CONNEXION PRESERVING SPRAY MAPS[1]

BY

N. J. HICKS

This study is concerned primarily with finding necessary and sufficient conditions for a spray map to be connexion preserving. We now define these terms. Let M and $'M$ be connected C^∞ manifolds with connexions[2] D and $'D$, respectively (see references [4] or [7] for undefined terms). Let f be a C^∞ map of M into $'M$ with Jacobian f_*. If X is an element of the tangent space M_m and Y is an arbitrary C^∞ vector field on a neighborhood of m, then f is *connexion preserving in the direction* X, or *connexion preserving with respect to* X, if

$$(1) \qquad f_*(D_X Y) = 'D_{f_* X}(f_* Y),$$

where the right side is computed as in [7, p. 142]. If f is connexion preserving with respect to all vectors X at all points of M, then we say f is a *connexion preserving map*. It is easy to show that f is connexion preserving if and only if f_* commutes with parallel translation along C^∞ curves. To obtain a spray map, we choose points m in M and $'m$ in $'M$, let U be a normal neighborhood of m, let F be a linear map of M_m into $'M_{'m}$, and define

$$(2) \qquad f = '\exp_{'m} \circ F \circ (\exp_m)^{-1}$$

on U. Thus in (2), f maps geodesics in U emanating from m into geodesics in $'M$ emanating from $'m$ via the association of initial tangent vectors provided by F. Any map of form (2) is called a *spray map*, and sometimes we may call it a *spray map from m*.

The local theorems which provide conditions implying a spray map is connexion preserving are found in Section 2. In Section 3, we use the method of Ambrose in [1] to obtain a global theorem. Among the applications in Section 4, we obtain a differential geometric proof of a classical theorem concerning homomorphisms of Lie groups. Finally in Section 5, we generalize a local result of E. Cartan in [2], and then follow a suggestion of R. Hermann to obtain an existence theorem for geodesic submanifolds of an affine connexion.

This work is a generalization of [6] where a similar study was made under the restriction that M and $'M$ have the same dimension. Besides extending the previous results, we believe the methods used in Sections 1 and 2 illustrate more clearly the connection between hypotheses and conclusions of the analytical parts of the theorems involved. In particular, we exhibit clearly the inference chain from hypotheses about curvature and torsion to Jacobi

Received August 4, 1965.

[1] This research was supported in part by the National Science Foundation.

[2] All connexions in this study will be C^∞.

fields to the Jacobian of the exponential to conclusions about parallel translation.

1. Jacobi fields for affine connexions

In this section let M be a C^∞ manifold with connexion D, curvature R, and torsion Tor. Recall, for C^∞ vector fields X, Y, and Z, on M,

$$\operatorname{Tor}(X, Y) = D_X Y - D_Y X - [X, Y] \tag{3}$$

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z. \tag{4}$$

Let σ be a geodesic in M with tangent field T. Following [8], we define a *Jacobi field* W *along* σ to be a C^∞ field W on σ such that

$$D_T^2 W = R(T, W)T + D_T \operatorname{Tor}(T, W). \tag{5}$$

To motivate this definition suppose $\alpha(t, w)$ is a 1-parameter family of geodesics in M with $\alpha_0 = \sigma$, i.e. $\alpha_0(t) = \alpha(t, 0) = \sigma(t)$, (see [7, p. 144]). Let $T = \alpha_*(\partial/\partial t)$ and $W = \alpha_*(\partial/\partial w)$. Then from Section 10.1 of [7], $D_T T = 0$, $[T, W] = 0$, and $\operatorname{Tor}(T, W) = D_T W - D_W T$. Thus

$$R(T, W)T = D_T D_W T = D_T(D_T W - \operatorname{Tor}(T, W)),$$

and W satisfies (5).

Let $e_1, \cdots, e_n$ be a set of linearly independent parallel fields along σ with $T = e_n$. Any C^∞ field W on σ can be represented uniquely in the form $W = \sum_{i=1}^n f_i e_i$ where f_i are C^∞ real-valued functions on the domain of σ. The torsion and curvature tensors induce C^∞ real-valued functions T_{ijk} and R_{ijks} on the domain of σ by letting

$$\operatorname{Tor}(e_i, e_j) = \textstyle\sum_{k=1}^n T_{ijk} e_k \tag{6}$$

$$R(e_i, e_j)e_k = \textstyle\sum_{s=1}^n R_{ijks} e_s . \tag{7}$$

Using this notation we prove the following proposition which is exactly analogous to the Riemannian case.

PROPOSITION 1. *A C^∞ field $W = \sum_1^n f_i e_i$ is Jacobi along σ if and only if*

$$f_i'' - \textstyle\sum_j T_{nji} f_j' - \sum_j (R_{njni} + T_{nji}')f_j = 0 \tag{8}$$

for $i = 1, \cdots, n$, where the summation index j goes from 1 to n, and the prime superscript indicates differentiation. A Jacobi field W is uniquely determined by the vectors W_m and $(D_T W)_m$ at one point m on σ. The set of all Jacobi fields along σ is a real vector space of dimension $2n$. The set of Jacobi fields along σ that vanish at one particular point is a real vector space of dimension n.

Proof. Since the e_i are parallel, we have $D_T W = \sum f_i' e_i$ and $D_T^2 W = \sum f_i'' e_i$. Since $T = e_n$, equation (5) becomes

$$\begin{aligned}\textstyle\sum_i f_i'' e_i &= R(e_n\,, \textstyle\sum_i f_i\, e_i)e_n + D_T \operatorname{Tor}\,(e_n\,, \textstyle\sum_i f_i\, e_i)\\ &= \textstyle\sum_{i,s} f_i\, R_{nins}\, e_s + D_T(\textstyle\sum_{i,s} f_i\, T_{nis}\, e_s)\\ &= \textstyle\sum_{j,i}(f_j\, R_{njni} + f_j' T_{nji} + f_j\, T'_{nji})e_i\,.\end{aligned}$$

Equating coefficients of e_i yields (8).

The existence and uniqueness theorems concerning solutions to second order linear ordinary differential equations when applied to (8) will prove the remaining sentences in the proposition.

In order to connect Jacobi fields with the exponential map in the usual way, we fix some notation. For m in M and A in M_m, let $\bar{A}$ denote the unique "constant" vector field induced on M_m by A (see [7, p. 132 and p. 145]).

PROPOSITION 2. *Let T and A be any vectors in M_m. Let*

$$Q = [(t, w) \text{ in } R^2 : \exp_m \text{ is defined on } t(T + wA)],$$

which is an open set in R^2. Let $\alpha : Q \rightarrow M$ by

$$\alpha(t, w) = \exp_m t(T + wA).$$

Then α is a one-parameter family of geodesics and $W = (\exp_m)_*(t\bar{A})$ is a Jacobi field along each geodesic such that $W(0, w) = 0$ and $D_T\, W(t, 0) = exp_*\, \bar{A}_0 = A$.

Proof. The definition of the exponential map implies α is a one-parameter family of geodesics which in turn implies W is Jacobi. Finally,

$$D_T\, W(t, 0) = D_T(t \exp_* \bar{A}_{tT}) = \exp_* \bar{A}_{tT} + tD_T(\exp_* \bar{A}),$$

and evaluating at $t = 0$ shows $D_T\, W(t, 0) = A$.

We remark that conjugate points may be defined as singularities of the exponential map or in terms of Jacobi fields, as in the Riemannian case. Indeed, Theorems 3, 4, and 5, in [7, p. 145], are true for general affine connexions and their proofs are identical to the Riemannian proofs given in [7].

2. Connexion preserving local maps

As in the introduction, we let M and $'M$ be C^∞ manifolds with connexions D and $'D$, respectively. In general, the prime superscript before a letter will denote an object associated with $'M$. This notation is fixed throughout this section.

DEFINITION. Let σ be a geodesic in M and let f be a C^∞ map from a neighborhood of the image of σ into $'M$. The map f is *curvature and torsion parallel invariant along σ* if the following condition is satisfied: Choose any real number b in the domain of σ and any base $e_1, \cdots, e_n$ of M_m where $m = \sigma(b)$. Let $P_t\, e_i$ denote the parallel translate of e_i along σ from m to $\sigma(t)$. Let $'e_i = (f_*)_m\, e_i$ and let $'P_t{'e_i}$ denote the parallel translate of $'e_i$ along $'\sigma = f \circ \sigma$

from $'m = '\sigma(b)$ to $'\sigma(t)$, for $i = 1, \cdots, n$. Define C^∞ functions T_{ijk} and R_{ijks} on the domain of σ by letting

$$\text{Tor}\,(P_t\, e_i\,,\, P_t\, e_j) = \textstyle\sum_{k=1}^n T_{ijk}(t)(P_t\, e_k) \tag{9}$$

$$R(P_t\, e_i\,,\, P_t\, e_j)P_t\, e_k = \textstyle\sum_{s=1}^n R_{ijks}(t)(P_t\, e_s). \tag{10}$$

The condition we impose is that the following equations hold on the domain of σ (and $'\sigma$):

$$'\text{Tor}\,('P_t{}'e_i\,,\, 'P_t{}'e_j) = \textstyle\sum_{k=1}^n T_{ijk}(t)('P_t{}'e_k) \tag{11}$$

$$'R('P_t{}'e_i\,,\, 'P_t{}'e_j)'P_t{}'e_k = \textstyle\sum_{s=1}^n R_{ijks}(t)('P_t{}'e_s). \tag{12}$$

Some remarks are in order. Notice $'e_1, \cdots, 'e_n$ need not be independent. The following lemma shows that the definition is independent of the base $e_1, \cdots, e_n$ at m, and Theorem 1 will show the definition is independent of the point m when f is geodesic preserving.

