SPECTRAL MANIFOLDS FOR A CLASS OF OPERATORS

BY

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Introduction

Various generalizations of the spectral theory for self-adjoint operators have been considered. Probably best known is the work of N. Dunford [3], [4] on spectral operators, which possess a countably additive resolution of the identity. Others, including R. G. Bartle [2], R. C. Sine [11] and F. Wolf [13], have considered classes of operators for which the uniform boundedness of the resolution of the identity is not required. In these latter papers the basic operators were assumed bounded. In this paper we consider spectral manifolds for a class of (in general) unbounded operators and obtain generalizations of results in [2] and in [9]. The operators are closed, are defined on dense subsets of a Banach space, have spectra on the real axis and their resolvent operators satisfy a growth condition considered by R. G. Bartle in [2] for bounded operators. The class includes the self-adjoint operators in Hilbert space and operators of the form $-iA$ where A is the infinitesimal generator of a strongly continuous group of bounded operators on a Banach space and has spectrum on the imaginary axis.

In Section 1 the basic properties of local spectra and resolvents for an operator T of the class are obtained. The principal result is the Approximation Theorem 1.13 which shows that the vectors with compact spectra are dense in the space X. The "Lorch Approximation" (Theorem 1.23) is derived for a first order growth rate. In Section 2 it is shown that T has invariant subspaces. It T lacks point spectrum, then certain spectral manifolds are shown to be quasi-complementary in reflexive spaces. In Section 3 under the further assumption that T has only continuous spectrum, these manifolds are shown to be quasi-complementary in arbitrary Banach spaces and a "resolution of the identity" for T consisting of closed projections is obtained.

Section 1. Properties of the local spectrum

Let T be a closed linear transformation defined on a dense subset $D(T)$ of a Banach space X. If the complex number μ is such that $\mu I - T$ has a range which is dense in X and has a bounded inverse we say that μ is in the *resolvent set* $\rho(T)$ of T and write $R(\mu) = (\mu I - T)^{-1}$. The complement $\sigma(T)$ to $\rho(T)$ is called the *spectrum* of T. The following properties are well known:

Received June 21, 1965.

[1] This paper is part of the author's dissertation submitted to Columbia University, written under the supervision of Professor E. R. Lorch.

0.1. $R(\mu)$ is an analytic function of μ on the open set $\rho(T)$ and satisfies $R(\mu) - R(\beta) = (\beta - \mu)R(\mu)R(\beta)$ [12, Th. 5. 1C].

0.2. If T is closed and $\rho(T)$ is non-empty and $p(\mu)$ is any polynomial then $p(T)$ is a closed operator [6, Th. 7, p. 602].

0.3. If the polynomial $p(\mu)$ is of degree k and if the vector x is in the domain of T^k and μ is in $\rho(T)$ then $R(\mu)^m x$ is in the domain of T^{m+k} and $p(T)R(\mu)^m x = R(\mu)^m p(T)x$. In particular, $R(\mu)$ commutes with each power of T [6, Th. 8, p. 603].

0.4. If T is closed, densely defined and has non-empty resolvent set $\rho(T)$ then T^m is densely defined for each positive integral m [6, Lemma 9, p. 648].

1.1. We shall further assume that the spectrum of T is on the real axis and that the resolvent operator $R(z)$ satisfies the n-th order growth condition:

$$(G_n) \qquad \begin{aligned} &|\operatorname{Im} z|^n \| R(z) \| \leq K \quad \text{for} \quad 0 < |\operatorname{Im} z| \leq 1 \quad \text{and} \\ &|\operatorname{Im} z| \| R(z) \| \leq K \quad \text{for} \quad |\operatorname{Im} z| > 1 \end{aligned}$$

for some positive integer n, some positive constant K and all non-real z.

For any x in X an X-valued analytic function $F(z)$ is called an *analytic extension* of $R(z)x$ if its domain $D(F)$ is open in the complex plane and the equation $(\mu I - T)F(\mu) = x$ holds for each μ in $D(F)$. $R(z)x$ is said to have the *single-valued extension property* if any two of its analytic extensions agree on their common domain. If this is the case then $R(z)x$ has a maximal analytic extension $\hat{x}(z)$ analytic on an open set $\rho(x)$ called the (local) *resolvent* of x. The complement $\sigma(x)$ to $\rho(x)$ is called the (local) *spectrum* of x. We note that the assumption that T has a real spectrum implies, since $\rho(T)$ is everywhere dense, that any two analytic extensions—in fact, any two continuous extensions—of $R(z)x$ must agree on their common domain. Sometimes $\hat{x}(z)$ itself will be called the resolvent of x.

