GENERALIZED LINEAR DIFFERENTIAL SYSTEMS AND RELATED RICCATI MATRIX INTEGRAL EQUATIONS

BY

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1. Introduction

In a previous paper [7], the author considered a generalized differential system that was equivalent to a type of linear vector Riemann-Stieltjes integral equation, and which included as special cases the real scalar generalized second-order differential equations occurring in the works of Sz.-Nagy, ([10] and [9, pp. 247–254]), and Feller [3], and also certain systems with "interface" conditions associated with the accessory differential equations for a simple integral non-parametric variational problem. The considered systems, however, did not include such modified accessory systems for variational problems of Lagrange or Bolza type. Moreover, the treatment of [7] was limited to first order systems of order $2n$, in which the component vector admitting possible discontinuities was of dimension n. The present paper deals with a generalized differential system whose form is inclusive enough to remove these two limitations. Moreover, attention is focused on the central role played by a non-linear matrix integral equation of Volterra type, which will be referred to as a "Riccati matrix integral equation" in view of its intimate relation to the Riccati matrix differential equation. Finally, for generality, and also for application in the "control formulation" of certain variational problems, the results are presented in a form which for accessory systems under classical variational assumptions, (see, for example, [1, §§39, 81]), would be in terms of canonical variables, but which far exceeds this particular instance.

The basic relationships between the considered generalized matrix differential system and certain functionals are derived in §2, while §3 is concerned with properties of such a system and its adjoint. The fundamental connections between such systems and Riccati matrix integral equations are presented in §4. The important instance of self-adjoint systems is treated in §5, with particular attention to the interrelations between criteria of non-oscillation, the existence of solutions of associated Riccati matrix integral equations, and the positive definiteness of certain hermitian functionals. In §6 some of the results of §5 are applied to a special scalar integral equation, concerning which Cameron [2] initially raised a question of solvability that was answered by Woodward [11]. Finally, in §7 there are presented two theorems that extend earlier results of Hestenes [3], Bliss [2, §87] and the author [6] on accessory systems for variational problems of Bolza type.

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Matrix notation is used throughout; in particular, matrices of one column are termed vectors, and for a vector (y_α), $(\alpha = 1, \cdots, n)$, the norm $|y|$ is given by $(|y_1|^2 + \cdots + |y_n|^2)^{1/2}$. The $n \times n$ identity matrix is denoted by E_n, or merely by E when there is no ambiguity, and 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix M is designated by M^*. If M is an $n \times m$ matrix the symbol $|M|$ is used for the supremum of $|My|$ on the unit sphere $\{y \mid |y| = 1\}$ of complex m-space. The relations $M \geq N$, $(M > N)$, are used to signify that M and N are hermitian matrices of the same dimensions and $M - N$ is a non-negative (positive) definite matrix. If an hermitian matrix function $M(x)$, $a \leq x \leq b$, is such that $M(s) - M(t) \geq 0$, (≤ 0), for $a \leq s < t \leq b$, then $M(x)$ is termed non-increasing (non-decreasing) hermitian on $[a, b]$. A matrix function is termed continuous, integrable, etc., when each element of the matrix possesses the specified property.

If a matrix function $M(x)$ is a.c., (absolutely continuous), on $[a, b]$ then $M'(x)$ signifies the matrix of derivatives at values where these derivatives exist, and zero elsewhere. Similarly, if $M(x)$ is (Lebesgue) integrable on $[a, b]$ then $\int_a^b M(x)\, dx$ denotes the matrix of integrals of respective elements of $M(x)$. For a given interval $[a, b]$ the symbols $\mathfrak{C}_{hk}$, $\mathfrak{L}_{hk}$, $\mathfrak{L}_{hk}^\infty$, $\mathfrak{BV}_{hk}$ and $\mathfrak{A}_{hk}$ are used to denote the class of $h \times k$ matrix functions on $[a, b]$ which are respectively continuous, (Lebesgue) integrable, (Lebesgue) measurable and essentially bounded, of bounded variation, and absolutely continuous; for brevity, we write $\mathfrak{C}_h$, $\mathfrak{L}_h$, $\mathfrak{L}_h^\infty$, $\mathfrak{BV}_h$ and $\mathfrak{A}_h$ for $\mathfrak{C}_{h1}$, $\mathfrak{L}_{h1}$, $\mathfrak{L}_{h1}^\infty$, $\mathfrak{BV}_{h1}$ and $\mathfrak{A}_{h1}$, respectively. If $M(x) \in \mathfrak{BV}_{hk}$, $S(x) \in \mathfrak{C}_{rh}$, and $T(x) \in \mathfrak{C}_{ks}$, then $\int_a^b S[dM]T$ denotes the $r \times s$ matrix with elements given by the Riemann-Stieltjes integrals

$$\sum_{\alpha=1}^{h} \sum_{\beta=1}^{k} \int_a^b S_{i\alpha}(x) T_{\beta j}(x)\, dM_{\alpha\beta}(x),$$

and $\int_a^b [dM]T$ and $\int_a^b S[dM]$ designate $\int_a^b E_h[dM]T$ and $\int_a^b S[dM]E_k$, respectively.

2. Generalized linear differential systems

Throughout most of the present section it will be supposed that $A(x)$, $B(x)$, $C(x)$, $D(x)$, $M(x)$ are matrix functions satisfying on a fixed interval $[a, b]$ the condition

$$(\mathfrak{H}) \qquad A \in \mathfrak{L}_{nn}, \quad B \in \mathfrak{L}_{nm}, \quad C \in \mathfrak{L}_{mn}, \quad D \in \mathfrak{L}_{mm}, \quad M \in \mathfrak{BV}_{mn}.$$

For $c \in [a, b]$, and $(U, V) \in \mathfrak{C}_{nk} \times \mathfrak{L}_{mk}^\infty$, the integral operator $\Omega(c, x \mid U, V)$ is defined on $[a, b]$ by

$$(2.1) \qquad \Omega(c, x \mid U, V) = \int_c^x [C(t)U(t) - D(t)V(t)]\, dt + \int_c^x [dM(t)]U(t);$$

for brevity, $\Omega(x \mid U, V)$ is written for $\Omega(a, x \mid U, V)$.

Corresponding to the system treated in [7], we shall consider here the generalized matrix differential system

$$(2.2_M)\qquad \begin{aligned} dV - [CU - DV]\,dx - [dM]U &= 0, \\ L(x \mid U, V) \equiv -U' + AU + BV &= 0, \end{aligned} \qquad a \le x \le b.$$

By a solution of (2.2_M) is meant a pair $(U, V) \in \mathfrak{A}_{nk} \times \mathfrak{BV}_{mk}$ for some positive integer k, such that

$$\Delta(x \mid U, V) \equiv V(x) - \Omega(x \mid U, V) = V(a), \qquad a \le x \le b,$$

and $L(x \mid U, V) = 0$ a.e., (almost everywhere), on $[a, b]$. It is to be remarked that the validity of the above equation is clearly equivalent to the existence of a $c \in [a, b]$, and a constant $m \times k$ matrix V_0 such that

$$V(x) - \Omega(c, x \mid U, V) = V_0, \qquad a \le x \le b.$$

For the special case of $k = 1$, in which (2.1) and (2.2_M) reduce to vector relations, corresponding small letters for U, V will be used. In particular, for frequent reference we exhibit the generalized vector differential system

$$(2.2)\qquad \begin{aligned} dv - [Cu - Dv]\,dx - [dM]u &= 0, \\ L(x \mid u, v) \equiv -u' + Au + Bv &= 0, \end{aligned} \qquad a \le x \le b.$$

If u_0 and v_0 are given vectors of respective dimensions n and m, and $c \in [a, b]$, there is a unique solution of (2.2) such that $u(a) = u_0$, $v(a) = v_0$. This result is a consequence of Theorem 2.3 below. It could also be derived from the existence of a unique solution of a Riemann-Stieltjes integral equation similar to that used by the author in [7, §2]. Specifically, if $U_1(x)$ is the solution of the differential system $U_1' = A(x)U_1$, $U_1(c) = E_n$, and $V_1(x)$ is the solution of the system $V_1' = -D(x)V_1$, $V_1(c) = E_m$, then (u, v) is the solution of (2.2) satisfying $u(c) = u_0$, $v(c) = v_0$, if and only if $u(x) = U_1(x)u_1(x)$, $v(x) = V_1(x)v_1(x)$, where $u_1(x)$ is the (unique) solution of the integral equation

$$(2.3)\qquad u_1(x) = u_0 + \left[\int_c^x B_1(t)\,dt\right] v_0 + \int_c^x B_1(t)\left\{\int_c^t [dM_1(s)]\,u_1(s)\right\} dt,$$

$v_1(x) = v_0 + \int_c^x [dM_1(s)]u_1(s)$, and $B_1(x)$, $M_1(x)$ are defined as

$$B_1(x) = U_1^{-1}(x)B(x)V_1(x), \quad M_1(x) = \int_c^x V_1^{-1}CU_1\,dt + \int_c^x V_1^{-1}\,[dM]\,U_1\,.$$

The symbol $\mathfrak{D} = \mathfrak{D}[a, b]$ will denote the class of vector functions $\eta \in \mathfrak{A}_n$ for which there exists a corresponding $\zeta \in \mathfrak{L}_m^\infty$ such that $L(x \mid \eta, \zeta) = 0$ a.e. on $[a, b]$, and the fact that ζ is such an associated vector function will be indicated by $\eta \in \mathfrak{D}:\zeta$. The subclass of $\mathfrak{D}$ on which $\eta(a) = 0 = \eta(b)$ will be denoted by $\mathfrak{D}^0 = \mathfrak{D}^0[a, b]$, with similar meaning for $\eta \in \mathfrak{D}^0:\zeta$. Correspondingly, $\mathfrak{D}_\star = \mathfrak{D}_\star[a, b]$ will designate the class of vector functions $\rho \in \mathfrak{A}_m$ for which there is a corresponding $\sigma \in \mathfrak{L}_n^\infty$ such that

$$L_{\star}(x \mid \rho, \sigma) \equiv -\rho' + D^*(x)\rho + B^*(x)\sigma = 0 \quad \text{a.e. on} \quad [a, b],$$

and $\mathfrak{D}_{\star}^0 = \mathfrak{D}_{\star}^0[a, b]$ the subclass of $\mathfrak{D}_{\star}$ on which $\rho(a) = 0 = \rho(b)$, with similar meanings for $\rho \in \mathfrak{D}_{\star}:\sigma$ and $\rho \in \mathfrak{D}_{\star}^0:\sigma$. Finally, for $(\eta, \zeta) \in \mathfrak{A}_n \times \mathfrak{L}_m^\infty$ and $(\rho, \sigma) \in \mathfrak{A}_m \times \mathfrak{L}_n^\infty$ we define $I[\eta, \zeta; \rho, \sigma] = I[\eta, \zeta; \rho, \sigma \mid a, b]$ as

$$I[\eta, \zeta; \rho, \sigma] = \int_a^b (\sigma^* B\zeta + \rho^* C\eta)\, dx + \int_a^b \rho^*[dM]\, \eta, \tag{2.4}$$

in terms of which we have the following functional result.