LEMMA. *The above definition is independent of the base $e_1, \cdots, e_n$ at m.*

Proof. Let $v_1, \cdots, v_n$ be any base at m and let (a_{ij}) be the non-singular n by n matrix of real numbers such that $v_j = \sum_{i=1}^n a_{ij}\, e_i$. Hence

$$'v_j = f_*(v_j) = \textstyle\sum_{i=1}^n a_{ij}{}'e_i\,.$$

Letting $\text{Tor}\,(P_t\, v_i\,,\, P_t\, v_j) = \sum_{k=1}^n S_{ijk}(t)(P_t\, v_k)$, we find

$$S_{ijk}(t) = \textstyle\sum_{r,s,u=1}^n a_{ri}\, a_{sj}\, a_{ku}^{-1}\, T_{rsu}(t),$$

where a_{ku}^{-1} denotes the k, u^{th} entry in the inverse matrix of the matrix (a_{ij}). We now check the condition (11) for the base $v_1, \cdots, v_n$,

$$\begin{aligned} '\text{Tor}\,('P_t{}'v_i\,,\, 'P_t{}'v_j) &= \textstyle\sum_{r,s} a_{ri}\, a_{sj}{}'\text{Tor}\,('P_t{}'e_r\,,\, 'P_t{}'e_s) \\ &= \textstyle\sum_{r,s,u} a_{ri}\, a_{sj}\, T_{rsu}(t)'P_t{}'e_u \\ &= \textstyle\sum_{u,k} S_{ijk}(t)a_{uk}('P_t{}'e_u) \\ &= \textstyle\sum_k S_{ijk}(t)('P_t{}'v_k). \end{aligned}$$

The computation needed to check (12) for $v_1, \cdots, v_n$ is similar.

THEOREM 1. *If f is geodesic preserving and also curvature and torsion parallel invariant along a geodesic σ, then f is connexion preserving in the direction of σ and f_* commutes with parallel translation along σ.*

Proof. In a normal neighborhood U of any point m on σ we can write

$$f = '\exp_{'m} \circ (f_*)_m \circ (\exp_m)^{-1} \tag{13}$$

since f is geodesic preserving. We restrict the domain of σ so σ lies in U and show f_* commutes with parallel translation along σ through this domain. A simple finite repetition of the argument will show f_* commutes with parallel translation along σ between any two points.

By Proposition 2, the fields $W_i = (\exp_m)_*(t\bar{e}_i)$ and the fields $'W_i = f_* W_i = ('\exp_{'m})_*(t'\bar{e}_i)$ are Jacobi along σ and $'\sigma = f \circ \sigma$, respectively, for $i = 1, \cdots, n$. Define C^∞ real-valued functions $g_{ij}(t)$ on the domain of σ by

$$W_j(t) = \sum_{i=1}^n g_{ij}(t)(P_t e_i). \tag{14}$$

Since there are no points $\sigma(t)$ conjugate to m in the normal neighborhood U, the matrix $(g_{ij}(t))$ is non-singular for $t \neq b(\sigma(b) = m)$. Assuming e_n is the tangent to σ at m, the Jacobi equations (8) imply for r and $s = 1, \cdots, n$ that

$$g''_{rs} - \sum_j T_{njr} g'_{js} - \sum_j (R_{njnr} + T'_{njr}) g_{js} = 0, \tag{15}$$

where

$$g_{rs}(b) = 0 \quad \text{and} \quad g'_{rs}(b) = \delta_{rs}. \tag{16}$$

Consider the fields $Y_j(t) = \sum_{i=1}^n g_{ij}(t)'P_t'e_i$ along $'\sigma$ for $j = 1, \cdots, n$. We show each Y_j is Jacobi along $'\sigma$ for

$$\begin{aligned} 'D^2_{'T} Y_s &= \sum_{r=1}^n g''_{rs}(t)('P_t'e_r) \\ &= \sum_{r,j} [T_{njr} g'_{js} + (R_{njnr} + T'_{njr}) g_{js}]('P_t'e_r) \\ &= 'R('T, Y_s)'T + 'D_{'T}'\mathrm{Tor}\,('T, Y_s) \end{aligned}$$

by (15), (11), and (12). But $Y_j(b) = 0$ and $('D_{'T} Y_j)(b) = 'e_j$, hence $Y_j = 'W_j$ for each j by the uniqueness of Jacobi fields (Proposition 1). Thus

$$f_*\left(\sum_{i=1}^n g_{ij}(t) P_t e_i\right) = Y_j = \sum_{i=1}^n g_{ij}(t)('P_t'e_i) \tag{17}$$

and since f_* is linear and $g_{ij}(t)$ is non-singular for $t \neq b$,

$$f_*(P_t e_i) = 'P_t'e_i = 'P_t(f_* e_i). \tag{18}$$

This proves the theorem.

COROLLARY 1. *Let M and $'M$ be C^∞ manifolds with connexions D and $'D$, respectively. A C^∞ map f of M into $'M$ is connexion preserving if and only if it is geodesic preserving and curvature and torsion parallel invariant along all geodesics.*

In the proof of Theorem 1 we only used the fact that f was geodesic preserving to obtain (13). Hence we can sharpen the result by making the assumption that (13) holds, i.e. that f is a spray map.

COROLLARY 2. *Let f be a spray map from a normal neighborhood U of m in M into $'M$. Let σ be a geodesic through m such that f is curvature and torsion parallel invariant along σ. Then f_* commutes with parallel translation along σ.*

We now investigate conditions under which a spray map is connexion preserving. The case in which $(f_*)_m$ is onto is treated in Theorem 2 while Theorem 3 covers the general case.

First some notation. If Z is a vector in M_m, let $\tilde{Z}$ be the C^∞ field induced on each normal neighborhood of m by parallel translating Z along geodesics emanating from m. If A is a subspace of M_m, let $\tilde{A}$ be the distribution (in the sense of Chevalley [3]) on each normal neighborhood of m generated by all fields $\tilde{Z}$ for Z in A. If A and B are C^∞ distributions with a common domain let $D_A B$ be the set of all fields $D_Z X$ for Z in A and X in B, and define sets $R(A, B)B$ and Tor (A, B) analogously.

THEOREM 2. *Let f be a C^∞ map from a normal neighborhood U of m in M into $'M$ such that $f(U)$ is contained in a normal neighborhood of $'m = f(m)$, and let K denote the kernel of f_* at m. Then f is connexion preserving onto a normal neighborhood of $'m$ if and only if the following three conditions hold:*

(a) *f is a spray map from m,*

(b) *f is curvature and torsion parallel invariant along all geodesics emanating from m and contained in U,*

(c) *there exists a complement N to K (i.e. N is a subspace of M_m with $M_m = N + K$ and $N \cap K = \emptyset$) such that $(f_*)_m$ maps N onto $'M_{'m}$ with $D_{\tilde{N}}(\tilde{K}) \subset \tilde{K}$, $R(\tilde{K}, \tilde{N})\tilde{N} \subset \tilde{K}$,* Tor $(\tilde{K}, \tilde{N}) \subset \tilde{K}$, *and $D_X \tilde{Y}$ in $\tilde{K}$ if Y in N and X in $\tilde{K}$, at all points of U.*

Proof. If f is connexion preserving then (a) and (b) follow from Corollary 1. Since f_* commutes with parallel translation, the kernel of $(f_*)_p$ is $\tilde{K}_p$ for any p in U. If X in $\tilde{K}$ and Y in $\tilde{N}$, where N is any complement to K, then $f_*(D_{Y_p} X) = {'D}_{f_* Y_p}(f_* X) = 0$ since $f_* X = 0$. Hence $D_Y X$ is in $\tilde{K}$.

