3 is satisfied, then we see, by a method of Tillmann [8], that

$$W(f) = \lim_{\epsilon \to 0+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(t)\{R(t - i\epsilon) - R(t + i\epsilon)\}\, dt \tag{7}$$

is defined for $f \in C_c^m(R)$ and W is a continuous linear mapping of $C_c^m(R)$ into $L(E)$. If $\mathrm{Sp}(S)$ is compact ($\subseteq R$), then we can prove that W is a $C^m(t)$-representation for S (Tillmann [9]; also see Remark 2.3). Now, we ask the following question for S such that $\mathrm{Sp}(S) \subseteq R$ and $\mathrm{Sp}(S)$ is *not* compact: Under what conditions on $R(\lambda)$, does W, defined by (7), become the $C^m(t)$-representation for S? If this question is solved under suitable conditions on $R(\lambda)$, we may be able to prove Theorem 2.7 directly without using Lemma 2.3 or Theorem 2.6.

2.5. Another characterization of C^m-real operators. If $\mathrm{Sp}(S)$ is compact, then e^{itS} $(-\infty < t < \infty)$ is defined by the operational calculus. Following Kantorovitz [2], we give here a characterization of a C^m-real operator whose spectrum is compact in terms of e^{itS}, which extends Corollary 2.11 of [2].

THEOREM 2.8. (i) *If S is a C_c^m-scalar operator such that* $\mathrm{Sp}(S)$ *is compact and contained in R (in particular, if S is a C^m-real operator with compact spectrum), then for each continuous semi-norm q on $L(E)$, there exists a non-negative integer m_q ($=m$, if m is finite) such that*

$$q(e^{itS}) \leq M_q \,|\, t \,|^{m_q} \tag{8}$$

for all real number t with $|\, t \,| \geqq 1$.

(ii) *Conversely, if $S \in L(E)$,* Sp (S) *is compact and if* (8) *is satisfied by m_q with $0 \leq m_q \leq m - 2 (m \geq 2)$, then S is a C^m-real operator.*

Proof. (i) Let $\varphi_d (d > 0)$ be the functions defined in the proof of Lemma 1.2 and let $\psi_t(\lambda) = \varphi_{1/|t|}(\lambda)e^{it\lambda}$ for $|\, t \,| \geq 1$. Then

$$\|\, \psi_t \,\|_{l,\Sigma} \leq K_l'' \,|\, t \,|^l$$

for all $l = 0, 1, 2, \cdots$; $|\, t \,| \geq 1$. If U is a C_c^m-spectral representation for S, then $e^{itS} = U(\psi_t)$. Hence, we obtain (8) by the continuity of U.

(ii) By Theorem 1.9, (ii), the condition (8) implies that $\mathrm{Sp}(e^{itS}) \subseteq \Gamma$ for each t. It follows from the spectral mapping theorem that $\mathrm{Sp}(S) \subseteq R$. Let

$$\tau \geq \sup \{|\, \lambda \,|; \lambda \in \mathrm{Sp}(S)\} \qquad (\tau > 0)$$

and let $t_0 = \pi/2\tau$. Then, $\lambda \to e^{it_0\lambda}$ is a one-to-one analytic mapping of a neighborhood of $\mathrm{Sp}(S)$ onto a neighborhood of the semi-circle on which $\mathrm{Sp}(e^{it_0S})$ lies. Hence, by Corollary 3 to Theorem 1.9, we see that S is a C^m-real operator.