THEOREM 2.1. *If* $(u, v) \in \mathfrak{C}_n \times \mathfrak{L}_m^\infty$, *then the following two conditions are equivalent*:

(2.5) $I[u, v; \rho, \sigma] = 0$ *for* $\rho \in \mathfrak{D}_{\star}^0:\sigma$;

(2.6) *there exist a constant vector* γ *and a* $v_0 \in \mathfrak{BV}_m$ *such that*

$$B[v - v_0] = 0 \quad \textit{a.e., and} \quad \Delta(x \mid u, v_0) = \gamma \quad \textit{on} \quad [a, b].$$

If $(u, v) \in \mathfrak{C}_n \times \mathfrak{L}_m^\infty$ and satisfies (2.6), then for $\rho \in \mathfrak{D}_{\star}^0:\sigma$ we have

$$I\,[u, v; \rho, \sigma] = \int_a^b (\rho^{*\prime} v_0\, dx + \rho^*\, dv_0) = \rho^* v_0 \big|_a^b = 0,$$

so that condition (2.5) holds. Conversely, if $(u, v) \in \mathfrak{C}_n \times \mathfrak{L}_m^\infty$ and (2.5) is satisfied, for $w(x) = \Omega(x \mid u, v)$ we have that $w \in \mathfrak{BV}_m$ and for $\rho \in \mathfrak{D}_{\star}^0:\sigma$,

$$\begin{aligned} 0 &= \int_a^b (\rho^{*\prime} v\, dx + \rho^* dw) = \rho^* w\big|_a^b + \int_a^b \rho^{*\prime}[v - w]\, dx \\ &= \int_a^b \rho^{*\prime}[v - w]\, dx. \end{aligned} \tag{2.7}$$

Now if $Y(x)$ is the solution of $Y' = D^*(x)Y$, $Y(a) = E_m$, then $(\rho, \sigma) \in \mathfrak{A}_m \times \mathfrak{L}_n^\infty$ with $L_{\star}(x \mid \rho, \sigma) = 0$ and $\rho(b) = 0$, if and only if

$$\rho(x) = -\int_x^b Y(x)Y^{-1}(t)B^*(t)\sigma(t)\, dt, \qquad a \le x \le b, \tag{2.8}$$

and consequently $\rho \in \mathfrak{D}_{\star}^0:\sigma$ if and only if $\sigma \in \mathfrak{L}_n^\infty$, ρ satisfies (2.8), and

$$0 = \int_a^b Y^{-1}(t)B^*(t)\sigma(t)\, dt. \tag{2.9}$$

The vector function $f = v - w$ is such that $f \in \mathfrak{L}_m^\infty$, and from (2.7), (2.8), and (2.9) it follows that if $\rho \in \mathfrak{D}_{\star}^0:\sigma$ then

$$g(x) = -Y^{*-1}(x) \int_a^x Y^*(t)D(t)f(t)\, dt + f(x)$$

is such that $g \in \mathfrak{L}_m^\infty$ and

$$\int_a^b \sigma^* Bg\, dx = 0 \textit{ if } \sigma \in \mathfrak{L}_n^\infty \textit{ and } \int_a^b \sigma^* BY^{*-1}\, dx = 0. \tag{2.10}$$

Condition (2.10) implies that there exists a constant m-dimensional vector λ such that

$$B(x)g(x) = B(x)Y^{*-1}(x)\lambda \quad \text{a.e. on} \quad [a, b].$$

Consequently, if $h(x) = g(x) - Y^{*-1}(x)\lambda$, then $h \in \mathfrak{L}_m^\infty$, $Bh = 0$ a.e. on $[a, b]$, and

$$(2.11)\quad f(x) - \int_a^x Y^{*-1}(x)Y^*(t)D(t)f(t)\,dt = h(x) + Y^{*-1}(x)\lambda, \quad a \leq x \leq b.$$

That is, f is determined as the solution of the Volterra vector integral equation (2.11) with kernel $Y^{*-1}(x)Y^*(t)D(t) = Y^{*-1}(x)Y^{*\prime}(t)$, $a \leq t \leq x \leq b$. It may be verified directly that the resolvent kernel for this equation is $-D(t)$, $a \leq t \leq x \leq b$, and hence

$$f(x) = h(x) + Y^{*-1}(x)\lambda + \int_a^x D(t)[h(t) + Y^{*-1}(t)\lambda]\,dt, \qquad a \leq x \leq b.$$

As $DY^{*-1} = -[Y^{*-1}]'$ a.e. it follows that $v - w = f$ satisfies

$$v(x) - w(x) = h(x) + \int_a^x D(t)h(t)\,dt + \gamma, \quad \textit{where } \gamma = Y^{*-1}(a)\lambda.$$

Consequently, if v_0 is defined as

$$v_0(x) = w(x) + \int_a^x D(t)h(t)\,dt + \gamma, \qquad a \leq x \leq b,$$

then $v_0 \in \mathfrak{B}\mathfrak{V}_m$, $B[v - v_0] = 0$ a.e., and

$$v_0(x) = \Omega(x \mid u, v) + \int_a^x D(t)h(t)\,dt + \gamma = \Omega(x \mid u, v - h) + \gamma = \Omega(x \mid u, v_0) + \gamma.$$

THEOREM 2.2. *If $u \in \mathfrak{A}_n$ there exists a v such that (u, v) is a solution of (2.2) if and only if there exists a v_1 such that $u \in \mathfrak{D}{:}v_1$ and $I[u, v_1; \rho, \sigma] = 0$ for $\rho \in \mathfrak{D}_{\star}^0{:}\sigma$.*

If $(\eta, \zeta) \in \mathfrak{A}_n \times \mathfrak{L}_m^\infty$ and $(\rho, \sigma) \in \mathfrak{A}_m \times \mathfrak{L}_n^\infty$, then

$$(2.12)\qquad I[\eta, \zeta; \rho, \sigma] = \int_a^b [\sigma^*B\zeta - \rho^{*\prime}M\eta - \rho^*M\eta' + \rho^*C\eta]\,dx + \rho^*M\eta\,|_a^b,$$

and consequently if $u \in \mathfrak{D}{:}v$ and $\rho \in \mathfrak{D}_{\star}^0{:}\sigma$ then

$$(2.13)\qquad I[u, v; \rho, \sigma] = \int_a^b [(\sigma - M^*\rho)^*B(v - Mu) + \rho^*(C - DM - MA - MBM)\eta]\,dx.$$

For brevity, let $\hat{A}(x)$, $\hat{B}(x)$, $\hat{C}(x)$, $\hat{D}(x)$ be defined on $[a, b]$ as

$$(2.14)\qquad \begin{gathered} \hat{A} = A + BM, \quad \hat{B} = B, \\ \hat{C} = C - DM - MA - MBM, \quad \hat{D} = D + MB, \end{gathered}$$

and denote by $\hat{\mathfrak{D}}$, $\hat{\mathfrak{D}}^0$, $\hat{\mathfrak{D}}_{\star}$, $\hat{\mathfrak{D}}^0_{\star}$ the above defined classes $\mathfrak{D}$, $\mathfrak{D}^0$, $\mathfrak{D}_{\star}$, $\mathfrak{D}^0_{\star}$ when A, B, C, D are replaced respectively by $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$. It follows readily that $\eta \in \mathfrak{D}:\zeta$ or $\eta \in \mathfrak{D}^0:\zeta$ if and only if $\hat{\zeta} = \zeta - M\eta$ is such that $\eta \in \hat{\mathfrak{D}}:\hat{\zeta}$ or $\eta \in \hat{\mathfrak{D}}^0:\hat{\zeta}$; correspondingly, $\rho \in \mathfrak{D}_{\star}:\sigma$ or $\rho \in \mathfrak{D}^0_{\star}:\sigma$ if and only if $\hat{\sigma} = \sigma - M^*\rho$ is such that $\rho \in \hat{\mathfrak{D}}_{\star}:\hat{\sigma}$ or $\rho \in \hat{\mathfrak{D}}^0_{\star}:\hat{\sigma}$. Moreover, if $u \in \mathfrak{D}:v$ then $(\hat{u}, \hat{v}) = (u, v - Mu)$ is such that $\hat{u} \in \hat{\mathfrak{D}}:\hat{v}$ and (2.5) holds if and only if

$$\hat{I}[\hat{u}, \hat{v}; \hat{\rho}, \hat{\sigma}] \equiv \int_a^b [\hat{\sigma}^*\hat{B}\hat{v} + \hat{\rho}^*\hat{C}\hat{u}]\, dx = 0 \quad \textit{for } \hat{\rho} \in \hat{\mathfrak{D}}^0_{\star} : \sigma^*. \tag{2.15}$$

Also, an integration by parts of $\int_c^x [dM(t)]U(t)$ in (2.1) provides the following precise derivative result, which is basic for future discussion.

THEOREM 2.3. *A pair* (u, v) *is a solution of* (2.2) *if and only if* $(\hat{u}, \hat{v}) = (u, v - Mu)$ *is such that* $(\hat{u}, \hat{v}) \in \mathfrak{A}_n \times \mathfrak{A}_m$ *and is a solution of the vector ordinary differential system*

$$\begin{aligned} \hat{v}' - \hat{C}(x)\hat{u} + \hat{D}(x)\hat{v} &= 0, \\ -\hat{u}' + \hat{A}(x)\hat{u} + \hat{B}(x)\hat{v} &= 0, \end{aligned} \qquad a \le x \le b. \tag{2.16}$$

It is to be emphasized that in order for the system (2.16) to be well defined and have integrable coefficients given by (2.14) it is not necessary that $M(x)$ be of bounded variation on $[a, b]$. In particular, (2.16) has integrable coefficients if A, B, C, D satisfy the conditions of ($\mathfrak{H}$), while it is supposed merely that $M(x) \in \mathfrak{L}^\infty_{mn}$.

3. Adjoint systems

For A, B, C, D and M matrix functions satisfying hypothesis ($\mathfrak{H}$) the generalized vector differential system

$$\begin{aligned} dz - [C^*y - A^*z]\, dx - [dM^*]y &= 0, \\ L_{\star}(x \mid y, z) \equiv -y' + D^*y + B^*z &= 0, \end{aligned} \qquad a \le x \le b, \tag{3.1}$$

is termed "adjoint to (2.2)". Corresponding to the notation of §2, the operators $\Omega_{\star}$ and $\Delta_{\star}$ are defined as

$$\Omega_{\star}(x \mid y,z) = \int_a^x [C^*(t)y(t) - A^*(t)z(t)]\, dt + \int_a^x [dM^*(t)]y(t),$$

$$\Delta_{\star}(x \mid y, z) = z(x) - \Omega_{\star}(x \mid y, z).$$

In particular, (3.1) is obtained from (2.2) upon replacing A, B, C, D, M by D^*, B^*, C^*, A^*, M^*, respectively, so that (2.2) is also adjoint to (3.1) in the sense thus defined. Moreover, under this substitution the classes $\mathfrak{D}$, $\mathfrak{D}^0$ are interchanged with the respective classes $\mathfrak{D}_{\star}$, $\mathfrak{D}^0_{\star}$, while for $(\eta, \zeta) \in \mathfrak{C}_n \times \mathfrak{L}^\infty_m$ and $(\rho, \sigma) \in \mathfrak{C}_m \times \mathfrak{L}^\infty_n$ the functional

$$I_{\star}[\rho, \sigma; \eta, \zeta] = \int_a^b [\zeta^*B^*\sigma + \eta^*C^*\rho]\, dx + \int_a^b \eta^*[dM^*]\rho \tag{3.2}$$

is such that $(I_{☆}[\rho, \sigma; \eta, \zeta])^* = I[\eta, \zeta; \rho, \sigma]$. Consequently, results for (3.1) corresponding to those of Theorems 2.1, 2.2, 2.3 for (2.2) may be stated as follows in terms of the functional $I[\eta, \zeta; \rho, \sigma]$ defined by (2.3).

