ON FULL EMBEDDINGS OF CATEGORIES OF ALGEBRAS

BY

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Introduction and summary

The aim of this paper is to describe some full embeddings of categories, especially full embeddings concerning categories of abstract algebras; e.g. we prove that every full category of algebras can be fully embedded into the category of algebras with two unary operations, which strengthens a result of J. Isbell [2, p. 15]. To summarize the results in a simple way we describe some concrete categories[1] that will be referred to:

$\mathfrak{R}$. The objects are couples (X, R), where X is a non-void set and $R \subset X \times X$ (a binary relation on X); the morphisms from (X, R) into (Y, S) are all the mappings $f : X \to Y$ such that $(x, y) \in R$ implies $(f(x), f(y)) \in S$ for all $(x, y) \in R$. The morphisms of $\mathfrak{R}$ are sometimes called compatible mappings.

$A\mathfrak{R}$ (A is a set). The objects are systems $(X, \{R_a \mid a \in A\})$ where X is a non-void set, $R_a \subset X \times X$ for every $a \in A$; the morphisms from $(X, \{R_a\})$ into $(Y, \{S_a\})$ are all the mappings $f : X \to Y$ such that, for every $a \in A$, $(x, y) \in R_a$ implies $(f(x), f(y)) \in S_a$.

Let γ be an ordinal number, let $\Delta = \{\kappa_\alpha \mid \alpha < \gamma\}$ be a sequence of ordinal numbers (we consider zero to be an ordinal number, too). Such a sequence Δ will be frequently called a type. The symbol $\sum \Delta$ denotes the sum of ordinals in the ordinary sense.

$\mathfrak{Q}(\Delta)$ (the category of quasi-algebras of the type Δ). The objects are quasi-algebras, i.e. systems $(X, \{F_\alpha \mid \alpha < \gamma\})$, where X is a non-void set and F_α, for every $\alpha < \gamma$, is a κ_α-ary partial operation on X, i.e. a mapping of a subset of X^{κ_α} into X for $\kappa_\alpha \neq 0$, an element of X for $\kappa_\alpha = 0$. The morphisms from $(X, \{F_\alpha \mid \alpha < \gamma\})$ into $(Y, \{G_\alpha \mid \alpha < \gamma\})$ are all the homomorphisms, i.e. mappings $f : X \to Y$ satisfying the following conditions:

(1) If $\alpha < \gamma$, $\kappa_\alpha \neq 0$ and if $F_\alpha(\{x_\iota \mid \iota < \kappa_\alpha\})$ is defined, then $G_\alpha(\{f(x_\iota) \mid \iota < \kappa_\alpha\})$ is defined and

$$f(F_\alpha(\{x_\iota\})) = G_\alpha(\{f(x_\iota)\}).$$

(2) If $\kappa_\alpha = 0$, then $f(F_\alpha) = G_\alpha$.

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[1] The assumptions that the objects of the following categories are non-void sets are not substantial. All the results of the present paper remain true if we admit void objects simultaneously in all the categories.

$\mathfrak{A}(\Delta)$ (the category of algebras of the type Δ). This is the full subcategory of $\mathfrak{Q}(\Delta)$ generated by the algebras, i.e. by the objects

$$(X, \{F_\alpha \mid \alpha < \gamma\})$$

such that F_α is a mapping of X^{κ_α} into X for every $\kappa_\alpha \neq 0$.

$A\mathfrak{A}$ (A is a set). The objects are systems $(X, \{\varphi_\alpha \mid \alpha \in A\})$, where X is a non-void set; φ_α are unary operations on X; the morphisms from $(X, \{\varphi_\alpha\})$ into $(Y, \{\varphi_\alpha\})$ are all homomorphisms.

$\mathfrak{R}(\Delta^*)$. (the category of relational systems of the type Δ^*;

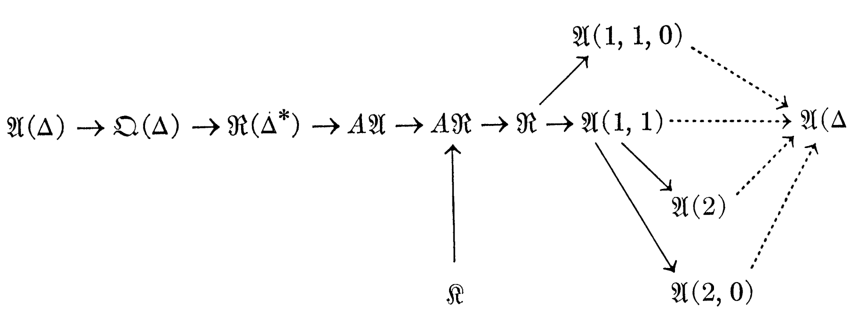
$$\Delta^* = \{\kappa_\alpha \mid \alpha < \gamma\}$$

such that $\kappa_\alpha > 0$ for every $\alpha < \gamma$; in general the asterisk over a type indicates always this fact.) The objects are systems $(X, \{R_\alpha \mid \alpha < \gamma\})$, where X is a non-void set and $R_\alpha \subset X^{\kappa_\alpha}$ for every $\alpha < \gamma$. The morphisms from $(X, \{R_\alpha\})$ into $(Y, \{S_\alpha\})$ are all mappings $f : X \to Y$ such that $\{f(x_\iota)\} \in S_\alpha$ for every $\alpha < \gamma$ and for every $\{x_\iota\} \in R_\alpha$.

If there is no danger of misunderstanding, some brackets will be sometimes omitted. We shall write e.g. $\mathfrak{A}(1, 1)$ instead of $\mathfrak{A}(\{1, 1\})$ etc. Let us remark that $A\mathfrak{R}$ is isomorphic with some $\mathfrak{R}(2, 2, \cdots)$, $A\mathfrak{A}$ with some $\mathfrak{A}(1, 1, \cdots)$. In the notation given above the mentioned theorem by J. Isbell may be formulated as follows:

Every full subcategory $\mathfrak{K}$ of some $\mathfrak{A}(\Delta)$ is isomorphic with a full subcategory of some $A\mathfrak{A}$.