The following properties are immediate consequences of the definition of local resolvent and spectrum:

1.2(i) $(zI - T)\hat{x}(z) = x$ for all z in $\rho(x)$ and $\hat{x}(z) = R(z)x$ for all z in $\rho(T)$,

1.2(ii) $\sigma(x + y) \subset \sigma(x) \cup \sigma(y)$ and $\sigma(kx) = \sigma(x)$ for any scalar $k \neq 0$ and

1.2(iii) $(ax + by)^\wedge(z) = a\hat{x}(z) + b\hat{y}(z)$ for all z in $\rho(x) \cap \rho(y)$ and for any scalars a and b.

1.3 LEMMA. *If A is any bounded linear operator which commutes with T then $\sigma(Ax) \subset \sigma(x)$.*

Proof. Since $(zI - T)A\hat{x}(z) = Ax$ for all z in $\rho(x)$, it follows that $\rho(x) \subset \rho(Ax)$.

1.4 LEMMA. *For each x in $D(T)$, $\sigma(Tx), \subset \sigma(x)$.*

Proof. Let $F(z) = z\,\hat{x}(z) - x$ for z in $\rho(x)$. Then F is analytic and a calculation shows that $(zI - T)\,F(z) = Tx$. If $\{z_j\}_j$ is a sequence in $\rho(T)$ converging to a real μ in $\rho(x)$, then $F(\mu) = \lim_j F(z_j)$ and a calculation shows that $Tx = \lim_j(\mu I - T)\,F(z_j)$. Since $\mu I - T$ is closed, it follows that $F(\mu)$ is in $D(T)$ and $(\mu I - T)\,F(\mu) = Tx$.

The lemma which follows can be proved as in [5, pp. 254-255], by replacing the continuity of T by its closure.

1.5 LEMMA. *Let J be any bounded open interval in $(-\infty, \infty)$ and let $\{x_j\}_j$ be a sequence of vectors such that $\rho(x_j) \supset J$ for each j. If $x = \lim_j x_j$, then $\rho(x) \supset J$.*

1.6 COROLLARY. *Let F be a closed set in $(-\infty, \infty)$. The set of all vectors whose spectra are in F is a closed linear manifold in X.*

1.7. For a development of further properties of local spectra we introduce integrals of a type first considered by E. R. Lorch [9] and used later by N. Dunford [5]. For each pair a, b of real numbers with $a < b$, let $C(a, b)$ denote a piecewise smooth Jordan curve meeting the real axis at the points a and b at non-zero angles (say, at right angles). Such a curve will be called an *admissible contour*. Let $K(a, b)$ be the operator defined on X by

$$1.7(\mathrm{i}) \qquad K(a, b) = \frac{1}{2\pi i}\int_{C(a,b)} (z - a)^n (z - b)^n R(z)\, dz.$$

The finite growth rate (G_n) of $R(z)$ near the real axis shows that the integrand in 1.7(i) is bounded on the contour $C(a, b)$. Thus the integral exists as an improper Riemann integral in the uniform operator topology and represents a bounded linear operator. The analyticity of the integrand off the real axis shows that the integral is independent of the contour $C(a, b)$ provided that it meets the requirements above.

More generally, we may form integrals of the type

$$1.7(\mathrm{ii}) \qquad J(a, b; f) = \int_{C(a,b)} f(z)(z - a)^n (z - b)^n R(z)\, dz$$

where $f(z)$ is a complex-valued function analytic in some neighborhood of the closed interval $[a, b]$ and $C(a, b)$ is an admissible contour which lies in this neighborhood. Properties of 1.7(i) were first studied by E. R. Lorch in the case where T is a (not-necessarily bounded) self-adjoint operator acting on a Hilbert space $(n = K = 1)$. The operator $K(a, b)$ here is the operator $-K_{ab}(n, n)$ of [9]. The proof of the following lemma given there [9, pp. 141–142] is valid for arbitrary Banach spaces and will not be repeated here.