THEOREM 3.1. *If $(y, z) \in \mathfrak{C}_m \times \mathfrak{L}_n^\infty$, then the following two conditions are equivalent:*

(3.3) $I[\eta, \zeta; y, z] = 0$ *for* $\eta \in \mathfrak{D}^0:\zeta$;

(3.4) *there exist a constant vector $\gamma_{☆}$ and a $z_0 \in \mathfrak{BB}_n$ such that*

$$B^*[z - z_0] = 0 \quad \text{a.e., and} \quad \Delta_{☆}(x \mid y, z_0) = \gamma_{☆} \quad \text{on} \quad [a, b].$$

THEOREM 3.2. *If $y \in \mathfrak{A}_m$ there exists a z such that (y, z) is a solution of (3.1) if and only if there exists a z_1 such that $y \in \mathfrak{D}_{☆}:z_1$ and $I[\eta, \zeta; y, z_1] = 0$ for $\eta \in \mathfrak{D}^0:\zeta$.*

THEOREM 3.3. *A pair (y, z) is a solution of (3.1) if and only if $(\hat{y}, \hat{z}) = (y, z - M^*y)$ is such that $(\hat{y}, \hat{z}) \in \mathfrak{A}_m \times \mathfrak{A}_n$ and is a solution of the vector ordinary differential system*

$$\begin{aligned} \hat{z}' - \hat{C}^*(x)\hat{y} + \hat{A}^*(x)\hat{z} &= 0, \\ -\hat{y}' + \hat{D}^*(x)\hat{y} + \hat{B}^*(x)\hat{z} &= 0, \end{aligned} \qquad a \leq x \leq b. \tag{3.5}$$

As (3.5) is adjoint to (2.16), the following result for solutions of (2.2) and (3.1) is a direct consequence of the corresponding result for ordinary differential systems.

COROLLARY. *If (u, v) and (y, z) are solutions of (2.2) and (3.1), respectively, then $z^*u - y^*v$ is constant on $[a, b]$.*

The result of the following theorem is of basic importance for the study of solutions of (2.2) and (3.1).

THEOREM 3.4. *Suppose that $(\eta, \zeta) \in \mathfrak{A}_n \times \mathfrak{L}_m^\infty$, $(\rho, \sigma) \in \mathfrak{A}_m \times \mathfrak{L}_n^\infty$, and there exist $(U, V) \in \mathfrak{A}_{np} \times \mathfrak{L}_{mp}^\infty$, $(Y, Z) \in \mathfrak{A}_{mq} \times \mathfrak{L}_{nq}^\infty$ and $h \in \mathfrak{A}_p$, $k \in \mathfrak{A}_q$ such that $\eta = Uh$ and $\rho = Yk$. Then for $a \leq c < d \leq b$ the value of the functional*

$$I[\eta, \zeta; \rho, \sigma \mid c, d] = \int_c^d \sigma^*B\zeta + \rho^*C\eta]\, dx + \int_c^d \rho^*[dM]\eta \tag{3.6}$$

is equal to each of the following expressions:

$$\begin{aligned} \int_c^d \{(\sigma - Zk)^*B(\zeta - Vh) - [L_{☆}(x \mid \rho, \sigma)]^*Vh \\ + k^*Z^*(L(x \mid U, V)h - L(x \mid \eta, \zeta)) \\ + k^*(Z^*U - Y^*V)h'\}\, dx \\ - \int_c^d k^*Y^*[d\Delta(x \mid U, V)]h + k^*Y^*Vh\,\Big|_c^d, \end{aligned} \tag{3.7}$$

$$\begin{aligned}(3.8)\qquad & \int_c^d \{(\sigma - Zk)^*B(\zeta - Vh) - k^*Z^*L(x \mid \eta, \zeta) \\ & \quad + (k^*[L_\star(x \mid Y, Z)]^* - [L_\star(x \mid \rho, \sigma)]^*)\, Vh \\ & \quad + k^{*\prime}(Y^*V - Z^*U)h\}\, dx \\ & \quad - \int_c^d k^*[d\Delta_\star^*(x \mid Y, Z)]\, Uh + k^*Z^*Uh \big|_c^d .\end{aligned}$$

The fact that $I[\eta, \zeta; \rho, \sigma \mid c, d]$ is given by (3.7) may be established by direct comparison of the involved integrand functions. In turn, the fact that (3.6) is also equal to (3.8) may be deduced from the preceding result applied to the adjoint functional $I_\star[\rho, \sigma; \eta, \zeta \mid c, d]$ and the fact that

$$I[\eta, \zeta; \rho, \sigma \mid c, d] = (I_\star[\rho, \sigma; \eta, \zeta \mid c, d])^*.$$

Of particular significance are the following results which are direct consequences of the above general results for the special instance of h and k arbitrary constant vector functions.

COROLLARY. *If* $(U, V) \in \mathfrak{A}_{np} \times \mathfrak{L}_{mp}^\infty$, $(Y, Z) \in \mathfrak{A}_{mq} \times \mathfrak{L}_{nq}^\infty$, *while* $L(x \mid U, V) = 0$ *and* $L_\star(x \mid Y, Z) = 0$ *a.e. on* $[a, b]$, *then for* $a \leq t \leq b$,

$$I[U, V; Y, Z \mid a, t] = -\int_a^t Y^*[d\Delta(x \mid U, V)] + Y^*V\big|_a^t, \tag{3.9}$$

$$Y^*V - Z^*U \big|_a^t = \int_a^t Y^*[d\Delta(x \mid U, V)] - \left(\int_a^t U^*[d\Delta_\star(x \mid Y, Z)]\right)^*. \tag{3.10}$$

4. A Riccati matrix integral equation

In view of Theorems 2.3 and 3.3, results concerning solutions of (2.2) or (3.1) are equivalent to corresponding results for the respective systems (2.16) and (3.5). Entirely analogous to the consideration for the special case $n = m$, (see, for example, Reid [8; §II]), one may show that there exists a $\hat{W} \in \mathfrak{A}_{mn}$ which is a solution of the Riccati matrix differential equation

$$\hat{K}[\hat{W}] \equiv \hat{W}' - \hat{F}(x, \hat{W}) = 0, \tag{4.1}$$

where

$$\hat{F}(x, \hat{W}) = \hat{C}(x) - \hat{W}\hat{A}(x) - \hat{D}(x)\hat{W} - \hat{W}\hat{B}(x)\hat{W}, \tag{4.2}$$

if and only if there exists a solution $(\hat{U}, \hat{V})$ of the differential system

$$\begin{aligned}\hat{V}' - \hat{C}(x)\hat{U} + \hat{D}(x)\hat{V} &= 0, \\ -\hat{U}' + \hat{A}(x)\hat{U} + \hat{B}(x)\hat{V} &= 0,\end{aligned} \qquad a \leq x \leq b \tag{4.3}$$

with $\hat{U}(x)$ non-singular on $[a, b]$ and $\hat{W}(x) = \hat{V}(x)\hat{U}^{-1}(x)$ on this interval. The general Riccati equation of the form (4.1) has been considered by J. J. Levin [5], and the reader is referred to his bibliography for references to other studies of related problems, especially of the form (4.1) with $n = 1, m > 1$.

In accord with the remark following the statement of Theorem 2.3, it is to be noted that the validity of the above comments does not depend upon $M(x)$ being of bounded variation, and these comments are true under the following weaker assumption:

$$(\mathfrak{H}_0) \qquad A \in \mathfrak{L}_{nn}, \quad B \in \mathfrak{L}_{nm}, \quad C \in \mathfrak{L}_{mn}, \quad D \in \mathfrak{L}_{mm}, \quad M \in \mathfrak{L}^{\infty}_{mn}.$$

Now for $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ related to A, B, C, D, M by (2.14), and

$$F(x, W) = C(x) - WA(x) - D(x)W - WB(x)W, \tag{4.4}$$

it may be verified directly that if W and $\hat{W}$ are $m \times n$ matrices satisfying

$$\hat{W}(x) = W(x) - M(x), \qquad a \leq x \leq b, \tag{4.5}$$

then $\hat{F}(x, \hat{W}) \equiv F(x, W)$, and hence $\hat{W}(x)$ is the solution of the differential system

$$\hat{K}[\hat{W}] = 0, \qquad \hat{W}(s) = \Psi, \tag{4.6}$$

on a subinterval containing $x = s$ if and only if $W = \hat{W} + M$ is the solution of the Riccati matrix integral equation

$$W(x) - \int_s^x F(t, W(t))\, dt = M(x) + \Psi, \qquad x \in X. \tag{4.7}$$

In particular, for X a subinterval containing $x = s$ of sufficiently small length there is a unique solution of (4.7) on X, and in the following discussion many criteria involve the existence of a solution of an equation of the form (4.7) on the entire given interval $[a, b]$.