If $\Delta = \{\kappa_\alpha \mid \alpha < \beta\}$, we denote $\Delta + 1 = \{\kappa_\alpha + 1 \mid \alpha < \beta\}$. The symbol $\mathfrak{K} \to \mathfrak{L}$ (where $\mathfrak{K}$, $\mathfrak{L}$ are categories) will mean that $\mathfrak{K}$ is isomorphic with a full subcategory of $\mathfrak{L}$ (the possibility of full embedding of $\mathfrak{K}$ into $\mathfrak{L}$). Obviously, $\mathfrak{K} \to \mathfrak{L}$ and $\mathfrak{L} \to \mathfrak{M}$ imply $\mathfrak{K} \to \mathfrak{M}$. We shall show in this paper that there are full embeddings described by the following diagram ($\mathfrak{R}$ is a small category, $\sum \Delta' \geq 2$):



$\mathfrak{A}(\Delta) \to \mathfrak{Q}(\Delta)$ follows immediately from the definitions.

$\mathfrak{Q}(\Delta) \to \mathfrak{R}(\Delta^*)$ means that there exists Δ^* such that $\mathfrak{Q}(\Delta) \to \mathfrak{R}(\Delta^*)$; it suffices to put $\Delta^* = \Delta + 1$.

$\mathfrak{R}(\Delta^*) \to A\mathfrak{A}$ means that there exists a set A such that $\mathfrak{R}(\Delta^*) \to A\mathfrak{A}$. The proof is given in paragraph 1.

The meaning of $\mathfrak{K} \to A\mathfrak{R}$ is similar ([4], see §4).

$A\mathfrak{A} \to A\mathfrak{R}$ follows easily from the definitions. The dotted arrows mean that, for any Δ', $\sum \Delta' \geq 2$, one of the categories $\mathfrak{A}(1, 1)$, $\mathfrak{A}(2)$, $\mathfrak{A}(1, 1, 0)$, $\mathfrak{A}(2, 0)$, can be fully embedded in $\mathfrak{A}(\Delta')$. Actually, any of them can be embedded in $\mathfrak{A}(\Delta')$; we describe it in this way only to indicate the proof in §1.

All assertions $\mathfrak{R} \to \mathfrak{A}(1, 1)$, $\mathfrak{R} \to \mathfrak{A}(1, 1, 0)$, $\mathfrak{R} \to \mathfrak{A}(2)$, $\mathfrak{R} \to \mathfrak{A}(2, 0)$, will be proved in §2.

$A\mathfrak{R} \to \mathfrak{R}$ has been proved in [4]. §4 contains some consequences of this assertion.

§3 contains some negative results. It is shown that the condition $\sum \Delta' \geq 2$ is not only sufficient, but also necessary.

Some results concerning representation of semigroups are given in §4. Actually, the research on representation of semigroups stimulated the problems concerning full embeddings of categories. It follows from $\mathfrak{K} \to \mathfrak{R}$, where $\mathfrak{K}$ is a small category of an accessible cardinal, and from the results of §3 that any semigroup with a unit element S^1 is isomorphic with a semigroup of all endomorphisms of an algebra of a type Δ if and only if $\sum \Delta \geq 2$. This assertion strengthens the result of M. Armbrust and J. Schmidt [1],[2] which states that every S^1 is isomorphic with a semigroup of all endomorphisms of an object of some $A\mathfrak{A}$.

§5 is devoted to some applications of the assertion $\mathfrak{R}(\Delta^*) \to \mathfrak{R}(\to\mathfrak{A}(1, 1)$ etc.). Choosing some special Δ^*, we get some results on full embeddings of categories of metric, uniform and topological spaces, and topological algebras.

1. Some embeddings

THEOREM 1. *Let $\Delta^* = \{\kappa_\alpha \mid \alpha < \beta\}$ be a type. Then there exists a set A such that $\mathfrak{R}(\Delta^*) \to A\mathfrak{A}$.*

Proof. Let $\hat{X} = (X, \{R_\alpha \mid \alpha < \beta\})$ be an object of $\mathfrak{R}(\Delta^*)$. Put

$$\Phi(\hat{X}) = (X \cup \textstyle\bigcup_{\alpha<\beta} ((\alpha) \times R_\alpha) \cup \{u(\hat{X}), v(\hat{X})\}, \{\varphi_{\alpha\gamma}, \varphi_1, \varphi_2, \varphi_3 \mid \alpha < \beta, \gamma \leq \kappa_\alpha\}),$$

where

$$\begin{aligned} \varphi_{\alpha\gamma}(\alpha, \{x_\iota \mid \iota < \kappa_\alpha\}) &= x_\gamma && \text{for all } \{x_\iota\} \in R_\alpha, \\ \varphi_{\alpha\gamma}(\xi) &= u && \text{otherwise}, \\ \varphi_1(\xi) &= u && \text{for all } \xi, \\ \varphi_2(\xi) &= v && \text{for all } \xi \neq v,\ \varphi_2(v) = u, \\ \varphi_3(\xi) &= u && \text{for all } \xi \neq u,\ \varphi_3(u) = v. \end{aligned}$$

[2] This result itself can be obtained as a corollary of the result of [2].

$u(\hat{X})$, $v(\hat{X})$ are some different elements, $u(\hat{X}), v(\hat{X}) \notin X \cup \bigcup((\alpha) \times R_\alpha)$. We may choose e.g. $u(\hat{X}) = (0, \hat{X})$, $v(\hat{X}) = (1, \hat{X})$. If

$$f : \hat{X} \to \hat{Y} = (Y, \{S_\alpha \mid \alpha < \beta\})$$

is a morphism, put

$$\Phi(f)(x) = f(x) \qquad \text{for} \quad x \in X$$

$$\Phi(f)(\alpha, \{x_\iota\}) = (\alpha, \{f(x_\iota)\}) \qquad \text{for} \quad \{x_\iota\} \in R_\alpha,$$

$$\Phi(f)(u) = u, \qquad \Phi(f)(v) = v.$$

It is easy to see that Φ is a 1-1 functor into $A\mathfrak{A}$, where

$$A = \{(\alpha, \gamma) \mid \alpha < \beta, \gamma \leq \kappa_\alpha\} \cup \{1, 2, 3\}.$$

Now, we are going to prove that Φ maps $\mathfrak{R}(\Delta^*)$ onto a full subcategory of $A\mathfrak{A}$.