If Y is in N, let $'Y = (f_*)_m Y$ and by Theorem 1, $'\tilde{Y}_{f(p)} = (f_*)_p \tilde{Y}_p$ since parallel translation commutes with f_*. Thus if X in $\tilde{K}_p$ then $f_*(D_X \tilde{Y}) = {'D}_{f_* X}{'\tilde{Y}} = 0$ since $f_* X = 0$ and $f_* \tilde{Y}$ *is imbedded in the field* $'\tilde{Y}$. (To facilitate understanding, we include the following example. Let $f : R^3 \to R^2$ by $f(a, b, c) = (a, b)$. Let $Y = z(\partial/\partial x)$ and $X = \partial/\partial z$. Then $D_X Y = \partial/\partial x$ (with the standard connexion on each R^n), and $f_*(D_X Y) = \partial/\partial x$. The map f is connexion preserving since f_* clearly commutes with parallel translation. Indeed, by equation (1), p. 142, in [7],

$$D_{f_* X}(f_* Y) = [(\partial/\partial z)z](\partial/\partial x) + zD_{f_* X}(\partial)\partial x) = \partial/\partial x$$

since $f_* X = 0$. Thus $D_X Y$ is *not* in the kernel of f_* even though X is; however, Y is not of the type $\tilde{Y}$ and $f_* Y$ depends on z.)

Furthermore, if f is connexion preserving then

$$f_*(R(X, Y)Z) = {'R}(f_* X, f_* Y)f_* Z$$

$$\text{and} \quad f_* \operatorname{Tor}(X, Y) = {'\operatorname{Tor}}(f_* X, f_* Y).$$

Hence if X in $\tilde{K}$ then $f_* X = 0$, and thus $R(X, Y)Z$ and Tor (X, Y) are in $\tilde{K}$ for all Y and Z. Since f is a spray map onto a neighborhood of $'m$, $(f_*)_m$ maps any complement N to K onto $'M_{'m}$.

Assume now f satisfies (a), (b), and (c). Since $(f_*)_m$ maps onto $'M_{'m}$ it is clear that f maps onto a normal neighborhood of $'m$.

Let $e_1, \cdots, e_{'n}$ be a base for N, where $'n$ is the dimension of $'M$, and let $e_{'n+1}, \cdots, e_n$ be a base for K. We write e_i for the fields $\tilde{e}_i$ on U for $i = 1, \cdots, n$. Let $T = \sum_{i=1}^n a_i e_i$ be a C^∞ "radial" field on $V = U - [m]$ such that T is a tangent field on V to geodesics emanating from m. To be more explicit, let $z_1, \cdots, z_n$ be the dual base of linear functionals to $e_1, \cdots, e_n$, then let

$$R = \left(\sum_1^n z_j^2\right)^{1/2}\left(\sum_1^n z_i\, \partial/\partial z_i\right)$$

on $M_m - [0]$ and $T = (\exp_m)_* R$ on V. By definition then we have $D_T T = 0$, $D_T e_i = 0$, $Ta_i = 0$, and $e_r a_i$ are known C^∞ functions on V.

Let Γ^i_{jk} denote the connexion functions on U associated with the base field $e_1, \cdots, e_n$; so

$$D_{e_j} e_i = \sum_{k=1}^n \Gamma^k_{ij} e_k . \tag{19}$$

We now develop a system of first order ordinary differential equations, (22), that determine the functions Γ^k_{ij} along a geodesic σ emanating from m. Define C^∞ functions R_{ijks} and T_{ijk} on U by equations (6) and (7). Then by (3), on V we have

$$\begin{aligned}[T, e_i] &= D_T e_i - D_{e_i} T - \mathrm{Tor}\,(T, e_i)\\ &= -D_{e_i}\left(\sum_k a_k e_k\right) - \sum_{k,j} a_k T_{kij} e_j\\ &= -\sum_k (e_i a_k) e_k - \sum_{j,k} a_j \Gamma^k_{ji} e_k - \sum_{j,k} a_j T_{jik} e_k\end{aligned} \tag{20}$$

where sums go from 1 to n. By (4) and (20),

$$\begin{aligned}R(T, e_i)e_j &= D_T D_{e_i} e_j - D_{e_i} D_T e_j - D_{[T,e_i]} e_j\\ &= \sum_{k,s} a_k R_{kijs} e_s\\ &= \sum_s (T\Gamma^s_{ji}) e_s + \sum_{k,r,s}(e_i a_k + a_r \Gamma^k_{ri} + a_r T_{rik})\Gamma^s_{jk} e_s .\end{aligned} \tag{21}$$

Thus on σ,

$$\frac{d}{dt}\Gamma^s_{ji} = \sum_k [a_k R_{kijs} - (e_i a_k + \sum_r a_r[\Gamma^k_{ri} + T_{rik}])\Gamma^s_{jk}] \tag{22}$$

where r and k go from 1 to n, $a_k(\sigma(t))$ is constant, and the functions $R_{kijs} \circ \sigma$, $(e_i a_k) \circ \sigma$, and $T_{rik} \circ \sigma$, are known. Furthermore, since $(D_{e_i} e_j)_m = 0$, at $\sigma(0)$,

$$\Gamma^s_{ji}(0) = 0. \tag{23}$$

Turning our attention to $'M$, let $f_* e_i = {}'e_i$ and $f_* T = {}'T$. Thus ${}'e_i = 0$ for $i > {}'n$ while ${}'e_i, \cdots, {}'e_{'n}$ is a C^∞ base field on a normal neighborhood of $'m$. Writing ${}'T = \sum_{i=1}^{'n} {}'a_i {}'e_i$, since $f_*(e_i)_p = ({}'e_i)_{f(p)}$, we have

$${}'T_{f(p)} = f_* T_p = \sum_i a_i(p)({}'e_i)_{f(p)}$$

for i summed from 1 to $'n$, so

$$a_i = {}'a_i \circ f \tag{24}$$

for $i = 1, \cdots, 'n$. Furthermore, if $1 \leq j \leq n$ and $1 \leq r \leq 'n$,

$$(e_j)_p\, a_r = (e_j)_p('a_r \circ f) = (f_*(e_j)_p)'a_r = ('e_j)_{f(p)}'a_r,$$

so

$$(e_j\, a_r) = ('e_j'a_r) \circ f. \tag{25}$$

For each $s, j, i = 1, \cdots, 'n$, we apply the analysis of the previous paragraph to $'M$ and use equations (11) and (12) to obtain, along the geodesic $'\sigma = f \circ \sigma$,

$$\frac{d'}{dt}\,\Gamma^s_{ji} = \sum\nolimits_k [a_k\, R_{kijs} - (e_i\, a_k + \sum\nolimits_r a_r['\Gamma^k_{ri} + T_{rik}])'\Gamma^s_{jk}], \tag{26}$$

where r and k are summed from 1 to $'n$ *only*, and $'\Gamma^s_{ji}(0) = 0$.

If either $i > 'n$ or $j > 'n$, then $'D_{f_* e_i} f_*\, e_j = 0$ on σ since either e_i or e_j is in K, but by (c), $D_{e_i}\, e_j$ is also in $\tilde{K}$ and $f_*(D_{e_i}\, e_j) = 0$. Thus to show f is connexion preserving at any point on σ it is sufficient to show $f_*(D_{e_i}\, e_j) = {}'D_{'e_i}\, 'e_j$ when $i \leq 'n$ and $j \leq 'n$. In this case,

$$f_*(D_{e_i}\, e_j) = f_*(\sum\nolimits_{k=1}^{n} \Gamma^k_{ji}\, e_k) = \sum\nolimits_{k=1}^{'n} \Gamma^k_{ji}\, 'e_k$$

while $'D_{'e_i}'e_j = \sum_{k=1}^{'n} {}'\Gamma^k_{ji}\,'e_k$, hence we show $\Gamma^s_{ji} \circ \sigma = {}'\Gamma^s_{ji} \circ {}'\sigma$ for i, j, and $s \leq 'n$. Since $\tilde{K}$, the kernel of f_*, is spanned by $e_{'n+1}, \cdots, e_n$, the condition (c) of the hypothesis implies,

$$R_{kijs} = 0 \quad \text{for} \quad k > 'n \quad \text{and} \quad i, j, s \leq 'n, \tag{27}$$

$$T_{rik} = 0 \quad \text{for} \quad r > 'n \quad \text{and} \quad i, k \leq 'n, \tag{28}$$

$$\Gamma^s_{jk} = 0 \quad \text{for} \quad k > 'n \quad \text{and} \quad s, j \leq 'n, \tag{29}$$

$$\Gamma^k_{ri} = 0 \quad \text{for} \quad r > 'n \quad \text{and} \quad k, i \leq 'n. \tag{30}$$

For example, to prove (29), $D_{e_k}\, e_j = \sum_{s=1}^{n} \Gamma^s_{jk}\, e_s$ which implies $\Gamma^s_{jk} = 0$ for $s \leq 'n$ if $D_{e_k}\, e_j$ in $\tilde{K}$.

Finally, the conditions (27) to (30) imply that the differential equations (22) for Γ^s_{ji} are valid when r and k are summed from 1 to $'n$, provided i, j, and s, are $\leq 'n$, and thus (22) becomes identical with (26). Thus $\Gamma^s_{ji}(\sigma(t)) = {}'\Gamma^s_{ji}('\sigma(t))$ for all i, j, and s, that are $\leq 'n$, which concludes the proof that f is connexion preserving.