THEOREM 4.1. *If hypothesis* $(\mathfrak{H}_0)$ *holds, and* $s \in [a, b]$, *then an* $m \times n$ *matrix* Ψ *is such that* (4.7) *has a solution on* $[a, b]$ *if and only if the solution* $(\hat{U}, \hat{V})$ *of* (4.3) *determined by the initial conditions* $\hat{U}(s) = E_n$, $\hat{V}(s) = \Psi$, *is such that* $\hat{U}(x)$ *is non-singular on* $[a, b]$, *and in this case*

$$W(x) = M(x) + \hat{V}(x)\hat{U}^{-1}(x).$$

If $M(x) \in \mathfrak{BV}_{mn}$, *then an equivalent condition is that the solution* (U, V) *of* (2.2_M) *determined by* $U(s) = E_n$, $V(s) = M(s) + \Psi$ *is such that* $U(x)$ *is non-singular on* $[a, b]$, *and in this case*

$$W(x) = V(x)U^{-1}(x).$$

If $\hat{W}_0(x)$ is a solution of $\hat{K}[\hat{W}_0] = 0$ on a subinterval X, and $s \in X$, let

$$\hat{G}(x) = \hat{G}(x, s \mid \hat{W}_0), \qquad \hat{H}(x) = \hat{H}(x, s \mid \hat{W}_0)$$

be defined as the solutions of the matrix differential systems

$$\begin{aligned} &\hat{G}' + (\hat{D} + \hat{W}_0 \hat{B})\hat{G} = 0, \qquad \hat{G}(s) = E_m, \\ &\hat{H}' + \hat{H}(\hat{A} + \hat{B}\hat{W}_0) = 0, \qquad \hat{H}(s) = E_n, \end{aligned} \tag{4.8}$$

and define the $n \times m$ matrix $\hat{\Theta}(x, s \mid \hat{W}_0)$ by

$$\hat{\Theta}(x, s \mid \hat{W}_0) = \int_s^x \hat{H}(t, s \mid \hat{W}_0)\hat{B}(t)\hat{G}(t, s \mid \hat{W}_0)\, dt, \qquad x \in X. \tag{4.9}$$

As in the case $n = m$, (see Reid [8]), it may be shown that $\hat{W}(x)$ is a solution of (4.1) on X if and only if the constant $m \times n$ matrix $\Gamma = \hat{W}(s) - \hat{W}_0(s)$ is such that the $n \times n$ matrix $E_n + \hat{\Theta}(x, s \mid \hat{W}_0)\Gamma$ is non-singular on X, and in this case

$$\hat{W}(x) = \hat{W}_0(x) + \hat{G}(x, s \mid \hat{W}_0)\Gamma[E_n + \hat{\Theta}(x, s \mid \hat{W}_0)\Gamma]^{-1}\hat{H}(x, s \mid \hat{W}_0). \tag{4.10}$$

Now for $\hat{\Theta}$ an $n \times m$ matrix and Γ an $m \times n$ matrix the identity $(E_m + \Gamma\hat{\Theta})\Gamma = \Gamma(E_n + \hat{\Theta}\Gamma)$ implies that the $m \times m$ matrix $E_m + \Gamma\hat{\Theta}$ is non-singular if and only if the $n \times n$ matrix $E_n + \hat{\Theta}\Gamma$ is non-singular, and $\Gamma[E_n + \hat{\Theta}\Gamma]^{-1} = [E_m + \Gamma\hat{\Theta}]^{-1}\Gamma$. Consequently, the non-singularity of the $n \times n$ matrix $E_n + \hat{\Theta}(x, s \mid \hat{W}_0)\Gamma$ on X is equivalent to the non-singularity of the $m \times m$ matrix $E_m + \Gamma\hat{\Theta}(x, s \mid \hat{W}_0)$ on this interval, and an alternate form for $\hat{W}(x)$ is

$$\hat{W}(x) = \hat{W}_0(x) + \hat{G}(x, s \mid \hat{W}_0)[E_m + \Gamma\hat{\Theta}(x, s \mid \hat{W}_0)]^{-1}\Gamma\hat{H}(x, s \mid \hat{W}_0). \tag{4.11}$$

From (2.14) it follows that if W, W_0 are related to respective matrices $\hat{W}$, $\hat{W}_0$ by (4.5) then

$$\hat{D} + \hat{W}_0\hat{B} = D + W_0 B, \quad \hat{A} + \hat{B}\hat{W}_0 = A + BW_0,$$
$$\hat{W}(s) - \hat{W}_0(s) = W(s) - W_0(s),$$

and in view of the above remarks we have the following theorem, which extends to systems (2.2) a result of Reid [8].

THEOREM 4.2. *Suppose that hypothesis* ($\mathfrak{H}_0$) *holds,* $s \in [a, b]$, *and* $W = W_0(x)$ *is a solution of* (4.7) *for* $\Psi = \Psi_0$ *on a subinterval* X *of* $[a, b]$ *containing* $x = s$. *If* $G(x) = G(x, s \mid W_0)$, $H(x) = H(x, s \mid W_0)$, *and* $\Theta(x, s \mid W_0)$ *are defined by the differential systems and integral relation*

$$\begin{aligned} &G' + (D + W_0 B)G = 0, \qquad G(s) = E_m, \\ &H' + H(A + BW_0) = 0, \qquad H(s) = E_n, \\ &\Theta(x,s \mid W_0) = \int_s^x H(t, s \mid W_0)B(t)G(t, s \mid W_0)\, dt, \qquad x \in X, \end{aligned} \tag{4.12}$$

then $W(x)$ *is a solution of* (4.7) *on* X *if and only if* $\Gamma = \Psi - \Psi_0$ *is such that* $E_n + \Theta(x, s \mid W_0)\Gamma$ *is non-singular on* X, *and in this case*

$$\begin{aligned} W(x) &= W_0(x) + G(x, s \mid W_0)\Gamma[E_n + \Theta(x, s \mid W_0)\Gamma]^{-1}H(x, s \mid W_0), \\ &= W_0(x) + G(x, s \mid W_0)[E_m + \Gamma\Theta(x, s \mid W_0)]^{-1}\Gamma H(x, s \mid W_0). \end{aligned}$$

Correspondingly, the results of Lemmas 2.2, 2.3 and 2.4 of Reid [8] are extensible to (2.2). In particular, if (U, V) is a solution of (2.2_M) with U non-singular on a subinterval X and $W = VU^{-1}$, then for $s \in X$ and

$$K(x, s \mid W) = H^{-1}(x, s \mid W)\Theta(x, s \mid W),$$
$$L(x, s \mid W) = W(x)K(x, s \mid W) + G(x, s \mid W),$$

the solution of (2.2) satisfying $u(s) = 0$, $v(s) = \xi$ is given by

$$(u, v) = (K(x, s \mid W)\xi, L(x, s \mid W)\xi).$$

Similar to the special case of $n = m$, $M(x) \equiv 0$, as treated in [8, §III], two distinct points s and t on $[a, b]$ are termed (mutually) *conjugate* with respect to (2.2) if there exists a solution (u, v) of this system with $u \not\equiv 0$ on the subinterval with endpoints s and t, while $u(s) = 0 = u(t)$. The system (2.2) is said to be *non-oscillatory* on a subinterval provided no two distinct points of this subinterval are conjugate. Moreover, for a non-degenerate subinterval X the order of abnormality of (2.2) is defined as the dimension of the linear space $\Lambda[X]$ of m-dimensional vector functions $v(x)$ which are solutions of $v' + Dv = 0$ and satisfy $Bv = 0$ a.e. on X; clearly, $v \in \Lambda[X]$ if and only if $(u, v) = (0, v(x))$ is a solution of (2.2) on X. If (2.2) has order of abnormality equal to d on a subinterval X_{st} with endpoints s and t, then there exists an $m \times d$ matrix $\Delta(s)$ such that the solution (U, V) of (2.2_M) with $U(s) = 0$, $V(s) = \Delta(s)$ has $U \equiv 0$ on X_{st} and the column vectors of $V(x)$ form a basis for $\Lambda[X_{st}]$. Moreover, if W is a solution of (4.7) which exists on X_{st}, then as in Lemma 3.4 of Reid [8] it may be shown that the column vectors of $G(x, s \mid W)\Delta(s)$ form a basis for $\Lambda[X_{st}]$ so that $\Theta(x, s \mid W)\Delta(s) \equiv 0$ for $x \in X_{st}$, and the $(n + d) \times m$ matrix

$$\begin{Vmatrix} \Theta(t, s \mid W) \\ \Delta^*(s) \end{Vmatrix}$$

has rank less than m if and only if t is conjugate to s with respect to (2.2). It is to be noted that if $n + d < m$, then t is conjugate to s.

5. Self-adjoint systems

In this section we shall consider generalized differential systems (2.2) and Riccati matrix integral equations (4.7) for which $n = m$, and the coefficient matrices satisfy on $[a, b]$ the following conditions:

$$(5.1) \quad A^*(x) \equiv D(x), \quad B(x) \equiv B^*(x), \quad C(x) \equiv C^*(x), \quad M(x) \equiv M^*(x).$$

The symbols $(5.1; \mathfrak{H})$ and $(5.1; \mathfrak{H}_0)$ will denote the hypothesis that (5.1) holds, together with the respective condition $\mathfrak{H}$ or $\mathfrak{H}_0$. In particular, (5.1) implies that $F(x, W)$ defined by (4.4) is now

$$F(x, W) = C(x) - WA(x) - A^*(x)W - WB(x)W,$$

so that $[F(x, W)]^* = F(x, W^*)$. Hence if $(5.1; \mathfrak{H}_0)$ holds and for $\Psi = \Psi_0$ the Riccati matrix integral equation

$$(5.2{:}s)\qquad W(x) - \int_s^x F(t, W(t))\,dt = M(x) + \Psi$$

has a solution $W = W_0(x)$ on a subinterval X containing $x = s$ then $W = W_0^*(x)$ is the solution of $(5.2{:}s)$ on X for $\Psi = \Psi_0^*$; in particular, if Ψ is hermitian then the solution $W(x)$ of $(5.2{:}s)$ is hermitian throughout its interval of existence. Moreover, $(5.1{:}\mathfrak{H}_0)$ implies that the classes $\mathfrak{D}$, $\mathfrak{D}^0$, $\hat{\mathfrak{D}}$, $\hat{\mathfrak{D}}^0$ are equal to the respective classes $\mathfrak{D}_{\star}$, $\mathfrak{D}_{\star}^0$, $\hat{\mathfrak{D}}_{\star}$, $\hat{\mathfrak{D}}_{\star}^0$, and the ordinary differential system (2.16) is identical with its adjoint (3.5). In case the stronger condition $(5.1{:}\mathfrak{H})$ holds, then the functional $I[\eta, \zeta; \rho, \sigma]$ defined by (2.4) is hermitian on $\mathfrak{D} \times \mathfrak{D}$, so that $I[\eta, \zeta] = I[\eta, \zeta; \eta, \zeta]$ is real-valued on $\mathfrak{D}$. *By* $\mathfrak{H}_+[\mathfrak{D}^0]$ *we shall denote the condition that* $I[\eta, \zeta]$ *is positive definite on* $\mathfrak{D}^0$; that is, $I[\eta, \zeta] \geq 0$ for $\eta \in \mathfrak{D}^0{:}\zeta$, and the equality sign holds only if $\eta \equiv 0$ and $B\zeta = 0$ a.e. on $[a, b]$. The basic criterion concerning non-oscillation of (2.2) on a compact interval $[a, b]$ is embodied in the following result, which extends Theorem 3.1 of Reid [7].