Let $g : \Phi(\hat{X}) \to \Phi(\hat{Y})$ be a homomorphism. Let $\psi_{\alpha\gamma}$, ψ_1, ψ_2, ψ_3 denote the operations in $\Phi(\hat{Y})$. We have

$$g(u) = g(\varphi_1 u) = \psi_1 g(u) = u$$

$$g(v) = g(\varphi_3 u) = \psi_3 g(u) = \psi_3 u = v.$$

Let $x \in X$. Since $\psi_{\alpha\gamma} g(x) = g(\varphi_{\alpha\gamma} x) = g(u) = u$, $g(x) \in Y \cup \{u, v\}$. If $g(x) = u$, we have $\psi_3 g(x) = v$, while $g(\varphi_3 x) = g(u) = u$; similarly, if $g(x) = v$, $\psi_2 g(x) = u \neq v = g(\varphi_2 x)$. Hence, $g(X) \subset Y$. Let $x_\iota \in X$, $\{x_\iota\} \in R_\alpha$. We have

$$\psi_{\alpha\gamma} g(\alpha, \{x_\iota\}) = g(\varphi_{\alpha\gamma}(\alpha, \{x_\iota\}) = g(x_\gamma) \in Y$$

and hence $g(\alpha, \{x_\iota\}) = (\alpha, \{y_\iota\})$ according to the definition of $\psi_{\alpha\gamma}$. Moreover, we get $y_\gamma = g(x_\gamma)$ and, hence, $f : X \to Y$ defined by $f(x) = g(x)$ $(x \in X)$ is a morphism and $g = \Phi(f)$.

LEMMA 1. *Let $\Delta_1 = \{\kappa_\alpha \mid \alpha < \beta\}$, $\Delta_2 = \{\lambda_\gamma \mid \gamma < \delta\}$ and let there be a 1-1 mapping φ of β into δ such that $\kappa_\alpha \leq \lambda_{\varphi(\alpha)}$ for every $\alpha < \beta$. Let at least one of the following two conditions be satisfied:*

(1) *there is an $\alpha < \beta$ such that $\kappa_\alpha = 0$;*

(2) *$\lambda_\gamma \neq 0$ for $\gamma \in \delta \backslash \varphi(\beta)$.*

Then the category $\mathfrak{A}(\Delta_1)$ is isomorphic with a full subcategory of $\mathfrak{A}(\Delta_2)$.

Proof. Let $\hat{X} = (X, \{F_\alpha \mid \alpha < \beta\})$ be an object of $\mathfrak{A}(\Delta_1)$. If the condition (1) is satisfied, let us choose an $\alpha_0 < \beta$, such that $\kappa_{\alpha_0} = 0$. We define the operations F'_γ (on X) as follows:

(1) if $\gamma \in \delta \backslash \varphi(\beta)$, $\lambda_\gamma = 0$, then $F'_\gamma = F_{\alpha_0}$;

(2) if $\gamma \in \delta \backslash \varphi(\beta)$, $\lambda_\gamma \neq 0$, then $F'_\gamma(\{x_\iota \mid \iota < \lambda_\gamma\}) = x_0$;

(3) if $\gamma = \varphi(\alpha)$, $\lambda_\gamma = \kappa_\alpha = 0$, then $F'_\gamma = F_\alpha$;

(4) if $\gamma = \varphi(\alpha)$, $\lambda_\gamma > \kappa_\alpha = 0$, then $F'_\gamma(\{x_\iota \mid \iota < \lambda_\gamma\}) = F_\alpha$;

(5) if $\gamma = \varphi(\alpha)$, $\kappa_\alpha \neq 0$, then $F'_\gamma(\{x_\iota \mid \iota < \lambda_\gamma\}) = F_\alpha(\{x_\iota \mid \iota < \kappa_\alpha\})$.

Put $\Phi(\hat{X}) = (X, \{F'_\gamma \mid \gamma < \delta\})$. Let $\hat{Y} = (Y, \{G_\alpha \mid \alpha < \beta\})$ be another object of $\mathfrak{A}(\Delta_1)$. We shall prove that a mapping $g : X \to Y$ is a homomorphism of $\hat{X}$ into $\hat{Y}$ if and only if it is a homomorphism of $\Phi(\hat{X})$ into $\Phi(\hat{Y})$. First, let g be a homomorphism of $\hat{X}$ into $\hat{Y}$.

(1) If $\gamma \in \delta \backslash \varphi(\beta)$, $\lambda_\gamma = 0$, then

$$g(F'_\gamma) = g(F_{\alpha_0}) = G_{\alpha_0} = G'_\gamma .$$

(2) If $\gamma \in \delta \backslash \varphi(\beta)$, $\lambda_\gamma \neq 0$, then

$$g(F'_\gamma(\{x_\iota \mid \iota < \lambda_\gamma\})) = g(x_0) = G'_\gamma(\{g(x_\iota) \mid \iota < \lambda_\gamma\}).$$

(3) If $\gamma = \varphi(\alpha)$, $\lambda_\gamma = \kappa_\alpha = 0$, then

$$g(F'_\gamma) = g(F_\alpha) = G_\alpha = G'_\gamma .$$

(4) If $\gamma = \varphi(\alpha)$, $\lambda_\gamma > \kappa_\alpha = 0$, then

$$g(F'_\gamma(\{x_\iota \mid \iota < \lambda_\gamma\})) = g(F_\alpha) = G_\alpha = G'_\gamma(\{g(x_\iota) \mid \iota < \lambda_\gamma).$$

(5) If $\gamma = \varphi(\alpha)$, $\kappa_\alpha \neq 0$, then

$$\begin{aligned} g(F'_\gamma(\{x_\iota \mid \iota < \lambda_\gamma\})) &= g(F_\alpha(\{x_\iota \mid \iota < \kappa_\alpha\})) \\ &= G_\alpha(\{g(x_\iota) \mid \iota < \kappa_\alpha\}) \\ &= G'_\gamma(\{g(x_\iota) \mid \iota < \lambda_\gamma\}). \end{aligned}$$

On the other hand, let g be a homomorphism of $\Phi(\hat{X})$ into $\Phi(\hat{Y})$.

(a) If $\kappa_\alpha = \lambda_{\varphi(\alpha)} = 0$, then $g(F_\alpha) = g(F'_{\varphi(\alpha)}) = G'_{\varphi(\alpha)} = G_\alpha$.