Remark 2.6. The second part of the previous theorem can also be proved as follows: The condition (8) implies that

$$q(R(\lambda)) \leq M'_q \,|\, \operatorname{Im} \lambda \,|^{-m_q-1} \quad \text{for} \quad 0 < |\, \operatorname{Im} \lambda \,| < 1. \tag{9}$$

Then, it follows that S is C^{m+2}-real (see Remark 2.5). The inequalities (9) can be seen from the facts that

$$R(\lambda) = i \int_0^\infty e^{-it\lambda} e^{itS}\, dt \quad \text{for} \quad \operatorname{Im} \lambda < 0;$$

$$R(\lambda) = -i \int_0^\infty e^{it\lambda} e^{-itS}\, dt \quad \text{for } \operatorname{Im} \lambda > 0$$

and that

$$\int_0^\infty e^{-t\eta} t^m\, dt = \frac{m!}{\eta^{m+1}} \quad \text{for any} \quad \eta > 0, \qquad m = 0, 1, 2, \cdots.$$

We can also show that (9) implies

$$q(e^{itS}) \leq M''_q \,|\, t \,|^{m_q+1}$$

for all t with $|\, t \,| \geq 1$, provided that $\operatorname{Sp}(S)$ is compact and contained in R.

Corollary (Cf. Corollary 2.12 of [2]). *Let S_1 and S_2 be commuting C^{m_1}- and C^{m_2}-real operators respectively and suppose* $\operatorname{Sp}(S_1)$ *and* $\operatorname{Sp}(S_2)$ *are both compact. Then $S_1 + S_2$ is a $C^{m_1+m_2+2}$-real operator and $S_1 \cdot S_2$ is a $C^{2m_1+2m_2+6}$-real operator.*

Appendix. Proofs of approximation theorems

We shall give proofs of Lemmas 1.1 and 2.1. If $m = 0$, then these lemmas are Theorem 7 and Theorem 8 of Walsh [10]. Our proofs for $m \geq 1$ are based on these theorems. We shall prove only the case m is finite, since, if these lemmas are true for any finite m, then it follows that they are true for $m = \infty$.

We remark that in order to have Lemma 1.1 (resp. Lemma 2.1), it is enough to assume that the representation γ is defined only on Γ (resp. on R) in such a way that γ is one-to-one on Γ (resp. on R), $\gamma \in C^m(\Gamma)$ (resp. $C^m(R)$) and $\tilde{\gamma}'(\theta) \neq 0$ everywhere on Γ (resp. $\gamma'(t) \neq 0$ everywhere on R).

I. *Proof of Lemma* 1.1. Without loss of generality, we may assume that the origin lies inside $\gamma(\Gamma)$, so that we can take $\lambda_0 = 0$. For any $f \in C^m(\Gamma)$, we use the notation $f(\theta)$ for $\tilde{f}(\theta) = f(e^{i\theta})$. Let

$$M = \sup_\theta |\, 1/\gamma(\theta)| \quad \text{and} \quad M' = \sup_\theta |\, \gamma'(\theta)|.$$

We know that $0 < M, M' < \infty$. Let

$$\mathcal{R} = \{P(z)/z^l;\, P : \text{polynomial} \quad \text{and} \quad l : \text{integer} \geq 0\}.$$

For $Q \in \mathcal{R}$, let $r(Q) =$ the residue of Q at $z = 0$.

We prove by induction on m. As remarked above, the case $m = 0$ is known to be true. Now, we assume that Lemma 1.1 is true for $m (\geq 0)$. If

$f \in C^{m+1}(\Gamma)$, then $g = f'/\gamma' \in C^{m}(\Gamma)$. ($\gamma$ is now a representation of a C^{m+1}-curve.) Hence, given $\varepsilon > 0$ there exists $Q_1 \in \mathfrak{R}$ such that

$$\left| g^{(l)}(\theta) - \frac{d^l}{d\theta^l} Q_1(\gamma(\theta)) \right| < \varepsilon \tag{A1}$$

for all θ and $l = 0, 1, 2, \cdots, m$. In particular,

$$|f'(\theta)/\gamma'(\theta) - Q_1(\gamma(\theta))| < \varepsilon. \tag{A2}$$

Let $Q(z)$ be a primitive of $Q_1(z) - r(Q_1)/z$ such that $Q(\gamma(0)) = f(0)$. Then $Q \in \mathfrak{R}$. Let $R(\theta) = f'(\theta)/\gamma'(\theta) - Q_1(\gamma(\theta))$. Then,

$$\begin{aligned} f'(\theta) &= \gamma'(\theta)Q_1(\gamma(\theta)) + \gamma'(\theta)R(\theta) \\ &= \gamma'(\theta)Q'(\gamma(\theta)) + \gamma'(\theta)\frac{r(Q_1)}{\gamma(\theta)} + \gamma'(\theta)R(\theta). \end{aligned}$$