THEOREM 5.1. *If condition* $(5.1{:}\mathfrak{H})$ *is satisfied, then* $\mathfrak{H}_+[\mathfrak{D}^0]$ *holds if and only if* $B(x) \geq 0$ *a.e. on* $[a, b]$, *together with one of the following*:

(i) (2.2) *is non-oscillatory on* $[a, b]$;

(ii) *there exists a solution* (U, V) *of* (2.2_M) *with* $U(x)$ *non-singular and* $U^*V - V^*U \equiv 0$ *on* $[a, b]$;

(iii) *there exists an hermitian* $n \times n$ *constant matrix* Ψ *such that* (5.2:a) *has a solution* $W(x)$ *on* $[a, b]$;

(iv) *there exists a non-increasing hermitian matrix* $\Phi(x)$ *on* $[a, b]$ *such that on this interval there is a solution* $W(x)$ *of*

$$(5.3)\qquad W(x) - \int_a^x F(t, W(t))\,dt = M(x) + \Phi(x).$$

For a system (2.2) the condition $(5.1{:}\mathfrak{H})$ implies that $\mathfrak{H}_+[\mathfrak{D}^0]$ is equivalent to the corresponding condition for the related ordinary differential system (2.16), and thus from the result of Theorem 5.1 of Reid [8] applied to (2.16) we have that $\mathfrak{H}_+[\mathfrak{D}^0]$ holds if and only if $B(x) \geq 0$ a.e. on $[a, b]$, together with either (i) or (ii) of Theorem 5.1. On the other hand, from Theorem 4.1 it follows that the existence of a solution (U, V) of (2.2_M) with U non-singular and $U^*V - V^*U \equiv 0$ on $[a, b]$ is equivalent to the existence of a constant matrix Ψ such that (5.2:a) has a solution $W(x)$ on $[a, b]$. Finally, (iii) implies (iv) with $\Phi(x) \equiv \Psi$, whereas if $B(x) \geq 0$ a.e. on $[a, b]$ and (iv) holds then the established results applied to (2.2) with $M(x)$ replaced by $M(x) + \Phi(x)$ imply that

$$\int_a^b [\zeta^* B\zeta + \eta^* C\eta]\,dx + \int_a^b \eta^*\,[d\{M(x) + \Phi(x)\}]\eta$$

is positive definite on $\mathfrak{D}^0$ and consequently, since $\Phi(x)$ is non-increasing hermitian, that $I[\eta, \zeta]$ is positive definite on $\mathfrak{D}^0$.

In particular, the basic relation in the proof that the positive definiteness of $I[\eta, \zeta] = I[\eta, \zeta; \eta, \zeta]$ on $\mathfrak{D}^0$ is implied by $B(x) \geq 0$ a.e. and condition (ii) for the related system (4.3) becomes the following result when translated into a condition on $I[\eta, \zeta]$. This result is a direct consequence of Theorem 3.4, and is presented here for specific later use.

LEMMA 5.1. *If* $(5.1:\mathfrak{H})$ *is satisfied, and* (U, V) *is a solution of* (2.2_M) *for which* $U^*V - V^*U \equiv 0$ *on* $[a, b]$, *while* $\eta \in \mathfrak{D}:\zeta$ *is such that there exists an* $h \in \mathfrak{A}_n$ *satisfying* $\eta(x) = U(x)h(x)$ *on a subinterval* $[c, d]$, *then*

$$(5.4)\qquad \begin{aligned} &\int_c^d [\zeta^* B\zeta + \eta^* C\eta]\, dx + \int_c^d \eta^* [dM]\eta \\ &\qquad = \int_c^d (\zeta - Vh)^* B(\zeta - Vh)\, dx + h^*U^*Vh \,\Big|_c^d. \end{aligned}$$

Application of Lemma 5.1 of Reid [8] to the system (2.16) yields the following result for (2.2).

THEOREM 5.2. *If condition* $(5.1:\mathfrak{H})$ *is satisfied, and* $\mathfrak{H}_+[\mathfrak{D}^0]$ *holds, then for* $\eta \in \mathfrak{D}:\zeta$ *there exists a solution* (u, v) *of* (2.2) *such that* $u(a) = \eta(a)$, $u(b) = \eta(b)$ *and* $I[\eta, \zeta] \geq I[u, v]$, *with equality if and only if* $\eta \equiv u$ *and* $B[\zeta - v] = 0$ *a.e. on* $[a, b]$.

THEOREM 5.3. *If condition* $(5.1:\mathfrak{H})$ *is satisfied, and* $I[\eta, \zeta] \geq 0$ *on* $\mathfrak{D}^0$, *then either there exists a solution* (u, v) *of* (2.2) *with* $u \not\equiv 0$ *on* $[a, b]$ *and* $u(a) = 0 = u(b)$, *or there exists a* $\kappa > 0$ *such that if* $\Pi(x)$ *is an* $n \times n$ *non-decreasing hermitian matrix function which is not constant on* $[a, b]$ *then*

$$(5.5)\qquad I[\eta, \zeta] \geq (\kappa/V[a, b:\Pi]) \int_a^b \eta^*[d\Pi]\eta \quad \textit{for } \eta \in \mathfrak{D}^0:\zeta,$$

where $V[a, b:\Pi]$ *is the supremum of* $\sum_{\alpha=1}^m \mid \Pi(t_\alpha) - \Pi(t_{\alpha-1}) \mid$ *for all partitions* $a = t_0 < \cdots < t_m = b$ *of* $[a, b]$.

If $I[\eta, \zeta] \geq 0$ on $\mathfrak{D}^0$ then for $\varepsilon > 0$ the functional $I[\eta, \zeta] + \varepsilon \int_a^b \mid \eta \mid^2 dx$ is positive definite on $\mathfrak{D}^0$, and therefore $B(x) \geq 0$ a.e. on $[a, b]$ by a result of Theorem 5.1. Consequently, from Theorem 5.1 it follows that if $I[\eta, \zeta] \geq 0$ on $\mathfrak{D}^0$, and $I[\eta, \zeta]$ is not positive definite on $\mathfrak{D}^0$, then (2.2) is oscillatory on $[a, b]$; that is, there exists a solution (u_0, v_0) of (2.2) and x_1, x_2 such that $a \leq x_1 < x_2 \leq b$, $u_0(x_1) = 0 = u_0(x_2)$, and $u_0(x) \not\equiv 0$ on $[x_1, x_2]$. Let $(\eta_0, \zeta_0) = (u_0, v_0)$ on $[x_1, x_2]$, $(\eta_0, \zeta_0) \equiv 0$ on $[a, x_1)$ and $(x_2, b]$. Then $\eta_0 \in \mathfrak{D}^0:\zeta_0$, and with the aid of Lemma 5.1 it follows that $I[\eta_0, \zeta_0] = 0$. As the non-negativeness of $I[\eta, \zeta]$ on $\mathfrak{D}^0$ implies the "Schwarz inequality" $\mid I[\eta_0, \zeta_0; \eta, \zeta]\mid^2 \leq I[\eta, \zeta]I[\eta_0, \zeta_0] = 0$ for $\eta \in \mathfrak{D}^0:\zeta$, it then follows from Theorem 2.2 that there exists a solution (u, v) of (2.2) such that $u \equiv \eta_0$ on $[a, b]$. On the other hand, if $I[\eta, \zeta]$ is positive definite on $\mathfrak{D}^0$ and (U, V) is a solution of (2.2_M) satisfying conditions (ii) of Theorem 5.1, then for $\eta \in \mathfrak{D}^0:\zeta$ the vector function $h(x) = U^{-1}(x)\eta(x)$ belongs to $\mathfrak{A}_n$, $h(a) = 0 = h(b)$, and with the aid of Lemma 5.1 it follows that for $\eta \in \mathfrak{D}^0:\zeta$,

$$I[\eta, \zeta] = \int_a^b \tau^* B\tau \, dx, \quad \textit{where } \tau(x) = \zeta(x) - V(x)h(x) \textit{ on } [a, b].$$

Since $L(x \mid \eta, \zeta) = 0$ and $L(x \mid U, V) = 0$ imply that $Uh' = B\tau$, it follows that

$$h(x) = -\int_x^b U^{-1}(t)B(t)\tau(t) \, dt,$$

and, in view of $B(x) \geq 0$ a.e., for arbitrary vectors ξ we have

$$|\, \xi^* h(x)|^2 \leq \left[\int_x^b \xi^* U^{-1} B U^{*-1} \xi \, dt\right] \cdot \left[\int_a^b \tau^* B\tau \, dt\right].$$

This inequality, combined with an analogous one for the interval $[a, x]$, implies

$$|\, h(x)|^2 \leq \tfrac{1}{2}\, \kappa_0 \int_a^b \tau^* B\tau \, dx, \quad \textit{where } \kappa_0 = \sup_{|\xi|=1} \xi^* \left[\int_a^b U^{-1} B U^{*-1} \, dx\right] \xi.$$

Consequently, for $\eta \in \mathfrak{D}^0:\zeta$,

$$(5.6) \qquad |\, \eta(x)|^2 \leq \tfrac{1}{2}\, \kappa_0\, \kappa_1^2 \int_a^b \tau^* B\tau \, dx, \quad \textit{where } \kappa_1 = \sup |\, U^{-1}(x)| \textit{ on } [a, b],$$

and for $\Pi(x)$ a non-decreasing hermitian $n \times n$ matrix on $[a, b]$,

$$\int_a^b \eta^*[d\Pi]\eta \leq \tfrac{1}{2}\, \kappa_0\, \kappa_1^2\, V[a, b:\Pi] \int_a^b \tau^* B\tau \, dx,$$

so that (5.5) holds with $\kappa = 2/(\kappa_0 \kappa_1^2)$.

COROLLARY. *Suppose that condition* $(5.1:\mathfrak{H})$ *holds, and* $\Pi(x)$ *is a non-decreasing hermitian matrix which is not constant on* $[a, b]$. *If the subclass* $\mathfrak{D}_1^0$ *of* $\mathfrak{D}^0$ *on which* $\int_a^b \eta^*[d\Pi]\eta = 1$ *is non-empty, and the infimum* λ_1 *of* $I[\eta, \zeta]$ *on* $\mathfrak{D}_1^0$ *is finite, then for* $\lambda = \lambda_1$ *there exists a solution* (u, v) *of the boundary value problem*

$$\begin{aligned} &d\Delta(x \mid u, v) + \lambda[d\Pi]u = 0, \\ (5.7) \qquad &L(x \mid u, v) \equiv -u' + Au + Bv = 0, \qquad a \leq x \leq b, \\ &u(a) = 0 = u(b), \end{aligned}$$

with $u(x) \not\equiv 0$ *on* $[a, b]$.

The corollary follows immediately from the application of Theorem 5.3 to the functional $I[\eta, \zeta] - \lambda_1 \int_a^b \eta^*[d\Pi]\eta$. By definition, a value λ is a *normal proper value* of (5.7) if there exists a corresponding solution (u, v) of this system with $u \not\equiv 0$ on $[a, b]$. Under the hypotheses of the corollary it may be established readily that all normal proper values are real, and that the value $\lambda = \lambda_1$ of the corollary is the smallest normal proper value.

Under the weakened assumption $(5.1:\mathfrak{H}_0)$, in view of Theorem 4.1 the application of Theorem 5.1 to (2.16) yields that $I[\eta, \zeta]$ is positive definite on $\mathfrak{D}^0$ if and only if $B(x) \equiv B^*(x) \geq 0$ a.e. on $[a, b]$ and either (iii) or (iv) of Theorem 5.1 holds. Consequently, for Riccati matrix integral equations one has the following comparison theorem.

THEOREM 5.4. *Suppose that* $(5.1:\mathfrak{H}_0)$ *holds,* $B(x) \geq 0$ *a.e. on* $[a, b]$, *and that for each* $n \times n$ *constant hermitian matrix* Ψ *the solution of* (5.2:a) *fails to exist throughout* $[a, b]$. *Then for* $\Phi(x)$ *an arbitrary monotone non-increasing hermitian matrix function on* $[a, b]$ *the solution of* (5.3) *fails to exist throughout* $[a, b]$.