(b) Let $\kappa_\alpha = 0 < \lambda_{\varphi(\alpha)}$. Let us choose an arbitrary system

$$\{x_\iota \mid \iota < \lambda_{\varphi(\alpha)}\}.$$

We have

$$g(F_\alpha) = g(F'_{\varphi(\alpha)}(\{x_\iota \mid \iota < \lambda_{\varphi(\alpha)}\})) = G'_{\varphi(\alpha)}(\{g(x_\iota) \mid \iota < \lambda_{\varphi(\alpha)}\}) = G_\alpha .$$

(c) Let $\kappa_\alpha \neq 0$. Let us take a system $\{x_\iota \mid \iota < \kappa_\alpha\}$, and let us choose x_ι's for $\kappa_\alpha \leq \iota < \lambda_{\varphi(\alpha)}$. We have

$$\begin{aligned} g(F_\alpha(\{x_\iota \mid \iota < \kappa_\alpha\})) &= g(F'_{\varphi(\alpha)}(\{x_\iota \mid \iota < \lambda_{\varphi(\alpha)}\})) \\ &= G'_{\varphi(\alpha)}(\{g(x_\iota) \mid \iota < \lambda_{\varphi(\alpha)}\}) \\ &= G_\alpha(\{g(x_\iota) \mid \iota < \kappa_\alpha\}). \end{aligned}$$

Consequently, defining $\Phi(g) : \Phi(\hat{X}) \to \Phi(\hat{Y})$ by $\Phi(g)(x) = g(x)$ for every homomorphism $g : \hat{X} \to \hat{Y}$, we get a 1-1 functor from $\mathfrak{A}(\Delta_1)$ onto a full subcategory of $\mathfrak{A}(\Delta_2)$.

As a consequence we get

THEOREM 2. *Let $\sum \Delta \geq 2$. Then at least one of the following statements holds:*

(1) $\mathfrak{A}(1, 1) \to \mathfrak{A}(\Delta)$
(2) $\mathfrak{A}(1, 1, 0) \to \mathfrak{A}(\Delta)$
(3) $\mathfrak{A}(2) \to \mathfrak{A}(\Delta)$
(4) $\mathfrak{A}(2, 0) \to \mathfrak{A}(\Delta)$.

2. Further embeddings

THEOREM 3. $\mathfrak{R} \to \mathfrak{A}(1, 1)$ *and* $\mathfrak{R} \to \mathfrak{A}(1, 1, 0)$.

Proof. Let $\bar{X} = (X, R)$ be an object of $\mathfrak{R}$ and $u_i(\bar{X})$, $i = 1, 2$, two elements none of them belonging to X or R. We define two unary operations F_0, F_1 (two unary operations F_0, F_1, and one nullary operation F_2, respectively) on the set $X \cup R \cup \{u_1, u_2\}$ as follows:

$$F_i(x) = u_{i+1} \quad \text{for every} \quad x \in X,\ i = 0, 1;$$

$$F_i((x_1, x_2)) = x_{i+1} \quad \text{for every} \quad (x_1, x_2) \in R,\ i = 0, 1;$$

$$F_0(u_1) = F_0(u_2) = u_2, \qquad F_1(u_1) = F_1(u_2) = u_1.$$

($F_2 = u_1$, resp.).

Let $\Phi(\bar{X})$ denote the algebra

$$(X \cup R \cup \{u_1, u_2\}; F_0, F_1)$$

($(X \cup R \cup \{u_1, u_2\}; F_0, F_1, F_2)$, resp.).

Let $\bar{X}$ and $\bar{Y} = (Y, S)$ be objects of $\mathfrak{R}$. Let $f : \bar{X} \to \bar{Y}$ be a morphism in $\mathfrak{R}$. $\Phi(f)$ denotes the mapping from

$$X \cup R \cup \{u_1(\bar{X}), u_2(\bar{X})\} \quad \text{into} \quad Y \cup S \cup \{u_1(\bar{Y}), u_2(\bar{Y})\}$$

defined by

$$\Phi(f)(x) = f(x) \quad \text{for every} \quad x \in X$$

$$\Phi(f)((x, y)) = (f(x), f(y)) \quad \text{for every} \quad (x, y) \in R$$

$$\Phi(f)(u_i(\bar{X})) = u_i(\bar{Y}) \quad \text{for} \quad i = 1, 2.$$

First, we are going to prove that $\Phi(f)$ is a morphism from $\Phi(\bar{X})$ into $\Phi(\bar{Y})$ in $\mathfrak{A}(1, 1)$ (in $\mathfrak{A}(1, 1, 0)$, resp.). Let G_i, $i = 0, 1$ (0, 1, 2, resp.) denote the operations in $\Phi(\bar{Y})$. We must prove that

$$\Phi(f)(F_i(\xi)) = G_i(\Phi(f)(\xi))$$

for $i = 0, 1$. In the respective case, $\Phi(f)(F_2) = G_2$ follows from the definition. Let $\xi = u_j$; then

$$\Phi(f)(F_i(u_j)) = \Phi(f)(u_{2-i}) = u_{2-i} = G_i(\Phi(f)(u_j)).$$

Let $\xi = x \in X$. Then

$$\Phi(f)(F_i(x)) = \Phi(f)(u_{i+1}) = u_{i+1} = G_i(\Phi(f)(x)).$$

Finally, let $\xi = (x_1, x_2)$. Then

$$\Phi(f)(F_i(x_1, x_2)) = \Phi(f)(x_{i+1}) = f(x_{i+1})$$
$$= G_i(f(x_1), f(x_2)) = G_i(\Phi(f)(x_1, x_2)).$$

Hence, Φ defines a functor from $\mathfrak{R}$ into $\mathfrak{A}(1, 1)(\mathfrak{A}(1, 1, 0)$ resp.), which is obviously 1-1. It remains to prove that its image is a full subcategory of the corresponding category.

Let $g : \Phi(\bar{X}) \rightarrow \Phi(\bar{Y})$ be a homomorphism. The proof will be completed, if we show that $g = \Phi(f)$ for some morphism $f \in \mathfrak{R}$. Since

$$g(u_1(\bar{X})) = g(F_1(u_1(\bar{X}))) = G_1(g(u_1(\bar{X}))),$$

we have $g(u_1(\bar{X})) = u_1(\bar{Y})$ for $u_1(\bar{Y})$ is the only element remaining fixed under G_1. Similarly, $g(u_2(\bar{X})) = u_2(\bar{Y})$. Let $x \in X$. If $g(x) = u_i$, we have $G_0(g(x)) = G_0(u_i) = u_2$, while $g(F_0(x)) = g(u_1) = u_1$; if $g(x) = (x_1, x_2)$, we have $G_0(g(x)) = G_0(x_1, x_2) = x_1$, while $g(F_0(x)) = g(u_1) = u_1$. Hence, $g(X) \subset Y$. Let $\xi = (x_1, x_2) \in R$; if $g(\xi) = u_i$, we have $G_0(g(\xi)) = G_0(u_i) = u_2$, while $g(F_0(\xi)) = g(x_1) \in Y$; if $g(\xi) = x \in Y$, we have $G_0(g(\xi)) = G_0(x) = u_1$, while $g(F_0(\xi)) = g(x_1) \in Y$. Hence, $g(R) \subset S$. Now, let $x_1 R x_2$. Hence, $(x_1, x_2) \in R$ and

$$g((x_1, x_2)) = (y_1, y_2) \in S.$$

We have

$$g(x_i) = g(F_{i-1}(x_1, x_2)) = G_{i-1}(g(x_1, x_2)) = y_i,$$

and, hence, $g(x_1) S_g(x_2)$ and $g((x_1, x_2)) = (g(x_1), g(x_2))$. Hence, $g = \Phi(f)$, where $f : \bar{X} \rightarrow \bar{Y}$ is defined by $f(x) = g(x)$. The proof is finished.