THEOREM 3. *Let f be a C^∞ map from a normal neighborhood U of m in M into $'M$ such that $f(U)$ is contained in a normal neighborhood of $'m = f(m)$, let K denote the kernel of f_* at m, and let $Q = (f_*)_m(M_m)$. Then f is connexion preserving if and only if the following four conditions are satisfied*:

(a) *f is a spray map from m,*

(b) *f is curvature and torsion parallel invariant along all geodesics emanating from m and contained in U,*

(c) *there exists a complement* N *to* K *such that* $(f_*)_m(N) = Q$ *with* $D_{\tilde N}\tilde K \subset \tilde K$, $R(\tilde K, \tilde N)\tilde N \subset \tilde K$, $\mathrm{Tor}\,(\tilde K, \tilde N) \subset \tilde K$, *and* $D_X \tilde Y$ *in* $\tilde K$ *if* Y *in* N *and* X *in* $\tilde K$, *at all points of* U,

(d) $'D_{\tilde Q}(\tilde Q) \subset \tilde Q$ *on* $f(U)$.

Proof. If f is connexion preserving then (a), (b), and (c) follow as in the proof of Theorem 2. To prove (d), we fix some notation. Let $e_1, \cdots, e_{'n}$ be base of N and $e_{'n+1}, \cdots, e_n$ be a base of K as above. Thus Q has dimension $'n$ with base $'e_1, \cdots, 'e_{'n}$. Let $v_j = 'e_j$ for $j = 1, \cdots, 'n$ and extend these independent vectors to a base $v_1, \cdots, v_d$ of $'M_{'m}$ where d is the dimension of $'M$. For i and $j \leq 'n$,

$$f_*(D_{e_i}\, e_j) = f_*(\textstyle\sum_{k=1}^{n} \Gamma_{ji}^k\, e_k) = \sum_{k=1}^{'n} \Gamma_{ji}^k\, v_k$$

while $'D_{f_* e_i}\, f_*\, e_j = \sum_{k=1}^{d} {'\Gamma}_{ji}^k\, v_k$. If f is connexion preserving then $'\Gamma_{ji}^k = 0$ for $k > 'n$ which implies $'D_{v_i}(v_j)$ in $\tilde Q$ for i and $j \leq 'n$. Thus $'D_{\tilde Q}\,\tilde Q \subset \tilde Q$ on $f(U)$.

Assume now (a), (b), (c), and (d) hold. As before, if either $i > 'n$ or $j > 'n$, then $f_*(D_{e_i}\, e_j) = 0 = 'D_{f_* e_i}(f_*\, e_j)$. Thus our only concern is with i and $j \leq 'n$. The condition (d) implies $'\Gamma_{ji}^k \circ f \circ \sigma = 0$ for $k > 'n$, so we must show $\Gamma_{ji}^k \circ \sigma = {'\Gamma}_{ji}^k \circ f \circ \sigma$ for i, j, and $k \leq 'n$ along each geodesic σ emanating from m and contained in U. This will follow by showing equations (22) and (35) define the same set of differential equations. The condition (c) again implies that the indices of summation r and k in (22) actually go only from 1 to $'n$.

Let $'T = \sum_1^d {'a}_i\, v_i$ be the C^∞ field defined on $'U - ['m]$, where $'U$ is a normal neighborhood of $'m$ with $f(U) \subset 'U$, such that $'T_{f(p)} = f_*(T_p)$ for p in U. Thus

$$(30) \qquad ['T, v_i] = -\textstyle\sum_{k=1}^{d} [(v_i{'a}_k) + \sum_{j=1}^{d} {'a}_j({'\Gamma}_{ji}^k + {'T}_{jik})]v_k$$

as in (20). Condition (d) implies

$$(31) \qquad 'T_{rik} = 0 \quad \text{for} \quad k > 'n \quad \text{and} \quad r, i \leq 'n, \quad \text{and}$$

$$(32) \qquad 'R_{rijk} = 0 \quad \text{for} \quad k > 'n \quad \text{and} \quad r, i, j \leq 'n.$$

To prove (31) notice that $'\mathrm{Tor}\,(v_r, v_i) = \sum_{k=1}^{d} {'T}_{rik}\, v_k$ is in $\tilde Q$ only if (31) holds, and $'\mathrm{Tor}\,(v_r, v_i) = 'D_{v_r} v_i - 'D_{v_i} v_r - [v_r, v_i]$ is in $\tilde Q$ for $r, i \leq 'n$ by (d) and Theorem 1 (which shows $\tilde Q$ is integrable). Similarly, (32) follows by (d) and Theorem 1. Finally, since $'\sigma = f \circ \sigma$ has its tangent $'T$ in $\tilde Q$ (and $f_*\, T$ is always in $\tilde Q$), we have $'a_k \circ \sigma = 0$ and $(v_i{'a}_k) \circ {'\sigma} = 0$ for $k > 'n$ and $i \leq 'n$. Thus on $'\sigma$, equation (30) holds for k and j summing from 1 to $'n$.

By (32) $'R('T, v_i)v_j$ in $\tilde Q$ for $i, j \leq 'n$, so on $'\sigma$,

$$(33) \qquad 'R('T, v_i)v_j = \textstyle\sum_{r,s=1}^{'n} {'a}_r {'R}_{rijs}\, v_s = \sum_{r,s=1}^{'n} a_r\, R_{rijs}\, v_s$$

by (b) and (12). Also on $'\sigma$, $(v_i{'a}_k) \circ {'\sigma} = (e_i\, a_k) \circ \sigma$ for $i, k \leq 'n$ by (25), thus

$$
\begin{aligned}
(34)\qquad {}'R('T, v_i)v_j &= {}'D_{'T}{}'D_{v_i} v_j - {}'D_{['T, v_i]} v_j \\
&= \textstyle\sum_s ('T'\Gamma^s_{ji})v_s + \sum_{k,r,s}(e_i a_k + a_r{}'\Gamma^k_{ri} + a_r T_{rik}){}'\Gamma^s_{jk} v_s
\end{aligned}
$$

where k, r, and s go from 1 to $'n$ and $i, j \leq 'n$. Thus

$$
(35)\qquad \frac{d}{dt}{}'\Gamma^s_{ji} = \textstyle\sum_k [a_k R_{kijs} - (e_i a_k + \sum_r a_r['\Gamma^k_{ri} + T_{rik}]){}'\Gamma^s_{jk}]
$$

on $'\sigma$, where k and r go from 1 to $'n$ for i, j, and $s \leq 'n$, and $'\Gamma^s_{ji}(0) = 0$.

The differential equations (22) and (35) imply $\Gamma^s_{ji} \circ \sigma = {}'\Gamma^s_{ji} \circ {}'\sigma$ for all i, j, and $s \leq 'n$. Hence f is connexion preserving and Theorem 3 is proved.

3. A global theorem

In this section we use the method of Ambrose (see [1]) to extend the results in [6]. We will try to keep the notation as much like [6] as possible.

Again let M (dimension n) and $'M$ (dimension d) be C^∞ connected manifolds with connexions D and $'D$, respectively, and assume both connexions are complete.

We now set up the necessary notation in order to relate broken geodesics in M to broken geodesics in $'M$. Let k be any integer >0. Let Y_k be the set of finite sequences $(r_1, \cdots, r_j)$—with j variable—where the r_i are arbitrary variable points of R^k. Let I_m be a linear map of R^k into M_m for a fixed point m in M. Corresponding to I_m we now define for each y in Y_k the following four objects:

(a) a C^∞ broken geodesic σ_y emanating from m,
(b) a point $m_y = m(y)$ in M,
(c) a map P_y of R^k into $M_{m(y)}$,
(d) a C^∞ map $\exp_y : R^k \to M$.

We proceed by induction on j. If $j = 1$ and $y = r_1$, then

$$
\sigma_{r_1}(t) = \exp_m [tI_m(r_1)]
$$

for t in $[0, |r_1|]$, $m_{r_1} = \sigma_{r_1}(|r_1|)$, P_{r_1} is the composite of I_m followed by parallel translation along σ_{r_1} of M_m into $M_{m(r_1)}$, and

$$
\exp_{r_1} = \exp_{m(r_1)} \circ P_{r_1}.
$$

If $j > 1$ and $y = (r_1, \cdots, r_j)$ is a point of Y_k let $y' = (r_1, \cdots, r_{j-1})$. Let σ_y be the broken geodesic defined on $[0, s + |r_j|]$, where $s = |r_1| + \cdots + |r_{j-1}|$, such that $\sigma_y(t) = \sigma_{y'}(t)$ if t in $[0, s]$ while

$$
\sigma_y(t) = \exp_{m(y')} [(t - s]P_{y'}(r_j)] = \exp_{y'} (t - s)r_j
$$

if t in $[s, s + |r_j|]$. Let $m_y = \sigma_y(s + |r_j|)$, P_y be the composite of I_m followed by parallel translation along σ_y of M_m into $M_{m(y)}$, and $\exp_y = \exp_{m(y)} \circ P_y$.