A more precise result on the solvability of the Riccati matrix integral equation is afforded by the study of the functional

$$J[\eta, \zeta] = \eta^*(a)\chi\eta(a) + I[\eta, \zeta], \tag{5.8}$$

where χ is a given $n \times n$ constant hermitian matrix, and η is restricted to the class $\mathfrak{D}^{*0}$ consisting of those η of $\mathfrak{D}$ satisfying $\eta(b) = 0$; $\mathfrak{H}_+[\mathfrak{D}^{*0}]$ will denote the condition that $J[\eta, \zeta]$ is positive definite on $\mathfrak{D}^{*0}$.

THEOREM 5.5. *If condition* $(5.1:\mathfrak{H})$ *is satisfied, then* $\mathfrak{H}_+[\mathfrak{D}^{*0}]$ *holds if and only if* $B(x) \geq 0$ *a.e. on* $[a, b]$, *and one of the following conditions*:

(i) *if* (U, V) *is the solution of* (2.2_M) *satisfying* $U(a) = E$, $V(a) = \chi$, *then* $U(x)$ *is non-singular on* $[a, b]$;

(ii) *the (necessarily hermitian) solution* $W(x)$ *of*

$$W(x) - \int_a^x F(t, W(t))\, dt = M(x) - M(a) + \chi \tag{5.9}$$

exists on $[a, b]$;

(iii) *there exists an hermitian* χ_0 *such that* $\chi - \chi_0 \geq 0$ *and a non-increasing hermitian* $\Phi(x)$ *such that*

$$W(x) - \int_a^x F(t, W(t))\, dt = M(x) - M(a) + \Phi(x) - \Phi(a) + \chi_0 \tag{5.10}$$

has a solution on $[a, b]$.

As $\mathfrak{D}^0 \subset \mathfrak{D}^{*0}$, it follows from Theorem 5.1 that $\mathfrak{H}_+[\mathfrak{D}^{*0}]$ implies $B(x) \geq 0$ a.e. on $[a, b]$. If (U, V) is the solution of (2.2_M) satisfying $U(a) = E$, $V(a) = \chi$, and for a c on $(a, b]$ we have $U(c)\xi = 0$, then $(\eta(x), \zeta(x)) = (U(x)\xi, V(x)\xi)$ on $[a, c]$, $(\eta(x), \zeta(x)) \equiv (0, 0)$ on $(c, b]$ is such that $\eta \in \mathfrak{D}^{*0}:\zeta$. With the aid of Lemma 5.1 it then follows that $J[\eta, \zeta] = 0$, and hence $\eta \equiv 0$ and $\xi = 0$ by $\mathfrak{H}_+[\mathfrak{D}^{*0}]$, so that U is non-singular on $[a, b]$.

Conversely, if (U, V) is determined by the conditions of (i) then

$$U^*V - V^*U \equiv U^*(a)V(a) - V^*(a)U(a) = 0,$$

and if U is non-singular on $[a, b]$ then for $\eta \in \mathfrak{D}^{*0}:\zeta$ the equation $\eta(x) = U(x)h(x)$ determines an $h \in \mathfrak{A}_n$ such that $h(b) = 0$. Consequently, Lemma 5.1 implies that

$$J[\eta, \zeta] = \int_a^b (\zeta - Vh)^*B(\zeta - Vh)\, dx,$$

and if $B(x) \geq 0$ a.e. then $J[\eta, \zeta] \geq 0$ for $\eta \in \mathfrak{D}^{*0}:\zeta$ and equality sign holds only if $0 = B(\zeta - Vh) = Uh'$ a.e. on $[a, b]$. This latter condition implies that $h(x) \equiv h(b) = 0$, and hence $\eta \equiv 0$ and $B\zeta = 0$ a.e., thus completing the proof that (i) and $B(x) \geq 0$ a.e. on $[a, b]$ imply $\mathfrak{H}_+[\mathfrak{D}^{*0}]$.

In turn, from Theorem 4.1 it follows that (i) is equivalent to (ii). In this connection it is to be noted that if $\eta \in \mathfrak{D}^{*0}:\zeta$ then

$$J[\eta, \zeta] = \hat{\eta}^*(a)[\chi - M(a)]\hat{\eta}(a) + \hat{I}[\hat{\eta}, \hat{\zeta}], \tag{5.11}$$

where $\hat{\eta} = \eta$, $\hat{\zeta} = \zeta - M\eta$ and $\hat{\eta} \in {}^{*0}:\hat{\zeta}$. For brevity, the right-hand member of (5.11) will be referred to as $\hat{J}[\hat{\eta}, \hat{\zeta}]$. Finally, condition (ii) implies (iii) with $\chi_0 = \chi$, $\Phi(x) \equiv 0$, whereas if $B(x) \geq 0$ a.e. on $[a, b]$ and (iii) holds, then the previously established results applied to (5.8) with χ replaced by χ_0 and $M(x)$ replaced by $M(x) + \Phi(x)$, imply

$$J[\eta, \zeta] \geq \eta^*(a)[\chi - \chi_0]\eta(a) - \int_a^b \eta^*[d\Phi]\eta, \tag{5.12}$$

with the equality sign holding only if $\eta \equiv 0$ and $B\zeta = 0$ a.e., and since the right-hand member of (5.12) is non-negative it follows that $\mathfrak{H}_+[\mathfrak{D}^{*0}]$ is satisfied, and hence (ii) also holds.

Corresponding to the deduction of Theorem 5.4 from Theorem 5.1, under the weakened assumption $(5.1:\mathfrak{H}_0)$ one obtains the following comparison theorem through the application of the criteria for the positive definiteness of the functional (5.11) on $\hat{\mathfrak{D}}^{*0}$.

THEOREM 5.6. *Suppose that* $(5.1:\mathfrak{H}_0)$ *holds, and* $B(x) \geq 0$ *a.e. on* $[a, b]$. *If* χ *is such that the solution of* (5.9) *does not exist throughout* $[a, b]$, *then for arbitrary* $\chi_0 \leq \chi$, *and non-increasing hermitian* $\Phi(x)$ *the solution of* (5.10) *does not exist throughout* $[a, b]$; *equivalently, if* χ *is such that the solution of* (5.9) *exists on* $[a, b]$ *then for arbitrary* $\chi_0 \geq \chi$ *and non-decreasing hermitian* $\Phi(x)$ *the solution of* (5.10) *also exists on* $[a, b]$.

THEOREM 5.7. *Suppose that condition* $(5.1:\mathfrak{H}_0)$ *is satisfied, that* $B(x) \geq 0$ *a.e. on* $[a, b]$, *and there exists an* $\hat{\eta} \in \hat{\mathfrak{D}}^{*0}:\hat{\zeta}$ *such that* $\hat{J}[\hat{\eta}, \hat{\zeta}] < 0$. *Then there exist constants* δ_1, δ *such that if* χ_0 *is an hermitian constant matrix and* $\Phi(x)$ *is an hermitian matrix function of class* $\mathfrak{L}_{nn}^{\infty}$ *satisfying* $|\chi - \chi_0 + \Phi(a)| \leq \delta_1$ *and the set* $\{x \mid |\Phi(x)| > \delta\}$ *has measure zero, then the solution of* (5.10) *does not exist on* $[a, b]$.

Let $\Pi(x)$ be a non-decreasing hermitian matrix function on $[a, b]$ for which there exist constants κ_α, $(\alpha = 1, 2, 3, 4)$, such that for arbitrary $\xi(x) \in \mathfrak{C}_n$,

$$(5.13)\quad \begin{aligned}
&\text{(a)}\quad -\xi^*(a)[\chi - M(a)]\xi(a) - \int_a^b \xi^* C\xi\, dx \leq \kappa_1 \int_a^b \xi^*[d\Pi]\xi,\\
&\text{(b)}\quad \int_a^b |B(x)|\cdot|\xi|^2\, dx \leq \kappa_2 \int_a^b \xi^*[d\Pi]\xi,\\
&\text{(c)}\quad \int_a^b |A(x)|\cdot|\xi|^2\, dx \leq \kappa_3 \int_a^b \xi^*[d\Pi]\xi,\\
&\text{(d)}\quad |\xi(a)|^2 \leq \kappa_4 \int_a^b \xi^*[d\Pi]\xi.
\end{aligned}$$

For example, if

$$\nu_0 = \operatorname{Max}\{1, |\chi - M(a)|\}, \quad \nu(x) = \operatorname{Max}\{|B(x)|, |A(x)|, |C(x)|\},$$

and $\Pi(x) = \pi(x)E$, where $\pi(0) = 0$, $\pi(x) = \nu_0 + \int_a^x \nu(s)\, ds$ on $(a, b]$, then the relations (5.13) hold with $\kappa_\alpha = 1$, $(\alpha = 1, 2, 3, 4)$. If $\hat{\mathfrak{D}}^{*0}(\Pi)$ denotes the subclass of $\mathfrak{D}^{*0}$ on which $\int_a^b \eta^*[d\Pi]\eta = 1$, then the hypotheses of the theorem and condition (5.13a) imply that if λ_1 is the infimum of $\hat{J}[\hat{\eta}, \hat{\zeta}]$ on $\hat{\mathfrak{D}}^{*0}(\Pi)$ then $0 < -\lambda_1 \leq \kappa_1$.