THEOREM 4. $\mathfrak{A}(1, 1) \rightarrow \mathfrak{A}(2), \mathfrak{A}(1, 1) \rightarrow \mathfrak{A}(2, 0)$.

Proof. Let $\hat{X} = (X; F_0, F_1)$ be an object of $\mathfrak{A}(1, 1)$. A binary operation F_0' on the set $X' = X \cup \{v_1(\hat{X}), v_2(\hat{X})\}$ (where $v_i(\hat{X})$ are some elements which are not in X) is defined as follows:

$$F_0'(x, v_1) = F_0(x), \quad F_0'(v_1, x) = F_1(x) \quad \text{for} \quad x \in X,$$
$$F_0'(v_2, v_2) = v_1,$$
$$F_0'(z, z') = v_2 \quad \text{otherwise}.$$

We put $F_1' = v_1$ in the case of the proof of the second assertion.

Let $\hat{X}$, $\hat{Y} = (Y; G_0, G_1)$ be objects of $\mathfrak{A}(1, 1)$, and let $f : \hat{X} \rightarrow \hat{Y}$ be a homomorphism. We define a mapping $\Phi(f) : X' \rightarrow Y'$ putting

$$\Phi(f)(x) = f(x) \qquad \text{for every} \quad x \in X,$$
$$\Phi(f)(v_i(\hat{X})) = v_i(\hat{Y}).$$

First, we shall prove that $\Phi(f)$ is a homomorphism of $\Phi(\hat{X})$ into $\Phi(\hat{Y})$. Really, for $x \in X$,

$$\Phi(f)(F_0'(x, v_1)) = \Phi(f)(F_0(x)) = G_0(\Phi(f)(x)) = G_0'(\Phi(f)(x), \Phi(f)(v_1)).$$

Similarly for $\Phi(f)(F_0'(v_1, x))$.

$$\Phi(f)(F_0'(v_2, v_2) = \Phi(f)(v_1) = v_1 = G_0'(\Phi(f)(v_2), \Phi(f)(v_2)).$$

$$\Phi(f)(F_0'(z, z')) = G_0'(\Phi(f)(z), \Phi(f)(z')) = v_2$$

in the remaining cases. Hence Φ defines a functor, which is evidently 1-1.

Let $g : \Phi(\hat{X}) \rightarrow \Phi(\hat{Y})$ be a homomorphism. Let us, to simplify the notation, designate the operations F_0', G_0' by juxtaposition. We get

$$\xi\xi \in \{v_1, v_2\} \quad \text{for any} \quad \xi \in X', \; Y' \text{ resp.}$$

As $v_1v_1 = v_2$ and $v_2v_2 = v_1$, the mapping g maps $\{v_1, v_2\}$ onto $\{v_1, v_2\}$. We have

$$g(v_2) = g(v_1 v_2) = g(v_1)g(v_2) = v_2,$$

since $v_1 v_2 = v_2 v_1 = v_2$. Let $g(x) = v_2$ for some $x \in X$. We get the following contradiction:

$$v_2 = g(v_2) = g(xv_2) = g(x)g(v_2) = v_2 v_2 = v_1.$$

If $g(x) = v_1$, then $v_2 = g(v_1)g(x) = g(v_1 x) \neq v_2$, hence, $g(v_i) = v_i$, $i = 1, 2$, and $g(X) \subset Y$. If we define a mapping $f : X \rightarrow Y$ by $f(x) = g(x)$, we get easily $f(F_i(x)) = G_i(f(x))$, i.e. f is a homomorphism of $\hat{X}$ into $\hat{Y}$. We have $g = \Phi(f)$.

3. Some groups of endomorphisms

Throughout this paragraph $\hat{X} = (X; \varphi, \{o_\alpha \mid \alpha \in A\})$ is a quasi-algebra with one partial unary operation φ, and with nullary operations o_α, $\alpha \in A$, where A is a set. Define a relation C on X as follows:

$(x, y) \in C$ if and only if there exist $i, j \geq 0$ such that $\varphi^i(x) = \varphi^j(x)$, where φ^0 is the identity mapping and $\varphi^n(x) = \varphi(\varphi^{n-1}(x))$ if the symbol on the right hand side is defined.

C is an equivalence relation, and if Y is a class of equivalence defined by C, then $\varphi(Y) \subset Y$. $\varphi \parallel Y : Y \rightarrow Y$ is defined by $\varphi \parallel Y(x) = \varphi(x)$. The quasi-algebra

$$\hat{Y} = (Y; \varphi \parallel Y, \{o_\alpha\} \cap Y)$$

is called a component of $\hat{X}$.

LEMMA 2. *Let $\{\hat{X}_b \mid b \in B\}$ be the family of all components of a quasi-algebra $\hat{X}$, and let $E(\hat{X})$—the semigroup of all endomorphisms of $\hat{X}$—be a group. Then* *every* $E(\hat{X}_b)$ *is a group and*

$$E(\hat{X}) \approx \prod \{E(\hat{X}_b),\ b \in B\},$$

where $\prod$ *denotes the direct product.*

Proof. Let $f : \hat{X} \to \hat{X}$ be a homomorphism. We know that the image of a component under φ is a subset of a component. We shall show that in the discussed case we have $f(X_b) \subset X_b$. Really, if $f(X_b) \subset X_c$, $b \neq$ c, the mapping $g : X \to X$ defined by

$$g(x) = f(x) \qquad \text{for} \quad x \in X_b,$$
$$g(x) = x \qquad \text{otherwise}$$

is a homomorphism. Since $E(\hat{X})$ is a group, g ought to possess an inverse, but g is not a 1-1 mapping.