1.8 LEMMA. (i)

$$K(a, b)K(a, b) = \frac{1}{2\pi i}\int_{C(a,b)} (z - a)^{2n} (z - b)^{2n} R(z)\, dz.$$

(ii) *If* (a, b) *and* (c, d) *are disjoint open intervals, then*

$$K(a, b)K(c, d) = 0.$$

1.9 LEMMA. *Let* $C(a, b)$ *be an admissible contour and let* $f(z)$ *be analytic inside and on* $C(a, b)$. $J(a, b; f)x$ *is in* $D(T)$ *for each* x *in* X *and*

$$TJ(a, b; f)x = \int_{C(a,b)} zf(z)(z-a)^n(z-b)^n R(z)x\, dz.$$

Proof. Approximate $J(a, b; f)x$ by Riemann sums and use the identity

$$TR(z)x = zR(z)x - x$$

for z in $\rho(T)$ and the closure of T to show that $J(a, b; f)x$ belongs to $D(T)$ and

$$TJ(a, b; f)x = \int_{C(a,b)} zf(z)(z-a)^n(z-b)^n R(z)x\, dz - \int_{C(a,b)} f(z)(z-a)^n(z-b)^n x\, dz.$$

Since the latter integrand is analytic, the assertion follows.

1.10 *Remark.* It follows from 1.9, by induction, that $J(a, b; f)x$ is in $D(T^m)$ for each positive m and that

$$T^m J(a, b; f)x = \int_{C(a,b)} z^m f(z)(z-a)^n(z-b)^n R(z)x\, dz.$$

Hence if $p(z)$ is any polynomial then $J(a, b; f)x$ is in the domain of $p(T)$ and

$$p(T)J(a, b; f)x = \int_{C(a,b)} p(z)f(z)(z-a)^n(z-b)^n R(z)x\, dz.$$

1.11 COROLLARY. *For each* x *in* X

(i) $K(a, b)x$ *is in the domain of* T^m *for each* $m \geq 1$ *and*

$$p(T)K(a, b)x = \frac{1}{2\pi i}\int_{C(a,b)} p(z)(z-a)^n(z-b)^n R(z)x\, dz$$

for any polynomial $p(z)$,

(ii) *if* μ *is outside* $C(a, b)$ *then the local resolvent of* $K(a, b)x$ *is given by*

$$\frac{1}{2\pi i}\int_{C(a,b)} \frac{(z-a)^n(z-b)^n}{\mu - z} R(z)x\, dz$$

and

(iii) $\sigma(K(a, b)x) \subset [a, b] \cap \sigma(x)$.

Proof. (i) and (ii) follow from Lemma 1.9 and Remark 1.10 following it (note that $f(z) = (\mu - z)^{-1}$ is analytic inside and on $C(a, b)$, for μ outside this contour). Thus, $\rho(K(a, b)x)$ contains the exterior of each admissible contour $C(a, b)$ so that $\sigma(K(a, b)x) \subset [a, b]$. (iii) now follows from Lemma 1.3.

1.12 LEMMA. *$\lim \| K(a, b) \| = 0$ as $| b - a |$ tends to zero.*

Proof. Choose any a and b with $b - a < 2$ and choose for $C(a, b)$ the circle of radius $(b - a)/2$ and center $(a + b)/2$. The change of variable $z - (a + b)/2 = re^{i\theta}$ with $r = (b - a)/2$ yields the estimate

$$\|(z - a)^n(z - b)^n R(z)\| \leq r^{2n}2^n \,| \sin \theta |^n \frac{K}{r^n \,| \sin \theta |^n} = 2^n K r^n.$$

Hence,

$$\| K(a, b) \| \leq (1/2\pi)(2^n K r^n)(2\pi r) = 2^n K r^{n+1} = O(| b - a |^{n+1}).$$

The theorem which follows is fundamental in our work on local spectra. It shows that the vectors with bounded (compact) spectra are dense in the Banach space X. In the case where T is bounded all local spectra are bounded. In this sense, relative to spectral analysis, Theorem 1.13 allows us to pass from bounded operators to unbounded operators.