Now if $\hat{J}_0[\hat{\eta}, \hat{\zeta}]$ denotes the functional (5.11) with χ replaced by χ_0, $M(x)$ replaced by $M_0(x) = M(x) + \Phi(x)$, and $\hat{\mathfrak{D}}_0^{*0}$ denotes the corresponding class for $\hat{J}_0[\hat{\eta}, \hat{\zeta}]$, then for $\eta \in \hat{\mathfrak{D}}^{*0}(\Pi):\zeta$ we have that $\hat{\eta}_0 = \hat{\eta}$, $\hat{\zeta}_0 = \hat{\zeta} - \Phi\hat{\eta}$ is such that $\hat{\eta}_0 \in \hat{\mathfrak{D}}_0^{*0}:\hat{\zeta}_0$ and

$$\hat{J}_0[\hat{\eta}_0, \hat{\zeta}_0]$$
$$= \hat{J}[\hat{\eta}, \hat{\zeta}] - \hat{\eta}^*(a)[(\chi - \chi_0) + \Phi(a)]\hat{\eta}(a) - 2\operatorname{Re}\int_a^b (\hat{\zeta}^* B\Phi\hat{\eta} + \hat{\eta}^*\Phi^*\hat{A}\hat{\eta})\, dx.$$

Since $B(x) \geq 0$ a.e. on $[a, b]$, it follows that

$$\left|\int_a^b \hat{\zeta}^* B\Phi\hat{\eta}\, dx\right|^2 \leq \left(\int_a^b \hat{\zeta}^* B\hat{\zeta}\, dx\right)\left(\int_a^b \hat{\eta}^*\Phi^* B\Phi\hat{\eta}\, dx\right),$$
$$\leq (\hat{J}[\hat{\eta}, \hat{\zeta}] + \kappa_1)\left(\int_a^b |\Phi|^2 |B|\, |\hat{\eta}|^2\, dx\right).$$

Moreover,

$$\left|\int_a^b \hat{\eta}^*\Phi\hat{A}\hat{\eta}\, dx\right| \leq \int_a^b |\Phi|\, |\hat{A}|\, |\hat{\eta}|^2\, dx.$$

Consequently, if $|\chi - \chi_0 + \Phi(a)| \leq \delta_1$, $|\Phi(x)| \leq \delta$ a.e. on $[a, b]$, and relations (5.13) hold, then in view of $\int_a^b \hat{\eta}^*[d\Pi]\hat{\eta} = 1$ we have

$$\hat{J}_0[\hat{\eta}_0, \hat{\zeta}_0] \leq \hat{J}[\hat{\eta}, \hat{\zeta}] + \delta_1\kappa_4 + 2\delta[\{\kappa_2(\hat{J}[\hat{\eta}, \hat{\zeta}] + \kappa_1)\}^{1/2} + \kappa_3].$$

Since λ_1 is the infimum of $\hat{J}[\hat{\eta}, \hat{\zeta}]$ on $\hat{\mathfrak{D}}^{*0}(\Pi)$, if δ_1 and δ are chosen so that

$$\delta_1 \kappa_4 + 2\delta[\{\kappa_2(\kappa_1 + \lambda_1)\}^{1/2} + \kappa_3] < -\lambda_1,$$

it follows that there exists an $\hat{\eta} \in \hat{\mathfrak{D}}^{*0}:\hat{\zeta}$ for which $\hat{J}_0[\hat{\eta}_0, \hat{\zeta}_0] < 0$. Indeed, by an argument similar to that employed in the proof of the Corollary to Theorem 5.3 one may show that $\lambda = \lambda_1$ is the smallest normal proper value of the boundary value problem

$$\begin{aligned} d\hat{v} - [\hat{C}\hat{u} - \hat{A}^*\hat{v}]\, dx + \lambda[d\Pi]\hat{u} &= 0, \\ -\hat{u}' + \hat{A}\hat{u} + \hat{B}\hat{v} &= 0, \qquad a \leq x \leq b, \\ \hat{u}(b) = 0, \qquad [\chi - M(a)]\hat{u}(a) - \hat{v}(a) &= 0, \end{aligned} \tag{5.14}$$

and that if $(\hat{u}, \hat{v})$ is a corresponding solution with $\hat{u} \not\equiv 0$ on $[a, b]$ then $\hat{J}[\hat{u}, \hat{v}] = \lambda_1 \int_a^b \hat{u}^*[d\Pi]\hat{u}$. Consequently, in view of Theorem 4.1, and the result of Theorem 5.6 applied to $\hat{J}_0[\hat{\eta}_0, \hat{\zeta}_0]$, it follows that the solution $W(x)$ of (5.10) does not exist throughout $[a, b]$.

It is to be noted that Theorems 5.1, 5.2 permit the immediate extension to systems (2.2) of the results of Reid [8, §5] on principal solutions.

6. A special scalar Riccati integral equation

For the scalar integral equation

$$w(x) + \int_0^x w^2(s)\, ds = m(x), \qquad 0 \leq x \leq 1, \tag{6.1}$$

Cameron [2] posed the question as to whether or not it has a solution for almost every choice of $m(x)$ in the class $\mathcal{C}$ of functions continuous on [0, 1], and which vanish at $x = 0$, where "almost every" means all but a set of Wiener measure zero. Woodward [11] answered this question in the negative, by showing that if $m \in \mathcal{C}$ and $|m(x) + 4x| < 0.1$ then (6.1) does not have a solution on [0, 1]. We shall proceed to apply to this particular equation some of the results of the preceding section. Firstly, if $m_0(x) \in \mathcal{C}$ is such that the solution of (6.1) does not exist on [0, 1] for $m = m_0$, then Theorem 5.6 implies that for $\phi(x)$ real-valued monotone non-increasing with $\phi(0) = 0$, and $m = m_0 + \phi$, the solution of (6.1) does not exist on [0, 1].

In the notation of §5, if $m(x)$ is real-valued then $A(x) \equiv C(x) \equiv 0$, $B(x) \equiv 1$, $\hat{A}(x) = m(x)$, $\hat{C}(x) = -m^2(x)$,

$$\hat{J}[\hat{\eta}, \hat{\zeta}] = \int_0^1 (|\hat{\zeta}|^2 - m^2(x)\,|\hat{\eta}|^2)\, dx, \tag{6.2}$$

and $\hat{\mathfrak{D}}^{*0}$ is the class of $\hat{\eta} \in \mathfrak{A}_1$ for which $\hat{\eta}(1) = 0$ and there exists a $\hat{\zeta} \in \mathfrak{L}_1^\infty$ such that $\hat{\eta}' = m(x)\hat{\eta} + \hat{\zeta}(x)$ a.e. on [0, 1]. If $\hat{\eta} \in \hat{\mathfrak{D}}^{*0}:\hat{\zeta}$ then

$$\hat{J}[\hat{\eta}, \hat{\zeta}] = \int_0^1 (|\hat{\eta}' - m\hat{\eta}|^2 - m^2(x)\,|\hat{\eta}|^2)\, dx, \tag{6.3}$$

and, in particular, if $m(x) \in \mathfrak{A}_1$ and $m(0) = 0$ then

$$\hat{J}[\hat{\eta}, \hat{\xi}] = \int_0^1 (|\hat{\eta}'|^2 + m'(x)|\hat{\eta}|^2)\, dx. \tag{6.3'}$$

As the minimum of $\int_0^1 |\hat{\eta}'|^2 dx$ on $\{\hat{\eta} \mid \hat{\eta} \in \hat{\mathfrak{D}}^{*0}, \int_0^1 |\hat{\eta}|^2 dx = 1\}$ is equal to $\pi^2/4$, the smallest proper value of the boundary value problem $u'' + \lambda u = 0$, $u'(0) = 0, u(1) = 0$, it follows that if $m(x) = -rx$, $(r > \pi^2/4)$, and $\Pi(x) = x$, $0 \leq x \leq 1$, then $\lambda_1 = -r + \pi^2/4$. Moreover, inequalities (a), (b), (c) of (5.13) hold with $\kappa_1 = r^2$, $\kappa_2 = 1$, $\kappa_3 = r$, and requirement (d) may be neglected when attention is limited to functions $m(x)$ which vanish at $x = 0$. From Theorem 5.7 it then follows that if $\phi \in \mathfrak{L}_1^\infty$ and $\phi(0) = 0$ then the solution of the integral equation

$$w(x) + \int_0^x w^2(s)\, ds = -rx + \phi(x) \tag{6.4}$$

does not exist on [0, 1] if

$$|\phi(x)| \leq (4r - \pi^2)/(4[(4r^2 - 4r + \pi^2)^{1/2} + 2r]).$$

In particular, for $r = 4$ the above bound reduces to 0.098, which is almost identical with the bound 0.1 derived by Woodward [11].

Actually, with a little additional attention one may obtain a greatly improved bound. For $m(x) = -rx + \phi(x)$, with $\phi \in \mathfrak{L}_1^\infty$, $\phi(0) = 0$, $|\phi(x)| \leq \delta$ a.e. on [0, 1], one has

$$\hat{J}[\hat{\eta}, \hat{\xi}] \leq \int_0^1 ([1 + \delta k]|\hat{\eta}'|^2 + [-r + \delta/k]|\hat{\eta}|^2)\, dx,$$

for arbitrary $k > 0$. Consequently, there exists an $\hat{\eta} \in \hat{\mathfrak{D}}^{*0}:\hat{\xi}$ with $\hat{J}[\hat{\eta}, \hat{\xi}] \leq 0$ if

$$[-r + \delta/k]/(1 + \delta k] \leq -\pi^2/4,$$

that is, if $\delta \leq (4r - \pi^2)/(k\pi^2 + 4/k)$, and the optimal bound $\delta \leq (4r - \pi^2)/(4\pi)$ is obtained for $k = 2/\pi$. That is, if $\phi(x) \in \mathfrak{L}_1^\infty$, $\phi(0) = 0$, $|\phi(x)| \leq (4r - \pi^2)/(4\pi)$ a.e., then the solution of the integral equation (6.4) does not exist on [0, 1]. For $r = 4$ this estimate provides the bound $|\phi(x)| \leq 0.488$, in place of the bound $|\phi(x)| \leq 0.1$ of Woodward [11].

7. Further results on Riccati matrix integral equations

In this section there will be presented two theorems which extend to generalized differential systems (2.2) results of Hestenes [3], Bliss [1, §87], and Reid [6, §§8–10] on accessory systems for variational problems of Bolza type.

THEOREM 7.1. *If condition* (5.1:$\mathfrak{H}$) *is satisfied, then* $\mathfrak{H}_+[\mathfrak{D}]$ *holds if and only if* $B(x) \geq 0$ *a.e. on* $[a, b]$ *and one of the following:*

(i) *there exists a solution* (U, V) *of* (2.2_M) *such that* U *is non-singular and* $U^*V - V^*U \equiv 0$ *on* $[a, b]$, *while* $-U^*(a)V(a) > 0$ *and* $U^*(b)V(b) > 0$;

(ii) *there exists an hermitian matrix* Ψ *such that on* $[a, b]$ *the integral equation* (5.2:a) *has a (necessarily hermitian) solution* $W(x)$ *satisfying* $-W(a) > 0$ *and* $W(b) > 0$.