For each y in Y_k and subspace A_y of $M_{m(y)}$, let $\tilde{A}_y$ denote the distribution induced on each normal neighborhood of $m(y)$ by parallel translating A_y along geodesics emanating from $m(y)$.

DEFINITION. A linear map F of M_m into $'M_{'m}$ is *curvature and torsion spray invariant* if the following condition is satisfied: Choose any base $e_1, \cdots, e_n$ of M_m and let

$$I_m : R^n \to M_m$$

by $I_m(\delta_i) = e_i$ where $\delta_i = (\delta_{i1}, \cdots, \delta_{in})$ is a unit point in R^n (and δ_{ij} is the Kronecker delta). For each y in Y_n let σ_y be the broken geodesic in M corresponding to I_m and let $'\sigma_y$ be the broken geodesic in $'M$ corresponding to $I_{'m} = F \circ I_m$. For $i = 1, \cdots, n$, let $P_t\, e_i$ denote the parallel translate of e_i along σ_y from m to $\sigma_y(t)$, and let $'P_t{'e_i}$ be the parallel translate of $'e_i = F(e_i)$ along $'\sigma_y$ from $'m$ to $'\sigma_y(t)$. Defining broken C^∞ functions T_{ijk} and R_{ijks} on the domain of σ_y so equations (9) and (10) hold, the condition we impose is that equations (11) and (12) should also hold on the domain of σ_y (and $'\sigma_y$).

A computation similar to the one in the lemma in Section 2 will show the above definition is independent of the base $e_1, \cdots, e_n$.

With I_m, $I_{'m}$, and $Y = Y_n$ fixed as in the above definition, we notice if y in Y, then P_y is an isomorphism of R^n onto $M_{m(y)}$, and we define the linear map $F_y = {'P_y} \circ P_y^{-1}$ with kernel K_y and image Q_y. Furthermore, let f_y be the spray map $'\exp_{'m(y)} \circ F_y \circ (\exp_{m(y)})^{-1}$ defined from a normal neighborhood U_y of $m(y)$ into a normal neighborhood of $'m(y)$. We use this notation in the following theorem.

THEOREM 4. *Let M and $'M$ be complete with M simply connected and let F be a linear map of M_m into $'M_{'m}$. There exists a connexion preserving map f of M into $'M$ such that $f(m) = {'m}$ and $(f_*)_m = F$ if and only if the following three conditions are satisfied*:

(a) *the map F is curvature and torsion spray invariant,*

(b) *for each y in Y there is a complement N_y to K_y in $M_{m(y)}$ such that $F_y(N_y) = Q_y$ with*

$$D_{\tilde{N}_y} \tilde{K}_y \subset \tilde{K}_y, \qquad R(\tilde{K}_y, \tilde{N}_y)\tilde{N}_y \subset \tilde{K}_y, \qquad \mathrm{Tor}\,(\tilde{K}_y, \tilde{N}_y) \subset \tilde{K}_y,$$

and $D_x \tilde{Z}$ in $\tilde{K}_y$ if Z in N_y and X in $\tilde{K}_y$, at all points of U_y,

(c) *for each y in Y, $'D_{\tilde{Q}_y}(\tilde{Q}_y) \subset \tilde{Q}_y$ on $f_y(U_y)$.*

Proof. If f is connexion preserving, then for each y in Y $f(m(y)) = {'m(y)}$ since $f \circ \sigma_y = {'\sigma_y}$, and hence (a), (b), and (c) follow from Theorem 3 since $(f_*)_{m(y)} = F_y$.

Assuming (a), (b), and (c), we slightly alter the method of Ambrose in [1] and Hicks in [6] to obtain the map f. We define an equivalence relation, $\sim$, on the points of Y as follows: $y_1 \sim y_2$ if all three of the following hold: (1) $m(y_1) = m(y_2)$, (2) $'m(y_1) = {'m(y_2)}$, and (3) $'P_{y_1} \circ P_{y_1}^{-1} = {'P_{y_2}} \circ P_{y_2}^{-1}$. Let W be the set of equivalence classes of this equivalence relation. Let I denote the natural map of Y into W, thus $I(y)$ is the equivalence class containing y. We define maps

$$e : W \to M \quad \text{and} \quad 'e : W \to {'M}$$

by $e(w) = m(y)$ and $'e(w) = {'m(y)}$ for any y with $I(y) = w$ (or y in w).

For each real number $\delta > 0$, let

$$B(\delta) = [p \quad \text{in} \quad R^n; \, | \, p \, | < \delta].$$

For each y in Y, let $\Delta(y)$ be a real number >0 such that $\exp_y$ maps $B(\Delta(y))$ diffeo onto a normal neighborhood of $m(y)$ contained in U_y. Indeed, we redefine U_y to be the image of $\exp_y$ on $B(\Delta(y))$. Let $I(y)$ denote the map of $B(\Delta(y))$ into W defined by $I_y(r) = I(y, r)$ where (y, r) is the element of Y obtained by placing r immediately after the last non-zero slot in y. A topology is now defined on W by requiring that each I_y, for all y in Y, be an open map of $B(\Delta(y))$ into W. Thus the topology has a sub-base consisting of sets of the form $I_y A$ where y in Y and A is an open subset of $B(\Delta(y))$. We define $B_y = I_y B(\Delta(y))$.

Since $\exp_y$ is a diffeo from $B(\Delta(y))$ onto U_y and $\exp_y = e \circ I_y$, we notice I_y is 1:1 from $B(\Delta(y))$ onto B_y and e is 1:1 from B_y onto U_y.

LEMMA 1. *Both e and $'e$ are continuous.*

Proof. Choose w in W, y in w, and an open neighborhood U of $e(W) = m(y)$. Let $A = (\exp_y^{-1} U) \cap B(\Delta(y))$. Since $\exp_y$ is continuous, the set A is open in R^n and the origin, 0, is in A. Hence $I_y A$ is an open set containing w and $e(I_y A) \subset U$. Thus e is continuous. If $'U$ is a neighborhood of $'e(w)$, then let

$$'A = ('\exp_y^{-1} {'U}) \cap B(\Delta(y))$$

and $I_y \, 'A$ is a neighborhood of w with $'e(I_y 'A) \subset {'U}$. Thus $'e$ is continuous.

LEMMA 2. *For each y in Y the map f_y is a connexion preserving map of U_y into $'M$.*

Proof. This follows from Theorem 3 and the hypothesis of Theorem 4.

LEMMA 3. *For y_1 and y_2 in Y and r_1 and r_2 in R^n, let $z_1 = (y_1, r_1)$ and $z_2 = (y_2, r_2)$ be points in Y with $| r_i | < \Delta(y_i)$, for $i = 1, 2$, and $z_1 \sim z_2$. Then there is a neighborhood A_i of r_i with $A_i \subset B(\Delta(y_i))$, for $i = 1, 2$, such that the following hold:*

(1) $\exp_{y_2}^{-1} \circ \exp_{y_1}$ *maps A_1 diffeo onto A_2,*

(2) *if one takes p_i in A_i, then* $\exp_{y_1} p_1 = \exp_{y_2} p_2$ *implies* $(y_1, p_1) \sim (y_2, p_2)$.

Proof. Let $q = e(z_i) = \exp_{y_i}(r_i)$. The set $U_{y_1} \cap U_{y_2}$ is open and includes q. By continuity there exists a real number $\delta > 0$ such that

$$\exp_{y_1} (B(r_1 ; \delta)) \subset U_{y_1} \cap U_{y_2},$$

where $B(r_1 ; \delta)$ is the usual metric open ball of radius δ about r_1. Let $A_1 = B(r_1 ; 0)$ and let $A_2 = \exp_{y_2}^{-1} \circ \exp_{y_1} (A_1)$. Thus conclusion (1) is satisfied since $\exp_{y_1}$ is a diffeo on $B(\Delta(y_1))$ and $\exp_{y_2}^{-1}$ is a diffeo on U_{y_2}.

We now show $f_{y_1} = f_{y_2}$ on $A = \exp_{y_1} (A_1)$. Since $f_{y_1}(q) = f_{y_2}(q) = {'e}(z_i)$ and f_{y_1} and f_{y_2} are connexion preserving, it suffices to show $(f_{y_1})_* = (f_{y_2})_*$

at q, or equivalently, $F_{z_1} = F_{z_2}$. But this follows from property (3) of the equivalence relation since $F_{z_1} = {}'P_{z_1} \circ P_{z_1}^{-1} = {}'P_{z_2} \circ P_{z_2}^{-1} = F_{z_2}$.

To prove (2) of the conclusion, we choose p_i in A_i with $\exp_{y_1} p_1 = \exp_{y_2} p_2$. Thus $m(y_1, p_1) = m(y_2, p_2)$. Since $f_y = {}'\exp_y \circ \exp_y^{-1}$ on U_y, we have

$$'m(y_1, p_1) = {}'\exp_{y_1} p_1 = f_{y_1}(\exp_{y_1} p_1) = f_{y_2}(\exp_{y_2} p_2) = {}'m(y_2, p_2).$$

Hence it remains to show ${}'P_{x_1} \circ P_{x_1}^{-1} = {}'P_{x_2} \circ P_{x_2}^{-1}$ where $x_i = (y_i, p_i)$ for $i = 1, 2$.

