UNKNOTTING LOCALLY FLAT CELL PAIRS[1]

BY

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Zeeman [6] shows that any proper combinatorial cell pair (n, k), where $n - \mathrm{k} \geqq 3$, is piecewise linearly homeomorphic to a standard pair. This implies the combinatorial unknotting of spheres in spheres, and of spheres in euclidean space, provided that the codimension is $\geqq 3$. If the codimension $= 2$, then the spheres can be knotted combinatorially. If the codimension $= 1$, the unknotting problem is the same as the combinatorial Schoenflies conjecture which is still unsolved.

In pure topology, with a hypothesis of topological local flatness, Brown [2] has proved the topological Schoenflies theorem in all dimensions. With a similar hypothesis, Stallings [5] has proved the topological unknotting of k-spheres in n-spheres for $n - k \geqq 3$ and $n \geqq 5$. Gluck [4] solved the case of a locally flat 1-sphere in S^4. Cantrell [3] has shown that if a $(n-1)$-sphere in S^n $(n \geqq 4)$ is locally flat except for perhaps one point then the sphere pair is also flat. Stallings' result [5] also follows for sphere pairs of codimension $\geqq 3$ where the lower dimensional sphere is locally flat except for perhaps one point. Gluck's result is valid if S^1 in S^4 is locally flat except perhaps at a countable number of points.

Here we consider the topological analogue of the combinatorial cell pair theory. The main results are that if a topological cell pair (A, B) of type (n, k) is locally flat except for perhaps one point, either in the interior of B or its boundary, then, if $n \geqq 5$ and $k \neq n - 2$, (A, B) is homeomorphic to the standard pair of type (n, k). For $k = n - 2$ and $n \geqq 6$, with additional appropriate hypothesis, the same conclusion follows.

The following notation and definitions are used throughout this paper. They are similar to those used in [1] and [5], except that the phrase locally flat is used instead of locally smooth, as in [5]. By an n-manifold with boundary we will mean a separable metric space each point of which has a closed neighborhood homeomorphic to I^n. The boundary of a manifold X is denoted by $X^{\cdot}$. A manifold pair (X, Y) of type (n, k), $n > k$, is an n-manifold X, and a subset Y which is a k-manifold. Furthermore, it is assumed that either both of X and Y have boundary or both do not have boundary. If both do have boundary it is assumed that $Y^{\cdot} \subset X^{\cdot}$. The boundary manifold pair $(X^{\cdot}, Y^{\cdot})$ of (X, Y) is denoted by $(X, Y)^{\cdot}$.

The following examples of manifold pairs will be of primary interest. When X is E^n and Y is a closed subset of X homeomorphic to E^k, then the pair (X, Y) is called a string of type (n, k). When X is an n-cell and Y is homeomorphic to a k-cell, then (X, Y) is called a cell pair of type (n, k). A pair

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(X, Y) is called a punctured cell pair if there exists a cell pair $(\hat{X}, \hat{Y})$ and a point $p \in \operatorname{int} \hat{Y}$ such that (X, Y) is homeomorphic to $(\hat{X} - \{p\}, \hat{Y} - \{p\})$. Finally, the pair (X, Y) is called a half string of type (n, k) if X is E_+^n and Y is a closed subset of X homeomorphic to E_+^k. The standard string of type (n, k), the standard cell pair of type (n, k), and the standard half string of type (n, k) are respectively (E^n, E^k), the unit cells of E^n and E^k, and (E_+^n, E_+^k), where $E^k = E^k \times \{0\} \subset E^n$.

If (X, Y) is a manifold pair of type (n, k) then Y is locally flat in X if each point of Y has a neighborhood U in X such that $(U, U \cap Y)$ is homeomorphic to (E^n, E^k) or (E_+^n, E_+^k) according as the point is or is not in the interior of Y. The pair $(X, Y)^{\cdot}$ is locally collared in (X, Y) if Y is locally flat at each point of $Y^{\cdot}$. The pair $(X, Y)^{\cdot}$ is collared in (X, Y) if there is a homeomorphism H from $(X, Y)^{\cdot} \times [0, 1)$ onto a neighborhood U of $(X, Y)^{\cdot}$ in (X, Y) such that $H(x, 0) = x$ for each $x \in X^{\cdot}$. A neighborhood V of $(X, Y)^{\cdot}$ in U is called a subcollar of U if

$$V = H((X, Y)^{\cdot} \times [0, t))$$

for some $0 < t < 1$. We will say that the mainfold pair (X, Y) of type (n, k) is unraveled at $p \in Y$, if for each compact $C \subset X - \{p\}$ there is a compact D, with $C \subset D \subset X - \{p\}$ such that the pair $(X - Y, X - (Y \cup D))$ is 2-connected.