1.13 THEOREM (The Approximation Theorem). *For each $r > 0$ let*

$$I_r = \frac{1}{2\pi i} \int_{C(-r,r)} \left[1 - \frac{z^2}{r^2}\right]^n R(z)\, dz.$$

The operators I_r converge strongly to the identity operator I on X as r becomes infinite.

Proof. Since $zR(z)x - x = TR(z)x$ we have

$$\begin{aligned} I_r x - x &= \frac{1}{2\pi i} \int_{C(-r,r)} \left[1 - \frac{z^2}{r^2}\right]^n R(z)x\, dz - \frac{1}{2\pi i} x \int_{C(-r,r)} \left[1 - \frac{z^2}{r^2}\right]^n \frac{1}{z}\, dz \\ &= \frac{1}{2\pi i} \int_{C(-r,r)} \left[1 - \frac{z^2}{r^2}\right]^n \frac{zR(z)x - x}{z}\, dz \\ &= \frac{1}{2\pi i} \int_{C(-r,r)} \left[1 - \frac{z^2}{r^2}\right]^n \frac{TR(z)x}{z}\, dz. \end{aligned}$$

If x is in $D(T)$, then $TR(z)x = R(z)Tx$ by 0.3 and hence $I_r x - x = J_r(Tx)$, where J_r is the operator defined by

$$J_r = (2\pi i)^{-1} \int_{C(-r,r)} [1 - (z/r)^2]^n (1/z) R(z)\, dz.$$

We shall show that $\| I_r \| = O(1)$ and $\| J_r \| = O(r^{-1})$. This will show that $x = \lim_r I_r x$ for x in the dense set $D(T)$ and, hence, (cf. [6, II.3.6]) for all x in X.

To prove the uniform boundedness of the I_r for $r \geq 1$ and the uniform convergence of the J_r to zero we choose for the contour $C(-r, r)$ the circle C_r with center at the origin and radius r. Let $\Delta(r)$ and $\Delta(-r)$ be the intersections at r and $-r$ respectively, of C_r with the horizontal strip $-1 \leq \operatorname{Im} z \leq 1$ and let $K+$ denote the circular arc above this strip and $K-$ the arc of C_r below this strip. On $\Delta(r)$ and $\Delta(-r)$ we have the estimate (using $z = re^{i\theta}$)

$$\left\| \left[1 - \frac{z^2}{r^2}\right]^n R(z) \right\| \leq 2^n \mid \sin \theta \mid^n \frac{K}{r^n \mid \sin \theta \mid^n} = 2^n K r^{-n}$$

while on $K+$ and on $K-$ we get the estimate

$$\left\| \left[1 - \frac{z^2}{r^2}\right]^n R(z) \right\| \leq 2^n \mid \sin \theta \mid^n \frac{K}{r \mid \sin \theta \mid} \leq 2^n K r^{-1}.$$

Thus, writing C_r as the sum of four contours none of which has length exceeding πr we get

$$\| I_r \| \leq (2/2\pi)(2^n K r^{-n})(\pi r) + (2/2\pi)(2^n K r^{-1})(\pi r) = 2^n K[1 + r^{1-n}].$$

Similarly,

$$\| J_r \| \leq (2/2\pi)(2^n K r^{-n-1})(\pi r) + (2/2\pi)(2^n K r^{-2})(\pi r) = 2^n K[r^{-1} + r^{-n}].$$

Thus for $r \geq 1$ we have $\| I_r \| \leq 2^{n+1} K$ and $\| J_r \| = O(r^{-1})$, QED.

1.14 COROLLARY (The Density Theorem). *The vectors with compact spectra are dense in X.*

The corollary to the Approximation Theorem which follows is basic to our later work. For a bounded operator many results on spectra are obtained from the operational calculus which is based on the formula

$$I = (2\pi i)^{-1} \int_C R(z)\, dz.$$

The next result allows us to write

$$x = (2\pi i)^{-1} \int_C R(z)x\, dz$$

for the dense set of vectors with compact spectra, where the integrand $R(z)x$ denotes the analytic function $\hat{x}(z)$.