Let $h_b : \hat{X}_b \to \hat{X}_b$, $b \in B$, be homomorphisms. The mapping $h : X \to X$ defined by $h(x) = h_b(x)$ for $x \in X_b$ is evidently a homomorphism of $\hat{X}$ into itself. In particular, we immediately see that the $E(\hat{X}_b)$ are groups, since $h^{-1} \parallel X_b$ forms the inverse homomorphism of h_b. Now, it is easy to see that the mapping

$$\Phi : E\ (\hat{X}) \to \prod E\ (\hat{X}_b)$$

defined by

$$\Phi(f) = \{f \parallel X_b \mid b \in B\}$$

is a group isomorphism.

Put $B(x) = \{y \mid \exists i \geq 0, \varphi^i y = x\}$. $B(x)$ is said to be simple if and only if

$$x = \varphi^i y = \varphi^i z \text{ implies } y = z.$$

LEMMA 3. *Let there be an element* $x_0 \in \hat{X}$ *possessing a non-simple* $B(x_0)$, *such that*

$$(B(x_0)\backslash\{\varphi^i x_0 \mid i = 1, 2, \cdots\} \cap \{o_\alpha\} = \emptyset.$$

Then $E(\hat{X})$ *is not a group.*

Proof. Let $B(x_0)\backslash\varphi B(x_0) \neq \emptyset$. We define, for $y \in B(x_0)\backslash\varphi B(x_0)$,

$$k(y) = \min\{k \mid \varphi^k y = \varphi z,\ z \neq \varphi^{k-1} y\}.$$

As $B(x_0)$ is not simple, such a k exists. Put

$$n = \min\{k(y) \mid y \in B(x_0)\backslash\varphi B(x_0)\},$$

and let us take a y such that $k(y) = n$; let us take an element $z_1 \neq \varphi^{n-1} y$ such that $\varphi^n y = \varphi z_1$. As n is minimal, there exists a sequence $\{z_i \mid i = 1, 2, \cdots\}$ such that $\varphi z_i = z_{i-1}$ for $i = 2, 3, \cdots$. The mapping $g : X \to X$ defined by

$$g(\varphi^i y) = z_{n-i}, \qquad i = 0, 1, \cdots, n-1,$$
$$g(x) = x \qquad \text{otherwise},$$

is evidently a homomorphism of $\hat{X}$ into itself possessing no inverse.

Now, let $B(x_0) \subset \varphi B(x_0)$. We consider two cases:

I. $x_0 = \varphi^i x_0$ for some $i > 0$. Let n be the least i with this property. Let us define $n(z)$, for $z \in B(x_0)\backslash\{\varphi^j x_0\}$, to be the least natural number such that $x_0 = \varphi^{n(z)} z$. The mapping $g : X \to X$ defined by $g(x) = \varphi^{k \cdot n - n(x)} x_0$ (k is such that $k \cdot n - n(x) > 0$) for $x \in B(x_0)\backslash\{\varphi^j x_0\}$, $g(x) = x$ otherwise, is a homomorphism of $\hat{X}$ into itself possessing no inverse.

II. $i > 0$ implies $\varphi^i x_0 \neq x_0$. Let us define $n(x)$ in the same way as in the case I. We take a sequence $\{a_i \mid i = 1, 2, \cdots\}$ such that $x_0 = \varphi a_1$, $a_i = \varphi a_{i+1}$ and put

$$g(x) = a_{n(x)} \qquad \text{for} \quad x \in B(x_0)\backslash(x_0)$$

$$g(x) = x \qquad \text{otherwise.}$$

The mapping $g : X \to X$ defined in this way is a homomorphism of $\hat{X}$ into itself possessing no inverse.

LEMMA 4. *Let $\hat{X}$ consist of one component. Let $E(\hat{X})$ be a non-trivial group. Then*

(1) $A = \emptyset$,
(2) φ *is a* 1-1 *mapping of X onto itself,*
(3) $X = \{\varphi^i x \mid i = \cdots, -1, 0, 1, \cdots\}$,
(4) $E(\hat{X}) \approx Z_n$ *if* card $X = n$,
$E(\hat{X}) \approx Z$ *if* X *is an infinite set.*

(*Z is the additive group of integers, Z_n is the additive group of integers* mod n.)

Proof. Let $A \neq \emptyset$. Put $Y = \{x \mid B(x) \cap \{o_\alpha\} = \emptyset\}$. As $E(\hat{X})$ is non-trivial, the set Y is non-void and, by Lemma 3, $B(x)$ is simple for any $x \in Y$. Moreover, there exists $y \in Y$ and a homomorphism $g : \hat{X} \to \hat{X}$ such that $g(y) \neq y$. Let k be the least natural number such that $\varphi^k y = \varphi^j o_\alpha$ for some j and α. $B(y_1)$, $y_1 = g^{k-1}(y)$, is simple and, hence, there is a uniquely defined sequence $\{y_i\}$ (finite or infinite) such that $y_i = \varphi y_{i+1}$ for $i = 1, 2, \cdots$. Let us define a mapping $f : X \to X$ as follows:

$$f(y_i) = g(y_i), \qquad f(x) = x \quad \text{otherwise.}$$

f is a homomorphism of $\hat{X}$ into itself and has no inverse. Hence, $A = \emptyset$.

Let φ be not defined on the whole X. Then, according to the definition of component, it is undefined in exactly one element $x_0 \in X$ and we have $X = B(x_0)$. The previous lemma shows that $X = \{x_0, x_1, \cdots\}$ (the sequence being finite or infinite) such that $\varphi x_{i+1} = x_i (i = 0, 1, 2, \cdots)$. Let g be a non-identical homomorphism of $\hat{X}$ into itself. Hence, there is $g(x_m) = x_n$ for some $m \neq n$. We get easily $n > m$, $g(x_k) = x_{n-m+k}$ and g is not mapping onto.

Now, since $A = \emptyset$ and φ is defined on the whole X, the mapping φ is a homomorphism of $\hat{X}$ into itself and therefore it has an inverse. The rest of the proof is evident.