Let γ be the geodesic in U_{y_1} from $m(x_1)$ to $m(y_1)$, and let $'\gamma$ be the geodesic in $'M$ from $'m(y_1)$ to $'m(x_1)$ which is the reverse of $f_{y_1} \circ \gamma$. Let P_γ and $'P_{'\gamma}$ denote parallel translation along γ and $'\gamma$, respectively. Since f_{y_1} is connexion preserving,

$$((f_{y_1})_*)_{m(x_1)} = {}'P_{'\gamma} \circ F_{y_1} \circ P_\gamma = {}'P_{'\gamma} \circ {}'P_{y_1} \circ P_{y_1}^{-1} \circ P_\gamma = {}'P_{x_1} \circ P_{x_1}^{-1},$$

and similarly, $((f_{y_2})_*)_{m(x_2)} = {}'P_{x_2} \circ P_{x_2}^{-1}$. From above $f_{y_1} = f_{y_2}$ on A, hence their Jacobians are equal at $m(x_1) = m(x_2)$, which completes the proof of the lemma.

LEMMA 4. *Each I_y is continuous.*

Proof. The proof is similar to that of Lemma 6 in [1, p. 358].

LEMMA 5. *For each y_1 and y_2 in Y, the mappings I_{y_1} and I_{y_2} are C^∞ related, i.e.* $(I_{y_2}^{-1} \mid (B_{y_1} \cap B_{y_2})) \circ I_{y_1}$ is C^∞.

Proof. Choose w in $B_{y_1} \cap B_{y_2}$ with $w = I_{y_1} r_1 = I_{y_2} r_2$. Let A_1 and A_2 be neighborhoods of r_1 and r_2, respectively, which are related as in Lemma 3. On A_1,

$$(I_{y_2}^{-1} \mid (B_{y_1} \cap B_{y_2})) \circ I_{y_1} = (\exp_{y_2}^{-1} \mid (U_{y_1} \cap U_{y_2})) \circ \exp_{y_1}$$

and the latter map is C^∞. Since w is arbitrary the lemma is proved.

LEMMA 6. *The space W is Hausdorff.*

Proof. Let w_1 and w_2 be arbitrary points of W with $w_1 \neq w_2$. Choose y_i in Y with $I(y_i) = w_i$ for $i = 1, 2$. Since $y_1 \nsim y_2$, one of the following three statements must be true: (1) $m(y_1) \neq m(y_2)$, (2) $'m(y_1) \neq {}'m(y_2)$, (3) ${}'P_{y_1} \circ P_{y_1}^{-1} \neq {}'P_{y_2} \circ P_{y_2}^{-1}$.

If (a) holds, we take disjoint open neighborhoods U_1 and U_2 of $m(y_1)$ and $m(y_2)$, respectively. Let $B_i = e^{-1}(U_i)$ for $i = 1, 2$, and then B_1 and B_2 are open disjoint neighborhoods of w_1 and w_2, respectively. If (2) holds, we separate w_1 and w_2 by a similar argument using the map $'e$.

Now suppose (1) and (2) are false and (3) is true. Let A be an open subset of 0 in R^n such that $A \subset B(\Delta(y_i))$ for $i = 1, 2$, and $\exp_{y_1} A$ is a normal neighborhood of $m(y_1)$. Letting $I_{y_i}(A) = B_i$, we obtain open neighborhoods B_1 and B_2 of w_1 and w_2, respectively, and we show $B_1 \cap B_2$ is empty. If z in $B_1 \cap B_2$, then there exists p_1 and p_2 in A such that $x_1 = (y_1, p_1)$ and

$x_2 = (y_2, p_2)$ are in z, i.e. $x_1 \sim x_2$. Since the point $m(x_1)$ is contained in the normal neighborhoods U_{y_i} $(i = 1, 2)$ of $m(y_i)$, we let σ denote the unique geodesic in $\exp_{y_1} A$ from $m(y_1)$ to $m(x_1)$. Also the point $'m(x_1)$ is contained in a normal neighborhood of $'m(y_1) = 'm(y_2)$ since $f_y(U_y)$ is contained in a normal neighborhood of $m(y)$ for each y in Y. Let $'\sigma$ be the unique geodesic $'\sigma = f_{y_1} \circ \sigma$ from $'m(y_1)$ to $'m(x_1)$. Let P_σ be parallel translation along σ from $m(y_1)$ to $m(x_1)$, and let $'P_{'\sigma}$ be the corresponding parallel translation along $'\sigma$. Notice $'P_{'\sigma}$ is an isomorphism. Thus we have,

$$P_{x_1} = P_\sigma \circ P_{y_1}, \; P_{x_2} = P_\sigma \circ P_{y_2}, \; 'P_{x_1} = 'P_{'\sigma} \circ 'P_{y_1}, \text{ and } 'P_{x_2} = 'P_{'\sigma} \circ 'P_{y_2},$$

as mappings on R^n. Since $x_1 \sim x_2$, $'P_{x_1} \circ P_{x_1}^{-1} = 'P_{x_2} \circ P_{x_2}^{-1}$ which implies

$$'P_{'o} \circ 'P_{y_1} \circ P_{y_1}^{-1} \circ P_\sigma^{-1} = 'P_{'\sigma} \circ 'P_{y_2} \circ P_{y_2}^{-1} \circ P_{\sigma_2}^{-1}.$$

Since P_σ and $'P_{'\sigma}$ are isomorphisms, we conclude

$$'P_{y_1} \circ P_{y_1}^{-1} = 'P_{y_2} \circ P_{y_2}^{-1}$$

which contradicts (3). Thus $B_1 \cap B_2$ is empty.

LEMMA 7. *The space W is arcwise connected.*

Proof. Let $w_0 = I(0)$, where 0 is the origin in R^n. We connect any point w in W to w_0 by a continuous curve. Let $y = (r_1, \cdots, r_k)$ be in w. Let

$$a_j : [0, 1] \to Y$$

by $a_j(t) = (r_1, \cdots, r_{j-1}, tr_j)$ for $j = 1, \cdots, k$, and let

$$a : [0, k] \to Y$$

by $a(t) = a_j(t - j)$ for $j \leq t < j + 1$. Then $b(t) = (I \circ a)(t)$ for t in $[0, k]$ is the desired curve with $b(0) = w_0$ and $b(k) = I(y) = w$. To see that b is continuous, we take t_0 with $j < t_0 < j + 1$ and notice $b(t) = I_{a(t_0)}((t - t_0)r_j)$ for t near t_0. If $t_0 = j$, then $b(t) = I_{a(j)}\, c(t)$ is the image of a broken straight line c through the origin in R^n under the homeo $I_{a(j)}$ for t near t_0.

The lemmas proved above imply that W becomes an n-dimensional connected Hausdorff C^∞ manifold by using the pairs (I_y^{-1}, B_y) as coordinate pairs (see [7]). The maps e and $'e$ become C^∞ maps of W into M and $'M$, respectively. Since e is a local diffeo (from B_y to U_y for each y), we define a C^∞ connexion $\bar{D}$ on W by requiring e to be connexion preserving.

LEMMA 8. *The map $'e : W \to 'M$ is connexion preserving.*

Proof. This follows since on B_y we have

$$'e = '\exp_y \circ I_y^{-1} = '\exp_y \circ \exp_y^{-1} \circ e = f_y \circ e$$

and f_y is connexion preserving.

LEMMA 9. *The connexion on W is complete.*

Proof. Take w in W and X in W_w. Let σ be the geodesic, defined for all t in R since M is complete, with $\sigma(0) = e(w)$ and $T_\sigma(0) = e_* X$. Since e is connexion preserving and a local diffeo, we can lift σ to a unique geodesic γ in W with $\sigma = e \circ \gamma$, and γ will be defined on all of R.

LEMMA 10. *W is a covering space of M.*

Proof. We apply Theorem 3, p. 249, in [6].

We now complete the proof of Theorem 4. Since M is simply connected, the map e is a diffeo and thus $f = {}'e \circ e^{-1}$ is a connexion preserving map of M into $'M$ with $f(m) = {}'m$ and $(f_*)_m = F_0 = \text{F}$.

4. Some applications

In this section we obtain some results using Theorem 4.