Integrating both sides from 0 to θ ($0 \leq \theta \leq 2\pi$), we have

$$f(\theta) - f(0) = Q(\gamma(\theta)) - Q(\gamma(0)) + r(Q_1)\int_0^\theta \frac{\gamma'(\theta)}{\gamma(\theta)}\, d\theta + \int_0^\theta \gamma'(\theta)R(\theta)\, d\theta.$$

If $\theta = 2\pi$, this equation becomes

$$0 = \pm 2\pi i\, r(Q_1) + \int_0^{2\pi} \gamma'(\theta)R(\theta)\, d\theta.$$

By (A2), $|R(\theta)| < \varepsilon$ for all θ. Hence,

$$|r(Q_1)| < \varepsilon M'. \tag{A3}$$

Then,

$$\begin{aligned} |f(\theta) - Q(\gamma(\theta))| &\leq |r(Q_1)| \left| \int_0^\theta \frac{\gamma'(\theta)}{\gamma(\theta)}\, d\theta \right| + \left| \int_0^\theta \gamma'(\theta)R(\theta)\, d\theta \right| \\ &\leq 2\pi M M'^2 \varepsilon + 2\pi M'\varepsilon = \varepsilon M_0 \qquad (M_0 > 0). \end{aligned}$$

Now,

$$\begin{aligned} f'(\theta) - \frac{d}{d\theta} Q(\gamma(\theta)) &= f'(\theta) - \gamma'(\theta)\left[Q_1(\gamma(\theta)) - \frac{r(Q_1)}{\gamma(\theta)} \right] \\ &= \gamma'(\theta)[g(\theta) - Q_1(\gamma(\theta))] + \frac{\gamma'(\theta)}{\gamma(\theta)} r(Q_1). \end{aligned}$$

Hence, for $1 \leq l \leq m + 1$,

$$\begin{aligned} &f^{(l)} - \frac{d^l}{d\theta^l}(Q \circ \gamma) \\ &\qquad = \sum_{k=0}^{l-1} \binom{l-1}{k} \gamma^{(l-k)} \left[g^{(k)} - \frac{d^k}{d\theta^k}(Q_1 \circ \gamma) \right] + r(Q_1) \frac{d^{l-1}}{d\theta^{l-1}} \left(\frac{\gamma'}{\gamma} \right). \end{aligned}$$

By (A1) and (A3), we see that there exists $M_l > 0$ such that

$$\left| f^{(l)}(\theta) - \frac{d^l}{d\theta^l} Q(\gamma(\theta)) \right| \leq \varepsilon M_l$$

for all θ, $l = 1, 2, \cdots, m+1$. Hence, we have the lemma for $m + 1$.

II. *Proof of Lemma* 1.2. It is enough to prove the following: Given $f \in C^m$, a compact interval $[-\tau, \tau]$ on R and $\varepsilon > 0$, there exists a polynomial P such that

$$\left| f^{(l)}(t) - \frac{d^l}{dt^l} P(\gamma(t)) \right| < \varepsilon$$

for all $t \in [-\tau, \tau]$ and $l = 0, 1, 2, \cdots, m$.

The proof goes in the same way as in I, replacing Q_1 and Q by polynomials, Theorem 7 of [10] by Theorem 8. Since we do not have to worry about the residue in this case, the proof becomes simpler.

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