THEOREM 5. *Let* $\hat{X} = (X; \varphi, \{o_\alpha, \alpha \in A\})$ *be a quasi-algebra with one partial unary operation* φ *and with nullary operations* $o_\alpha (\alpha \in A)$. *Let* $E(\hat{X})$ *be a non-trivial group. Then either*

(1) $E(\hat{X})$ *is the infinite cyclic group, or*

(2) $E(\hat{X})$ *is a direct product of at most a countable number of finite cyclic groups with orders which mutually do not divide each other.*

Proof. By Lemma 2, $E(\hat{X}_b)$ is a group for every component $\hat{X}_b$ of $\hat{X}$. Evidently, every homomorphism must map every component into itself. Considering Lemma 4, we obtain: if there is a component

$$\hat{X}_1 = \{\varphi^i x \mid i = \cdots, -1, 0, 1, \cdots\}$$

such that $\varphi^i x$ are different for different i, there is no other $\hat{X}_b$ with a non-trivial $E(\hat{X}_b)$, since $\hat{X}_1$ may be homomorphically mapped onto any such $\hat{X}_b$. Similarly, a component with a non-trival group consisting of n elements, n being a natural number, may be mapped on such a component consisting of k elements, if k divides n.

4. Main theorems

The following definitions play an important role in this paragraph.

A couple (X, R), where X is a set and $R \subset X \times X$, is called rigid if there is only one compatible mapping of (X, R) into itself, namely the identity.

The symbol $\mathfrak{F}(\mathfrak{a})$, where $\mathfrak{a}$ is a cardinal, denotes the following assertion:

$$\text{There is a rigid} \quad (X, R) \quad \text{such that} \quad \text{card } X \geq \mathfrak{a}.$$

We shall use the following assertions:

THEOREM 6. $\mathfrak{F}(\text{card } A) \Rightarrow (A\mathfrak{R} \rightarrow \mathfrak{R})$.

For the proof see [4].

THEOREM 7. $\mathfrak{F}(\mathfrak{a})$ *holds for every cardinal* $\mathfrak{a}$.

Proof. The assertion is an immediate consequence of the result of [5].

THEOREM 8. *Let* $\mathfrak{K}$ *be a small category; let* K *be the set of its morphisms. Then* $\mathfrak{K} \rightarrow K\mathfrak{R}$.

A very simple proof is given in [4].

Now, we shall prove a theorem concerning embeddings of small categories into the categories of algebras and representation of semigroups by semigroups of endomorphisms of algebras of a given type.

THEOREM 9. *The following assertions are equivalent:*

(1) $\mathfrak{K} \rightarrow \mathfrak{A}(\Delta)$ *for any small category* $\mathfrak{K}$.

(2) $\mathfrak{K} \rightarrow \mathfrak{Q}(\Delta)$ *for any small category* $\mathfrak{K}$.

(3) *If S^1 is a semigroup with a unity element, there exists an algebra X of the type Δ such that S^1 is isomorphic with $E(X)$.*

(4) *If S^1 is a semigroup with a unity element, there exists a quasi-algebra X of the type Δ such that S^1 is isomorphic with $E(X)$.*

(5) $\sum \Delta \geq 2$.

Proof. Evidently, (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4). (4) $\Rightarrow$ (5), by the results of paragraph 3. Let (5) hold. Then $\mathfrak{R} \to \mathfrak{A}(\Delta)$, by Theorems 2, 3 and 4. As $\mathfrak{F}(\text{card}\,\mathfrak{K})$ holds, we have $K\mathfrak{R} \to \mathfrak{R}$ (where K is the set of morphisms of $\mathfrak{K}$), by Theorem 6. By Theorem 8, we get $\mathfrak{K} \to K\mathfrak{R}$. Hence, (5) $\Rightarrow$ (1).

We remark that the previous result contains as a corollary the statement that the category $\mathfrak{A}(\Delta)$ ($\mathfrak{Q}(\Delta)$, resp.) is universal if and only if $\sum \Delta \geq 2$. The definition of a universal category is given in [4].

THEOREM 10. *Let Δ, Δ' be types, $\sum \Delta' \geq 2$. Let $\mathfrak{K}$ be a full subcategory of $\mathfrak{R}(\Delta)$. Then*

$$\mathfrak{K} \to \mathfrak{A}(\Delta').$$

In particular, $\mathfrak{A} \to \mathfrak{A}(1, 1)$ for any full category of algebras $\mathfrak{A}$.

If $\sum \Delta' < 2$, then $\mathfrak{A}(\Delta) \to \mathfrak{A}(\Delta')$ does not hold for any Δ such that $\sum \Delta \geq 2$.

Proof. By Theorem 1, $\mathfrak{R}(\Delta) \to A\mathfrak{A}$. By Theorems 6 and 7, $A\mathfrak{R} \to \mathfrak{R}$. Since $\mathfrak{R} \to \mathfrak{A}(\Delta')$ (by Theorems 2, 3 and 4), we obtain $\mathfrak{R}(\Delta) \to \mathfrak{A}(\Delta')$, and, hence, $\mathfrak{K} \to \mathfrak{A}(\Delta')$.

Now, let $\sum \Delta' < 2$, $\sum \Delta \geq 2$. Consider an arbitrary non-abelian group G. By Theorem 9, there exists an algebra X of the type Δ such that $E(X)$ is isomorphic with G. Let Φ be a full embedding of $\mathfrak{A}(\Delta)$ into $\mathfrak{A}(\Delta')$. Then $E(\Phi(X))$ is isomorphic with $E(X)$ and, hence, with G, which is not possible by Theorem 9.

5. Applications

We shall apply previous results to some concrete categories.

(A) Let $X = (X, \tau)$ be a topological space. We designate

$$\chi(X) = \sup \{\chi(x) \mid x \in X\},$$

where $\chi(x)$ is the character of the point x in (X, τ), i.e. the least cardinality of a set of neighbourhoods of x which is confinal in the directed system of all neighbourhoods of x.

Designate by $\mathfrak{T}(\mathfrak{a})$ the category of topological spaces X with $\chi(X) \leq \mathfrak{a}$ and all their continuous mappings.

LEMMA 5. $\mathfrak{T}(\mathfrak{a}) \to \mathfrak{R}(\Delta)$ *for some* Δ.

Proof. The idea of the proof is based on replacing the topology by an equivalent convergence structure.

Let A be a set, card $A = \mathfrak{a}$. Evidently, there exists a set C with the following properties:

(1) the elements of C are directed sets $(B, <)$, where $B \subset A$;

(2) if $(D, <')$, card $D \leq \mathfrak{a}$, is a directed set, then there exists $(B, <) \in C$ isomorphic with $(D, <')$;

(3) if $(B_1, <_1)$ and $(B_2, <_2)$ are isomorphic elements of C, then $(B_1, <_1) = (B_2, <_2)$.

Evidently, card $C \leq 2^{2^{\mathfrak{a}}}$ for infinite cardinals.