LEMMA 1. *Let (M, N) be a manifold pair with nonempty boundary. If the boundary pair $(M, N)^{\cdot}$ is locally collared in (M, N), then it is collared in (M, N).*

Proof. The proof is exactly the same as in M. Brown's original proof in [1] that the boundary of a manifold with nonempty boundary is collared. It is only necessary to check that the collar of $M^{\cdot}$ can be restricted to be a collar of $N^{\cdot}$ in N. This follows from the construction of the collar plus the fact that the local collaring, by definition, restricts to be a local collar on $N^{\cdot}$ in N.

LEMMA 2. *Let (X, Y) be either a locally flat half string or a locally flat punctured cell pair of type (n, k). Let U be a collar of $(X, Y)^{\cdot}$ in (X, Y) and let V be a subcollar of U. Suppose A is a compact subset of X; then there exists a homeomorphism $g : (X, Y) \to (X, Y)$ and a compact set $B \subset \operatorname{int}(X - V)$ such that:*

(i) $g|_{X-B} =$ *identity (hence* $g|_V =$ *identity), and*

(ii) $A \cap Y \subset g(U)$.

Proof. The proof is similar to the proof of Lemma 5.3 of [5]. We construct a finite sequence $h_1, h_2, \cdots, h_p$ of homeomorphisms of $Y - V$ onto itself, each of which can be extended to a homeomorphism of X onto itself by using the local flatness. The h_i's are each to be the identity except on a small compact subset of $Y - \bar{V}$.

THEOREM 1. *Let (X, Y) be either a locally flat half string or a locally flat* *punctured cell pair of type* (n, k) *with* $n \geqq 5$. *If* $(X - Y, X^{\cdot} - Y^{\cdot})$ *is* $(n-3)$-*connected and*

(i) $k \leqq n - 3$ *or*

(ii) $k = n - 2$ *and* (X, Y) *is unraveled at infinity,*

then (X, Y) *is homeomorphic to* $(X, Y)^{\cdot} \times [0, 1)$.

Proof. By Lemma 1 there exists a collar U of $(X, Y)^{\cdot}$ in (X, Y). We shall now construct a homeomorphism h taking $(U, U \cap Y)$ onto (X, Y). The construction is similar to that carried out in proving Theorem 9.2 of [5]. We shall now give some details.

Since U is a collar, there is a homeomorphism

$$H : (X, Y)^{\cdot} \times [0, 1) \to (U, U \cap Y).$$

Let

$$U_i = H\left((X, Y)^{\cdot} \times \left[0, \frac{i}{i+1}\right)\right) \qquad \text{for } i = 1, 2, 3, 4, \cdots.$$

Then U_i is a subcollar of U and $(X - U_i, Y - U_i)$ is homeomorphic to (X, Y). Let $\{E_i\}$, $i = 1, 2, 3, \cdots$ be a monotone sequence of compact subsets of X such that $X = \bigcup_{i=1}^{\infty} E_i$. We shall construct a sequence of homeomorphisms

$$f_i : (X, Y) \to (X, Y), \qquad i = 1, 2, 3, \cdots$$

such that $E_i \subset f_i(U_i)$ and $f_i |_{U_{i-1}} = f_{i-1} |_{U_{i-1}}$ $(i \geqq 2)$.

Since (X, Y) is unraveled at infinity, either by hypothesis if $k = n - 2$ or by making the necessary modifications of Proposition 4.2 of [5], there exists a compact set $D_i \subset X$ such that $E_i \subset D_i$ and $(X - Y, X - (Y \cup D_i))$ is 2-connected. In fact we will choose D_i so that

$$(X - (Y \cup f_{i-1}(U_{i-1})), X - (D_i \cup Y \cup f_{i-1}(U_{i-1})))$$

is 2-connected. (Recall $(X - U_{i-1}, Y - U_{i-1})$ is homeomorphic to (X, Y) and hence $(X - f_{i-1}(U_{i-1}), Y - f_{i-1}(U_{i-1}))$ is also homeomorphic to (X, Y), $i \geqq 2$.)

We will construct the f_i's inductively. Suppose we have already obtained $f_1, f_2, \cdots, f_{i-1}$. Lemma 2 gives us a homeomorphism $g : (X, Y) \to (X, Y)$ such that $D_i \cap Y \subset g \circ f_{i-1}(U_i)$ and $g |_{f_{i-1}(U_{i-1})} = \text{identity}$. We now proceed as in Section 7 of [5].