As $\mathfrak{H}_+[\mathfrak{D}]$ implies $\mathfrak{H}_+[\mathfrak{D}^0]$, it follows that $B(x) \geq 0$ a.e. on $[a, b]$ is a consequence of $\mathfrak{H}_+[\mathfrak{D}]$. Moreover, for $\mathfrak{D}^{*0}$ defined as in §5, $\mathfrak{H}_+[\mathfrak{D}]$ implies $\mathfrak{H}_+[\mathfrak{D}^{*0}]$, and if (U_a, V_a) is a solution of (2.2_M) such that $V_a(a) = 0$, $U_a(a)$ non-singular, then Theorem 5.6 implies that $U_a(x)$ is non-singular on $[a, b]$. Correspondingly, if (U_b, V_b) is a solution of (2.2_M) for which $V_b(b) = 0$ and $U_b(b)$ is non-singular, then $U_b(x)$ is non-singular on $[a, b]$. Now $U_a^* V_a - V_a^* U_a \equiv 0$, $U_b^* V_b - V_b^* U_b \equiv 0$, and the constant matrix P such that $U_a^* V_b - V_a^* U_b \equiv P$ is non-singular. Indeed, if $P\xi = 0$ then

$$(u(x), v(x)) = (U_b(x)\xi, V_b(x)\xi)$$

is a solution of (2.2) such that $U_a^*(b)v(b) - V_a^*(b)u(b) = 0$, and as the $n \times 2n$ matrix $\| U_a^*(b) \quad -V_a^*(b) \|$ is of rank n and

$$U_a^*(b)V_a(b) - V_a^*(b)U_a(b) = 0$$

there exists a ξ_0 such that

$$(u_0(x), v_0(x)) = (U_a(x)\xi_0, V_a(x)\xi_0)$$

is a solution of (2.2) satisfying $u_0(b) = u(b)$, $v_0(b) = v(b)$, and thus $(u_0(x), v_0(x)) \equiv (u(x), v(x))$. In particular, $v(a) = v_0(a) = 0$ and $v(b) = 0$, so that $I[u, v] = u^*v \,|_a^b = 0$. Hence $\mathfrak{H}_+[\mathfrak{D}]$ implies $0 \equiv u(x) = U_b(x)\xi$, so that $\xi = 0$ and P is non-singular. If the values of $U_a(a)$ and $U_b(b)$ are so chosen that $P = -E$, and $(U, V) = (U_a + U_b, V_a + V_b)$, it may be verified directly that $U^*V - V^*U \equiv 0$. Moreover, if $c \in [a, b]$ and $U(c)\xi = 0$, then $(\eta, \zeta) = (U_a\xi, V_a\xi)$ on $[a, c]$, $(\eta, \zeta) = (-U_b\xi, -V_b\xi)$ on $(c, b]$, is such that $\eta \in \mathfrak{D}:\zeta$ and application of Lemma 5.1 to the individual intervals $[a, c]$ and $[c, b]$, with the (U, V) of the lemma equal to (U_a, V_a) and (U_b, V_b), respectively, yields

$$\begin{aligned} I[\eta, \zeta] &= \xi^*V_a^*(c)U_a(c)\xi - \xi^*U_b^*(c)V_b(c)\xi, \\ &= -\xi^*V_a(c)U_b(c)\xi + \xi^*U_a^*(c)V_b(c)\xi = -\xi^*\xi; \end{aligned}$$

therefore, $\mathfrak{H}_+[\mathfrak{D}]$ implies $\xi = 0$ and hence the non-singularity of $U(c)$. Moreover,

$$\begin{aligned} U^*(a)V(a) &= U_a^*(a)V_b(a) + U_b^*(a)V_b(a) = -E + U_b^*(a)V_b(a), \\ U^*(b)V(b) &= U_a^*(b)V_a(b) + U_b^*(b)V_a(b) = U_a^*(b)V_a(b) + E, \end{aligned}$$

and hence

$$\begin{aligned} \xi^*U^*(a)V(a)\xi &= -\xi^*\xi - I[U_b\xi, V_b\xi] \leq -\xi^*\xi, \\ \xi^*U^*(b)V(b)\xi &= I[U_a\xi, V_a\xi] + \xi^*\xi \geq \xi^*\xi, \end{aligned}$$

for arbitrary ξ, so that $-U^*(a)V(a) > 0$ and $U^*(b)V(b) > 0$.

Conversely, if $B(x) \geq 0$ a.e. on $[a, b]$ and (i) holds, then for $\eta \in \mathfrak{D}:\zeta$ and h defined by $\eta = Uh$ it follows from Lemma 5.1 that

$$\begin{aligned} I[\eta, \zeta] &= \int_a^b (\zeta - Vh)^* B(\zeta - Vh)\,dx + h^*U^*Vh|_a^b, \\ &\geq \int_a^b (\zeta - Vh)^* B(\zeta - Vh)\,dx, \end{aligned} \tag{7.1}$$

with the equality sign in (7.1) holding only if $h(a) = 0 = h(b)$, that is, if and only if $\eta \in \mathfrak{D}^0:\zeta$, in which case Theorem 5.1 implies that $I[\eta, \zeta] \geq 0$, with equality only if $\eta \equiv 0$ and $B\zeta = 0$ a.e. on $[a, b]$.

Finally, (ii) is equivalent to (i) in view of Theorem 4.1 and the fact that if $W(x)$ is a solution of (5.2:a) on $[a, b]$, and (U, V) is a corresponding solution of (2.2_M) such that $W = VU^{-1}$, then $U^*V - V^*U \equiv U^*[W - W^*]U$.

Let θ_a and θ_b be $n \times r_a$ and $n \times r_b$ matrices with $0 \leq r_a \leq n$, $0 \leq r_b \leq n$, it being understood that if either $r_a = 0$ or $r_b = 0$ the respective matrix θ_a or θ_b does not occur, and that if $0 < r_a \leq n$ or $0 < r_b \leq n$ then the corresponding θ_a or θ_b has rank r_a or r_b. For M_a and M_b given $n \times n$ hermitian matrices we shall now consider the condition $\mathfrak{H}_+[\mathfrak{D}^\theta]$ that the functional

$$\eta^*(a)M_a\eta(a) + \eta^*(b)M_b\eta(b) + I[\eta, \zeta] \tag{7.2}$$

be positive definite on the class $\mathfrak{D}^\theta$ consisting of (η, ζ) such that $\eta \in \mathfrak{D}:\zeta$ and $\theta_a^*\,\eta(a) = 0$, $\theta_b^*\,\eta(b) = 0$. In particular, $\mathfrak{D}^\theta = \mathfrak{D}^0$ if $\theta_a = \theta_b = E_n$, $\mathfrak{D}^\theta = \mathfrak{D}$ if θ_a and θ_b are non-existent, and $\mathfrak{D}^\theta = \mathfrak{D}^{*0}$ if $\theta_b = E_n$ and θ_a is non-existent.

THEOREM 7.2. *If condition $(5.1{:}\mathfrak{H})$ is satisfied, then $\mathfrak{H}_+[\mathfrak{D}^\theta]$ holds if and only if $B(x) \geq 0$ a.e. on $[a, b]$ and there exists an hermitian matrix Ψ such that on $[a, b]$ the integral equation (5.2:a) has a (necessarily hermitian) solution $W(x)$ for which there is an associated constant k satisfying*

$$M_a + k\,\theta_a\,\theta_a^* - W(a) > 0, \qquad M_b + k\,\theta_b\,\theta_b^* + W(b) > 0.$$

Since $\mathfrak{H}_+[\mathfrak{D}^\theta]$ implies $\mathfrak{H}_+[\mathfrak{D}^0]$, condition $\mathfrak{H}_+[\mathfrak{D}^\theta]$ implies that $x = b$ is not conjugate to $x = a$, and if the order of abnormality of (2.2) on $[a, b]$ is equal to d then independent solutions $(u^{(j)}(x), v^{(j)}(x))$, $(j = 1, \cdots, 2n)$ of (2.2) may be chosen so that $u^{(\alpha)}(x) \equiv 0$ on $[a, b]$ for $\alpha = n - d + 1, \cdots, 2n$, and the $2n \times (2n - d)$ matrix

$$\left\| \begin{matrix} u_i^{(j)}(a) \\ u_i^{(j)}(b) \end{matrix} \right\|, \qquad (i = 1, \cdots, n; j = 1, \cdots, 2n - d)$$

is of rank $2n - d$. In view of Theorem 5.2, if $\eta \in \mathfrak{D}:\zeta$ there is a solution

$$(u_\mu(x), v_\mu(x)) = \left(\textstyle\sum_{j=1}^{2n-d} \mu_j u^{(j)}(x),\ \sum_{j=1}^{2n-d} \mu_j v^{(j)}(x)\right)$$

of (2.2) such that

$$(u_\mu(a), u_\mu(b)) = (\eta(a), \eta(b)) \quad \textit{and} \quad I[\eta, \zeta] \geq I[u_\mu, v_\mu],$$

with equality if and only if $\eta - u_\mu \equiv 0$ and $B(\zeta - v_\mu) = 0$ a.e. on $[a, b]$. As in the special instance of the accessory system for a variational problem of Bolza type, (see, for example, Reid [6, §9]), application of a theorem on pairs of hermitian forms and Theorem 5.2 yields the result that there is a real constant k such that the functional

$$(7.3) \quad \eta^*(a)[M_a + k\,\theta_a\,\theta_a^*]\eta(a) + \eta^*(b)[M_b + k\,\theta_b\,\theta_b^*]\eta(b) + I[\eta, \zeta]$$

is positive definite on $\mathfrak{D}$. Conversely, if (7.3) is positive definite on $\mathfrak{D}$ then condition $\mathfrak{H}_+[\mathfrak{D}^\theta]$ holds, so that the considered problem is reduced to the positive definiteness of (7.3) on $\mathfrak{D}$. Now (7.3) may also be written as a functional $I_0[\eta, \zeta]$ of the same form as $I[\eta, \zeta]$, with $M(x)$ replaced by the $M_0(x)$ defined as: $M_0(a) = M(a) - M_a$, $M_0(x) \equiv M(x)$ on (a, b), $M_0(b) = M(b) + M_b$. It may be verified readily that if (U_0, V_0) is a solution of the matrix system (2.2_M) associated with $I_0[\eta, \zeta]$ then (U, V) defined as:

$$\begin{aligned} U(x) &\equiv U_0(x), \\ V(a) &= V_0(a) + [M_a + k\,\theta_a\,\theta_a^*]U_0(a), \\ V(x) &\equiv V_0(x) \quad \text{on } (a, b), \\ V(b) &= V_0(b) - [M_b + k\theta_b\,\theta_b^*]U_0(b) \end{aligned}$$

is a solution of the system (2.2_M) for the original $I[\eta, \zeta]$, and consequently Theorem 7.2 follows from the result of Theorem 7.1 for the functional $I_0[\eta, \zeta]$.

BIBLIOGRAPHY

1. G. A. BLISS, *Lectures on the calculus of variations*, Chicago, Univ. of Chicago Press, 1946.
2. R. H. CAMERON, *Non-linear Volterra functional equations and linear parabolic differential systems*, J. d'Analyse Math., vol. 5 (1956–57), pp. 136–181.
3. W. FELLER, *Generalized second order differential operators and their lateral conditions*, Illinois J. Math., vol. 1 (1957), pp. 459–504.
4. M. R. HESTENES, *Sufficient conditions for the problem of Bolza in the calculus of variations*, Trans. Amer. Math. Soc., vol. 36 (1934), pp. 793–818.
5. J. J. LEVIN, *On the matrix Riccati equation*, Proc. Amer. Math. Soc., vol. 10 (1959), pp. 519–524.
6. W. T. REID, *Expansion methods for the isoperimetric problem of Bolza in non-parametric form*, Amer. J. Math., vol. 61 (1949), pp. 946–975.
7. ———, *Generalized linear differential systems*, J. Math. Mech., vol. 8 (1959), pp. 705–726.
8. ———, *Principal solutions of non-oscillatory linear differential systems*, J. Math. Analysis and Appl., vol. 9 (1964), pp. 397–423.
9. F. RIESZ AND B. SZ.-NAGY, *Functional Analysis*, New York, F. Ungar Publishing Co., 1955.
10. B. SZ.-NAGY, *Vibrations d'une corde non-homogène*, Bull. Soc. Math. France, vol. 75 (1947), pp. 193–209.
11. D. A. WOODWARD, *On a special integral equation*, Proc. Amer. Math. Soc., vol. 10 (1959), pp. 853–854.

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