Let the set C be well ordered, say by an ordinal β. If $(B_\alpha, <'_\alpha)$ is the α-th element of C according to the well ordering, we choose an ordinal κ_α with card $\kappa_\alpha =$ card B_α. Let us direct every set κ_α in such a way that $(\kappa_\alpha, <_\alpha)$ is isomorphic with $(B_\alpha, <'_\alpha)$.

If (X, τ) is an object of $\mathfrak{T}(\mathfrak{a})$, put

$$\Phi(X, \tau) = (X, \{R_\alpha(\tau) \mid \alpha < \beta\}),$$

where $R_\alpha(\tau)$ (abbreviated R_α) is the set of all those systems

$$\{x_\iota \mid \iota < \kappa_\alpha + 1\}$$

with the following property: for every neighbourhood U of the point x_{κ_α} there is $\iota_0 \in \kappa_\alpha$ such that $\iota_0 < \iota$ implies $x_\iota \in U$.

The lemma will be proved if one shows that a mapping $f : X \to Y$ is a continuous mapping of (X, τ) into (Y, σ) if and only if it is a morphism from $\Phi(X, \tau)$ into $\Phi(Y, \sigma)$ in $\mathfrak{R}(\Delta)$, which is almost evident. ($\Delta = \{\kappa_\alpha + 1 \mid \alpha < \beta\}$.)

Let the symbol $\mathfrak{T}(\mathfrak{a}, \Delta)$ denote the category, the objects of which are sets X endowed simultaneously by a topology (such that $\chi(X) \leq \mathfrak{a}$) and by a relational structure of the type Δ, and morphisms are all the continuous mappings satisfying the condition required for morphisms of $\mathfrak{R}(\Delta)$. In particular, if the relational structures on two objects are structures of algebra, the morphisms are continuous homomorphisms.

LEMMA 6. $\mathfrak{T}(\mathfrak{a}, \Delta) \to \mathfrak{R}(\Delta')$ *for some* Δ'.

The proof can be made similarly to the proof of Lemma 5. We must only modify it by adding the relational systems.

COROLLARY. *Let* $\mathfrak{K}$ *be a full subcategory of* $\mathfrak{T}(\mathfrak{a}, \Delta)$. *Then* $\mathfrak{K} \to \mathfrak{R}$ $(\to \mathfrak{A}(1, 1)$ *etc.*).

In particular, the assertion holds for the following categories:

objects:	*morphisms:*
metric spaces	*continuous mappings*
metric linear spaces	*continuous linear mappings*
normed linear spaces	*bounded linear mappings*
Banach algebras	*continuous homomorphisms.*

(B) Denote by $\mathfrak{U}(\mathfrak{a})$ the category, the objects of which are uniform spaces

$(X, \mathfrak{U})$ such that $\mathfrak{U}$ contains a confinal subsystem of a cardinality less than or equal to $\mathfrak{a}$; the morphisms are all their uniformly continuous mappings. Further, denote by $\mathfrak{U}(\mathfrak{a}, \Delta)$ the category of uniform spaces X having the mentioned property, and endowed by relational structures of the type Δ, where the morphisms are uniformly continuous mappings which are morphisms of $\mathfrak{R}(\Delta)$.

LEMMA 7. $\mathfrak{U}(\mathfrak{a}, \Delta) \to \mathfrak{R}(\Delta')$ *for some* Δ'.

Proof. The proof will be given for $\mathfrak{U}(\mathfrak{a})$, as the generalisation is obvious.

We find a system $(\kappa_\alpha, \prec_\alpha)$ similarly as in the proof of Lemma 5. Here we put $\Delta' = \{2\kappa_\alpha \mid \alpha < \beta\}$ and define

$$\Phi(X, \mathfrak{U}) = (X, \{R_\alpha(\mathfrak{U}) \mid \alpha < \beta\}),$$

where $\{x_\iota \mid \iota < 2\kappa_\alpha\} \in R_\alpha(\mathfrak{U})$ if and only if, for every $U \in \mathfrak{U}$, there is a ι_0 such that $[x_\iota, x_{\kappa_\alpha+\iota}] \in U$ for every $\iota > \iota_0$.

Finally, if $f : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ is a uniformly continuous mapping, we define $\Phi(f) : \Phi(X, \mathfrak{U}) \to \Phi(Y, \mathfrak{V})$ by $\Phi(f)(x) = f(x)$. It is easy to see that $\Phi(f)$ is a morphism of $\mathfrak{R}(\Delta')$, and that Φ is a 1-1 functor onto a full subcategory of $\mathfrak{R}(\Delta')$.

COROLLARY. *Let $\mathfrak{K}$ be a full subcategory of* $\mathfrak{U}(\mathfrak{a}, \Delta)$. *Then*

$$\mathfrak{K} \to \mathfrak{R} \qquad (\to \mathfrak{A}(1, 1) \quad \textit{etc.}).$$

THEOREM 11. *The category of metric spaces and their uniformly continuous mappings is isomorphic with a full subcategory of* $\mathfrak{R}$ (*and* $\mathfrak{A}(1, 1)$ *etc.*).

The proof follows from the fact that the uniformity defined by a metric contains a countable confinal subsystem.

(C) Let (X, ρ), (Y, σ) be metric spaces, $f : X \to Y$. f is called a contraction, if

$$\sigma(f(x), f(y)) \leq \rho(x, y) \qquad \text{for all} \quad x, y \in X.$$

THEOREM 12. *The category of metric spaces and their contractions is isomorphic with a full subcategory of* $\mathfrak{R}$ ($\mathfrak{A}(1, 1)$ *etc.*).

Proof. Let A be the set of all non-negative real numbers. We shall prove that the category under consideration is isomorphic with a full subcategory of $A\mathfrak{R}$. For a metric space (X, ρ), put

$$\Phi(X, \rho) = (X, \{R_a \mid a \in A\}),$$

where $[x, y] \in R_a$ if and only if $\rho(x, y) \leq a$; $\Phi(f) = f$. Obviously, Φ is a full embedding.

(D) We state explicitly a corollary concerning representation of semigroups by commuting mappings.

COROLLARY. *Let S^1 be a semigroup with a unity element. Then there exist* *a set X and two transformations f_1, f_2 of X such that S^1 is isomorphic with the semigroup (under composition)*

$$\{\varphi \mid \varphi : X \to X, \varphi \circ f_i = f_i \circ \varphi, i = 1, 2\}.$$

The proof follows immediately from Theorem 9.

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