Let T be a piecewise linear triangulation of $X - Y$ and $\hat{T}$ a triangulation of

$$X - (Y \cup \overline{f_{i-1}(U_{i-1})})$$

compatible with T so that if $\Delta \in \hat{T}$, then

$$\operatorname{diam} \Delta \leqq \tfrac{1}{2}\rho(\Delta, Y \cup \overline{f_{i-1}(U_{i-1})}).$$

Let K be the union of all closed simplexes Δ of $\hat{T}$ such that dimension $(\Delta) \leqq 2$ and

$$\Delta \subset \operatorname{st}_{\hat{T}}(X - (Y \cup g \circ f_{i-1}(U_i))).$$

We can suppose that K is a subcomplex of T also, since K misses $\overline{f_{i-1}(U_{i-1})}$. Now since

$$D_i \cup Y \subset g \circ f_{i-1}(U_i),$$

it follows that the compact set $D_i - g \circ f_{i-1}(U_i)$ is contained in

$$X - (Y \cup \overline{f_{i-1}(U_{i-1})}).$$

Hence

$$K \cap D_i \subset \mathrm{st}_{\hat{T}}(D_i - g \circ f_{i-1}(U_i)),$$

which is the union of finitely many simplexes of $\hat{T}$. We now want to apply the Engulfing Theorem [Theorem 6.1 of 5], making the following substitutions:

For	Substitute	For	Substitute
M^n	$X - (Y \cup \overline{f_{i-1}(U_{i-1})})$	C	$\emptyset$
U	$X - (Y \cup D_i \cup \overline{f_{i-1}(U_{i-1})})$	P	K
E	F_1	p	2
h	h_1		

We have that

$$(M^n - C, U - C) = (X - (Y \cup \overline{f_{i-1}(U_{i-1})}), X - (Y \cup D_i \cup \overline{f_{i-1}(U_{i-1})}))$$

is 2-connected by our remarks above and hence we can apply the Engulfing Theorem as indicated in the above table. We obtain a piecewise linear homeomorphism h_1 of

$$X - (Y \cup \overline{f_{i-1}(U_{i-1})})$$

onto itself and a compact set

$$F_1 \subset X - (Y \cup \overline{f_{i-1}(U_{i-1})})$$

such that:

(i) $h_1 |_{X-(Y \cup \overline{f_{i-1}(U_{i-1})} \cup F_1)} =$ identity and
(ii) $K \subset h_1(X - (Y \cup D_i \cup \overline{f_{i-1}(U_{i-1})}))$.

Because of (i) we can extend h_1 to take (X, Y) onto (X, Y) by defining it to be the identity on $Y \cup \overline{f_{i-1}(U_{i-1})}$. We will continue to call the extended homeomorphism h_1 also.

Let K_1 be the union of K and all those closed simplexes Δ of T such that $\Delta \subset X - (F_1 \cup D_i)$. Now $K_1 \subset h_1(X - (Y \cup D_i))$. Let L be the complementary skeleton of K_1 in T. Then $L - g \circ f_{i-1}(U_i)$ is compact and

$$\text{dimension } (L - g \circ f_{i-1}(U_i)) \leqq n - 3.$$

Now we apply the Engulfing Theorem to engulf L. The table corresponding to the above situation is:

For	Substitute	For	Substitute
M^n	$X - Y$	C	$\overline{f_{i-1}(U_{i-1})} - Y$
U	$g \circ f_{j-1}(U_i) - Y$	P	L
E	F_2	p	$n - 3$
h	h_2		

Except, possibly, for the hypothesis that $(M - C, U - C)$ is p-connected, all the hypotheses of the Engulfing Theorem are clear from the above construction. Now

$$(M - C, U - C) = (X - Y \cup \overline{f_{i-1}(U_{i-1})}), g \circ f_{i-1}(U_i) - (Y \cup \overline{f_{i-1}(U_{i-1})})).$$

But this is homeomorphic to $(\text{int}\, X - Y, U - (Y \cup X^{\cdot}))$ and is of the same homotopy type as $(X - Y, X^{\cdot} - Y^{\cdot})$ which is $(n - 3)$-connected by hypothesis. The Engulfing Theorem then gives that

$$h_2 |_{X-(Y \cup F_2)} = \text{identity},$$

$$F_2 \cap C = \emptyset \quad \text{and} \quad L \subset h_2(g \circ f_{i-1}(U_i) - Y).$$

Again we extend h_2 by the identity to take (X, Y) onto (X, Y).