THEOREM 5. *Let G and H be connected C^∞ Lie groups with Lie algebras g and h, respectively. Let F be a Lie algebra homomorphism of g into h. Then there exists a local homomorphism $f : U \to H$, where U is a normal neighborhood of the identity e in G with $(f_*)_e = F$. If G is simply connected, then there exists a unique homomorphism f of G into H with $(f_*)_e = F$.*

Proof. Let $G = M$, $H = {}'M$, and let D and $'D$ be the left invariant connexions. Thus D and $'D$ are both C^∞ complete connexions with zero curvature and torsion constant under parallel translation. Let $X_1, \cdots, X_n$ be a base of g, i.e. each X_i is a C^∞ left invariant vector field on G. Since $D_T X_i = 0$ for each vector T tangent to G, each X_i is a globally parallel field. Letting

$$\text{Tor}\,(X_i, X_j) = -[X_i, X_j] = \textstyle\sum_{k=1}^n T_{ijk} X_k,$$

we define *constant* functions T_{ijk} for i, j, and $k \leq n$. Since F is a Lie algebra homomorphism,

$$\text{Tor}\,('X_i, 'X_j) = -['X_i, 'X_j] = \textstyle\sum_{k=1}^n T_{ijk}\, 'X_k,$$

where $'X_i = F(X_i)$ for $i = 1, \cdots, n$, and this holds along each broken geodesic emanating from the identity in H. Thus F is curvature and torsion spray invariant.

We may assume the base $X_1, \cdots, X_n$ has been chosen so $X_{'n+1}, \cdots, X_n$ span the kernel K of F. Let N be the subspace spanned by $X_1, \cdots, X_{'n}$. If Y in N then $Y = \sum_{i=1}^{'n} a_i X_i$ for a_i in R, and hence $\tilde{Y}$ is a left invariant field. Thus $D_X \tilde{Y} = 0$ is in $\tilde{K}$ for any X. In this case, if Z is a field in $\tilde{K}$ and X is any vector, then $D_X Z$ is in $\tilde{K}$ for $Z = \sum_{'n+1}^{n} b_i X_i$ and $D_X Z = \sum_{'n+1}^{n} (Xb_i)X_i$. If Z_p in $\tilde{K}_p$ and Y_p in $\tilde{N}_p$, let Z and Y be the left invariant fields generated by Z_p and Y_p, respectively. Thus

$$\text{Tor}\,(Z_p, Y_p) = (D_Z Y - D_Y Z - [Z, Y])_p = -[Z, Y]_p.$$

But F is a Lie algebra homomorphism, hence $F([Z, Y]) = [FZ, FY] = 0$ since Z is in K. Thus Tor (Z_p, Y_p) is in $\tilde{K}_p$. Since $\tilde{N}_y$ and $\tilde{K}_y$ have bases

$$X_1(m(y)), \cdots, X_{'n}(m(y)) \quad \text{and} \quad X_{'n+1}(m(y)), \cdots, X_n(m(y)),$$

respectively, the conditions of part (b) in Theorem 4 are satisfied.

Let $Q = F(g)$ and let the left invariant vector fields $Y_1, \cdots, Y_k$ be a base for Q (and $\tilde{Q}_p = (\tilde{Q}_y)_p$ for any p in H and y in Y). If Z and W are fields in $\tilde{Q}_y$, then $Z = \sum_1^k a_i Y_i$, $W = \sum_1^k b_j Y_j$, and $D_Z W = \sum_{i,j=1}^k a_i(Y_i b_j)Y_j$ is in $\tilde{Q}_y$.

If G is not simply connected, we apply Theorem 3 to obtain a connexion preserving map $f = \exp_H \circ F \circ \exp_G^{-1}$ on a normal neighborhood U of e in G. If U is not convex, we may choose a normal convex neighborhood of e contained in U and redefine U to be this neighborhood. We show f is a local homomorphism relative to U. If x, y, and xy are in U, let $Y = \exp_G^{-1} y$ be the field in g lying in the normal neighborhood of 0 that corresponds to U. Thus $\sigma(t) = x \exp(tY)$ is the geodesic in U from x to xy, and $'\sigma(t) = (f \circ \sigma)(t)$ is a geodesic in H with $'\sigma(0) = f(x)$ and $T_{'\sigma}(0) = f_*(Y_x)$. But $f_*(Y_x) = F(Y)_{f(x)}$ since f is connexion preserving. Hence $'\sigma(t) = f(x) \exp(tF(Y))$ and $'\sigma(1) = f(xy) = f(x) \exp F(Y) = f(x)f(y)$.

If G is simply connected, we apply Theorem 4 to obtain a connexion preserving map f of G into H which is a local homomorphism by the above paragraph. For arbitrary x and y in G, we let $\sigma(t)$ be a broken geodesic from e to y in G. Replacing $\exp(tY)$ in the above argument by $\sigma(t)$, we can show $f(xy) = f(x)f(y)$.

We now list three theorems that follow immediately from Theorem 4. *In all three we assume M and $'M$ are connected C^∞ manifolds with complete connexions D and $'D$, respectively, and M is simply connected.*

THEOREM 6. *Let F be a linear injection of M_m into $'M_{'m}$. There exists a connexion preserving immersion f of M into $'M$ such that $f(m) = 'm$ and $(f_*)_m = F$ if and only if the map F is curvature and torsion spray invariant and for each y in $Y_{\dim (M)}$, $'D_{\tilde{Q}_y}(\tilde{Q}_y) \subset \tilde{Q}_y$ on $f_y(U_y)$ where $Q_0 = F(M_m)$.*

Proof. In this case K is zero so condition (b) of Theorem 4 is trivially satisfied.

THEOREM 7. *Let F be a linear bijection of M_m onto $'M_{'m}$. There exists a connexion preserving map f of M onto $'M$ such that $f(m) = 'm$ and $(f_*)_m = F$ if and only if the map F is curvature and torsion spray invariant, and in either case, f is a covering map.*

Proof. In this case, K is zero and $Q = 'M_{'m}$ so Theorem 4 can be applied to obtain f. The fact that f is a covering map follows from Theorem 3, p. 249, in [6].

The next theorem is essentially the main theorem of [6] and can be used to prove the other theorems in [6].

THEOREM 8. *Let F be a linear bijection of M_m onto $'M_{'m}$, and let $'M$ be simply connected. There exists a connexion preserving diffeo f of M onto $'M$*

with $f(m) = {'m}$ *and* $(f_*)_m = F$ *if and only if* F *is curvature and torsion spray invariant.*

Proof. Apply Theorem 7 to get both f and f^{-1}.

5. Geodesic submanifolds

In this section we generalize a theorem due to E. Cartan in [2] and a theorem of R. Hermann in [5]. Our methods are similar to those used in Sections 2 and 3.

DEFINITION. Let M be a C^∞ manifold with connexion D. A submanifold $'M$ of M is a *geodesic submanifold* if and only if there is a connexion $'D$ on $'M$ with $'D_X Y = D_X Y$ for all C^∞ fields X and Y tangent to $'M$.

COROLLARY. *A submanifold* $'M$ *of* M *is a geodesic submanifold if and only if* $D_{'\bar{M}}\, '\bar{M} \subset {'\bar{M}}$ *where* $'\bar{M}$ *is the tangent distribution to* $'M$, *i.e.* $D_X Y$ *is tangent to* $'M$ *if* X *and* Y *are fields tangent to* $'M$.

Proof. This follows immediately by letting $'D_X Y = D_X Y$.

Notice the image $f(M)$ in Theorems 4, 5, and 6, is an immersed geodesic submanifold. The Riemannian case of the following theorem is due to Cartan.

THEOREM 9. *Let* M *be a* C^∞ *manifold with connexion* D. *Choose* m *in* M, Q *a subspace of* M_m, U *a normal neighborhood of* m, *let* W *be the normal neighborhood of* 0 *in* M_m *such that* $\exp_m$ *maps* W *diffeo onto* U, *and let* $'M = \exp_m(W \cap Q)$. *Let* $\tilde{Q}$ *be the* C^∞ *distribution defined on* U *by parallel translating* Q *along geodesics emanating from* m. *Then* $'M$ *is a geodesic submanifold of* M *if and only if* $R(\tilde{Q}, \tilde{Q})\tilde{Q} \subset \tilde{Q}$ *and* $\mathrm{Tor}\,(\tilde{Q}, \tilde{Q}) \subset \tilde{Q}$ *on* $'M$.

Proof. Assume first that $'M$ is a geodesic submanifold and let $'D$ be the connexion on $'M$ such that $'D_X Y = D_X Y$ for C^∞ fields X and Y tangent to $'M$. We show $\tilde{Q}_p = {'M_p}$ for p in $'M$. Let Z be in Q, let σ be a geodesic in $'M$ emanating from m and passing through p, and let T be the tangent to σ. Extend Z to be a parallel field along σ with respect to $'D$, i.e. $'D_T Z = 0$, but then $D_T Z = {'D_T Z} = 0$ also, which implies Z is in $\tilde{Q}$ along σ. Thus $\tilde{Q}_p = {'M_p}$ for p in $'M$. Since $'M$ is geodesic, $D_Q \tilde{Q} = D_{'\bar{M}}\, '\bar{M} \subset \tilde{Q}$. Moreover, if X, Y, and Z, are in $\tilde{Q}_p$ for p in $'M$, we can extend them to tangent fields on $'M$ such that $[X, Y]$ is tangent to $'M$. Hence

$$R(X, Y)Z = {'D_X}\, {'D_Y} Z - {'D_Y}\, {'D_X} Z - {'D_{[X,Y]}} Z$$

is in $\tilde{Q}$ and $\mathrm{Tor}\,(X, Y) = {'D_X Y} - {'D_Y X} - [X, Y]$ is in $\tilde{Q}$.