We now have

$$L \subset h_2 \circ g \circ f_{i-1}(U_i) - Y, K_1 \subset h_1(X - (Y \cup D_i))$$

and K_1 and L are complementary complexes of T. Now applying Lemma 8.1 of [5] in the appropriate manner there exists a homeomorphism

$$h_3 : X - Y \to X - Y$$

so that

$$h_3 |_{f_{i-1}(U_{i-1})} = \text{identity},$$

$$h_3(h_2 \circ g \circ f_{i-1}(U_i) - Y) \cup h_1(X - (Y \cup D_i)) = X - Y,$$

and so that h_3 can be extended by the identity to all of X. If we define f_i to be $h_1^{-1} \circ h_3 \circ h_2 \circ g \circ f_{i-1}$, it then follows that

$$E_i \subset D_i \subset f_i(U_i) \quad \text{and} \quad f_i |_{f_{i-1}(U_{i-1})} = f_{i-1} |_{f_{i-1}(U_{i-1})} .$$

The latter statement follows since each of h_1, h_2, h_3 and g is the identity on $f_{i-1}(U_{i-1})$. Thus f_i satisfies the necessary requirements and inductively we get our desired sequence of homeomorphisms $\{f_i\}, i = 1, 2, \cdots$. If we define $f = \lim_{i \to \infty} f_i |_U$, then f is a homeomorphism of $(U, U \cap Y)$ onto (X, Y) and hence (X, Y) is homeomorphic to $(X, Y)^{\cdot} \times [0, 1)$.

COROLLARY 1. *Let (X, Y) be a cell pair of type (n, k) which is locally flat except possibly at one point $p \in Y$. If $n \geqq 5$ and $k \leqq n - 3$, then (X, Y) is homeomorphic to the standard cell pair of type (n, k).*

Proof. If $p \in Y^{\cdot}$, then $(X - \{p\}, Y - \{p\})$ is a locally flat half string and if $p \notin Y^{\cdot}$, then $(X - \{p\}, Y - \{p\})$ is a punctured cell pair. If $n \geqq 6$, $(X, Y)^{\cdot}$ is a locally flat string or sphere pair and by Theorem 9.2 or Corollary 9.3 of [5] it is homeomorphic to (E^{n-1}, E^{k-1}) or (S^{n-1}, S^{k-1}) respectively. For $n = 5$, if $(X, Y)^{\cdot}$ is a locally flat string or sphere pair of type (4, 1) then it follows from [4] and the remark in the introduction that $(X, Y)^{\cdot}$ is also homeomorphic to (E^4, E^1) or (S^4, S^1). The case where $(X, Y)^{\cdot}$ is of type (4, 0) is trivial. Similar arguments as those used in Proposition 4.2 of [5] will give that $\pi_i(X - Y, X^{\cdot} - Y^{\cdot}) = 0$ for all i and hence Theorem 1 applies. We then get that $(X - \{p\}, Y - \{p\})$ is homeomorphic to

$$(E^{n-1}, E^{k-1}) \times [0, 1) \quad \text{or to} \quad (S^{n-1}, S^{k-1}) \times [0, 1)$$

and hence (X, Y) is homeomorphic to standard cell pair of type (n, k).

COROLLARY 2. *Let (X, Y) be a cell pair to type $(n, n - 2)$ such that each of $X - Y$ and $X^{\cdot} - Y^{\cdot}$ have the homotopy type of S^1. If $n \geqq 6$ and either*

(i) *the pair (X, Y) is locally flat, or*

(ii) *the pair (X, Y) is locally flat except possibly at one point $p \in$ int Y and (X, Y) is unraveled at p, or*

(iii) *the pair (X, Y) is locally flat except possibly at one point $p \in Y^{\cdot}$ and each of (X, Y) and $(X, Y)^{\cdot}$ is unraveled at p,*

then (X, Y) is homeomorphic to the standard cell pair of type $(n, n - 2)$.

Proof. The proof proceeds exactly as in the proof of Corollary 1. The hypotheses guarantee that $(X, Y)^{\cdot}$ is homeomorphic to (E^{n-1}, E^{n-3}) or (S^{n-1}, S^{n-3}).

PROPOSITION 3. *Let (X, Y) be a cell pair of type $(n, n - 1)$ which is locally flat except possibly at one point $p \in Y$. If $p \in$ int Y and $n \geqq 4$ or if $p \in Y^{\cdot}$ and $n \geqq 5$, then (X, Y) is homeomorphic to the standard pair of type $(n, n - 1)$.*

Proof. The proof is just a simple application of the results of [1], [2], [3].

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