Now suppose $R(\tilde{Q}, \tilde{Q})\tilde{Q} \subset \tilde{Q}$ and $\mathrm{Tor}\,(\tilde{Q}, \tilde{Q}) \subset \tilde{Q}$ on $'M$. Let $e_1, \cdots, e_k$ be a base for Q and extend it to a base $e_1, \cdots, e_n$ of M_m. Extend each e_i to be a C^∞ field on U by parallel translating e_i along geodesics emanating from m. We show again that $\tilde{Q}_p = {'M_p}$ for p in $'M$. Let $\sigma(t) = \exp_m tX$ be the geodesic in $'M$ from m through p. For each $i = 1, \cdots, n$ let $\bar{e}_i$ be the constant field on M_m induced by e_i and let $W_i(t) = (\exp_m)_*\, t(\bar{e}_i)_{tX}$ be the Jacobi field along

σ such that $W_i(0) = 0$ and $D_X W_i = e_i$. Since $(\exp_m)_*$ is an isomorphism at points in W, we can define C^∞ real-valued functions $g_{ij}(t)$ on the domain of σ by

$$W_j(t) = \sum_{i=1}^{n} g_{ij}(t)e_i(\sigma(t)), \tag{36}$$

and the n by n matrix $(g_{ij}(t))$ is non-singular for $t \neq 0$. Let T be the tangent to σ, and let T_{ijr} and R_{ijrs} be defined by equations (6) and (7). The Jacobi equations (8) imply, for each $u, j = 1, \cdots n$,

$$g''_{uj} - \sum a_r\, a_s\, R_{risu}\, g_{ij} - \sum a_s\, T_{sru}\, g'_{rj} - \sum a_s\, T'_{sru}\, g_{rj} = 0 \tag{37}$$

where r, s, and i are summed from 1 to n, and $T = \sum a_r\, e_r$. By hypothesis

$$R_{ijrs} = 0 \quad \text{if } i, j, r \leq k \quad \text{and} \quad s > k, \tag{38}$$

and

$$T_{ijr} = 0 \quad \text{if } i, j \leq k \quad \text{and} \quad r > k. \tag{39}$$

Since σ is in $'M$, $a_r = 0$ if $r > k$, and thus for $j \leq k$ and $u > k$,

$$g''_{uj} - \sum a_r\, a_s\, R_{risu}\, g_{ij} - \sum a_s\, T_{siu}\, g'_{ij} - \sum a_s\, T'_{siu}\, g_{ij} = 0 \tag{40}$$

where r and s run from 1 to k while i is summed from $k + 1$ to n. Since for $j \leq k$ and $u > k$, $g_{uj}(0) = 0$ and $g'_{uj}(0) = 0$, and $g_{uj}(t)$ satisfy the linear system (40) which depends only on g_{rs} for $r > k$ and $s \leq k$, we conclude $g_{uj}(t) \equiv 0$. Thus for $j \leq k$, $W_j(t)$ lies in $\tilde{Q}_{\sigma(t)}$, and since $W_1(t), \cdots, W_k(t)$ form a base for $'M_{\sigma(t)}$ if $t \neq 0$, we know $'M_{\sigma(t)} = \tilde{Q}_{\sigma(t)}$.

The next (and final) theorem of this study was motivated by a suggestion of R. Hermann in [5].

We fix some notation. Let M be a C^∞ manifold of dimension n and let D be a complete connexion on M. Let m be in M, let Q be a subspace of M_m, and let $I_m = P_0$ be a linear injection of R^k onto Q, where 0 is the origin in R^k and $2 \leq k < n$. Using this integer k, we let $Y = Y_k$ be defined as in the second paragraph of Section 3, and we use the other notation introduced there. For each y in Y, we let $Q_y = P_y(R^k)$, let U_y be a normal neighborhood of $m(y)$, and let $\tilde{Q}_y$ be the usual distribution induced by Q_y on U_y.

THEOREM 10. *If for each y in Y, $R(\tilde{Q}_y, \tilde{Q}_y)\tilde{Q}_y \subset \tilde{Q}_y$ and* Tor $(\tilde{Q}_y, \tilde{Q}_y) \subset \tilde{Q}_y$, *then there exists an immersed complete geodesic submanifold $'M$ of M such that $'M_{m(y)} = Q_y$ for all y in Y.*

Proof. We use a slight modification of the method used to prove Theorem 4. The lemmas we refer to will be the lemmas occurring in the proof of Theorem 4. Let $\sim$ be an equivalence relation on Y defined by: $y_1 \sim y_2$ if $m(y_1) = m(y_2)$ and $Q_{y_1} = Q_{y_2}$. Let W be the set of equivalence classes of this equivalence relation, and let I denote the natural map of Y into W. Define the map $e : W \to M$ by $e(w) = m(y)$ for any y such that $I(y) = w$.

For each real number $\delta > 0$, let $B(\delta) = [p \text{ in } R^k : |\, p\,| < \delta]$. For each y in Y, let $\Delta(y)$ be a real number >0 such that $\exp_y$ maps $B(\Delta(y))$ diffeo onto its image M_y, and we demand $M_y \subset U_y$. Let I_y be the map of $B(\Delta(y))$ into W defined by $I_y(r) = I(y, r)$. Again define a topology on W by requiring that each I_y be an open map of $B(\Delta(y))$ onto its image B_y. Since $\exp_y$ is a diffeo from $B(\Delta(y))$ onto M_y and $\exp_y = e \circ I_y$, we know I_y is 1:1 from $B(\Delta(y))$ onto B_y and e is 1:1 from B_y onto M_y, for each y in Y.

The map e is continuous, and this is proved exactly as in Lemma 1 (but here A is a subset of R^k). By Theorem 9 and our hypothesis, for each y in Y, M_y is a geodesic submanifold of M. Changing n to k in Lemma 3, that lemma is valid in the present case and the new proof need only use the first paragraph of the old proof. Again, Lemma 3 is used to show each I_y is continuous. Then for each y_1 and y_2 in Y, the mappings I_{y_1} and I_{y_2} are C^∞ related since

$$(I_{y_2}^{-1} \mid (B_{y_1} \cap B_{y_2})) \circ I_{y_1} = (\exp_{y_2}^{-1} \mid (M_{y_2} \cap M_{y_1})) \circ \exp_{y_1}$$

on the neighborhood A_1 (see the proof of Lemma 5). The space W is Hausdorff by the second paragraph of the proof of Lemma 6. The space W is arcwise connected by the proof of Lemma 7 (with n replaced by k throughout).

Thus W becomes a k-dimensional connected Hausdorff C^∞ manifold by using the pairs (I_y^{-1}, B_y) as coordinate pairs, and e is then a C^∞ map of W into M since $e \circ I_y = \mathrm{exy}_y$ on $B(\Delta(y))$. Since e is a local diffeo (from B_y onto M_y), e is an immersion of W into M. The image $'M = e(W)$ is an immersed geodesic submanifold of M since each M_y is a geodesic submanifold. If we define a connexion on W by letting e be connexion preserving then the proof of Lemma 9 shows W is complete. This proves Theorem 10.

In the Riemannian case, Theorem 4 and Theorem 10 (due to Hermann) can be modified so the hypothesis only involves a subset Z of Y where $Z = [(r_1, r_2)$ in Y such that $|r_2| < \Delta(r_1)]$.

Finally, we remark that Theorem 9 implies the condition (d) in Theorem 4 can be replaced by the condition (d′), where (d′) states for each y in Y,

$$'R(\tilde{Q}_y, \tilde{Q}_y)\tilde{Q}_y \subset \tilde{Q}_y \quad \text{and} \quad '\mathrm{Tor}\,(\tilde{Q}_y, \tilde{Q}_y) \subset \tilde{Q}_y .$$

Bibliography

1. W. Ambrose, *Parallel translation of Riemannian curvature*, Ann. of Math., vol. 64 (1956), pp. 337–363.
2. E. Cartan, *Geometrie des espaces de Riemann*, Paris, Gauthier-Villars, 1946.
3. C. Chevalley, *Theory of Lie groups*, Princeton, Princeton University Press, 1946.
4. S. Helgason, *Differential geometry and symmetric spaces*, New York, Academic Press, 1962.
5. R. Hermann, *Existence in the large of totally geodesic submanifolds of Riemannian spaces*, Bull. Amer. Math. Soc., vol. 66 (1960), pp. 59–61.
6. N. Hicks, *A theorem on affine connexions*, Illinois J. Math., vol. 3 (1959), pp. 242–254.
7. ———, *Notes on differential geometry*, Van Nostrand Math. Stud. 3, 1965.
8. M. Kikkawa, *On affinely connected spaces without conjugate points*, J. Sci. Hiroshima Univ. Ser. A-I, vol. 28 (1964), pp. 33–38.

The University of Michigan
Ann Arbor, Michigan