1.15. COROLLARY. *Let x be any vector with bounded spectrum. Let C be any piece-wise smooth rectifiable Jordan curve containing $\sigma(x)$ in its interior.*

(i) $$x = \frac{1}{2\pi i} \int_C R(z)x\, dz \qquad \left(\text{or } \frac{1}{2\pi i} \int_C \hat{x}(z)\, dz \right)$$

(ii) $$\hat{x}(\mu) = \frac{1}{2\pi i} \int_C \frac{R(z)x}{\mu - z}\, dz$$

for each μ outside C and

(iii) $\hat{x}(\mu)$ *is analytic at infinity and has the value* 0 *there.*

Proof. The analyticity of $\hat{x}(z)$ outside $\sigma(x)$ shows that any integral of the form $\int_C f(z)\hat{x}(z)\,dz$ where $f(z)$ is entire is independent of the path C as long as C contains $\sigma(x)$ in its interior. Choosing such a C and then any positive r such that the circle C_r of radius r and center at the origin contains C in its interior we get

$$I_r\,x = \frac{1}{2\pi i}\int_{C_r}\left[1 - \frac{z^2}{r^2}\right]^n R(z)x\,dz = \frac{1}{2\pi i}\int_C\left[1 - \frac{z^2}{r^2}\right]^n \hat{x}(z)\,dz$$
$$= \frac{1}{2\pi i}\int_C \hat{x}(z)\,dz + \sum_{j=1}^{n}(-1)^j\binom{n}{j}\frac{1}{2\pi i}\int_C \frac{z^{2j}}{r^{2j}}\hat{x}(z)\,dz.$$

Letting $r \to \infty$ we get $x = (2\pi i)^{-1}\int_C \hat{x}(z)\,dz$, proving (i). The proof of (ii) is similar to that of Lemma 1.9 and its Corollary 1.11. Applying $\mu I - T$ to the integral in (ii) we get

$$(\mu I - T)\frac{1}{2\pi i}\int_C \frac{R(z)x}{\mu - z}\,dz = \frac{1}{2\pi i}\int_C R(z)x\,dz = x$$

for μ outside C. Thus, this integral is an analytic extension of $R(z)x$ to the exterior of C.

To prove (iii), choose for C a circle C_r with r fixed and sufficiently large so that $\sigma(x)$ is in the interior of C_r. Choose any $\rho > r$ and consider z and μ such that $|z| = r < \rho < |\mu|$. The series

$$1 + z^1/\mu^1 + z^2/\mu^2 + z^3/\mu^3 + \cdots$$

converges to $\mu(\mu - z)^{-1}$ uniformly for all such z and μ. Hence, the series $\sum_{j=0}^{\infty}\hat{x}(z)(z/\mu)^j$ converges to $\mu\hat{x}(z)(\mu - z)^{-1}$ uniformly for z and μ. Integrating the latter series term-by-term over the circle C_r and dividing through by $2\pi i$ we get, using (ii),

$$(\nabla)\qquad \mu\hat{x}(\mu) = \sum_{j=0}^{\infty}\frac{1}{2\pi i}\int_{C_r}\frac{z^j\hat{x}(z)}{\mu^j}\,dz = \sum_{j=0}^{\infty}\frac{T^jx}{\mu^j}.$$

Here we have used that $p(T)x = (1/2\pi i)\int_C p(z)R(z)x\,dz$ for any polynomial $p(z)$, the proof being identical to that of Lemma 1.9 and Remark 1.10. The convergence of the series in (∇) for all μ with $|\mu| > \rho$ shows that the series $S(\beta) = \sum_{j=0}^{\infty}T^jx\beta^j$ has a positive radius of convergence and thus is analytic at $\beta = 0$ and has the value zero there. Hence, from (∇) we have

$$\hat{x}(\mu) = \frac{x}{\mu} + \frac{Tx}{\mu^2} + \frac{T^2x}{\mu^3} + \frac{T^3x}{\mu^4} + \cdots$$

is analytic at infinity and has the value zero there, QED.

1.16 LEMMA. $\sigma(x)$ *is empty if and only if* $x = 0$.

Proof. If $x = 0$ then $\hat{x}(z) = 0$ for all z. Conversely, if $\sigma(x) = \emptyset$, let

$M = \max \| \hat{x}(z) \|$ for $| z | \leq 1$. If $0 < r < 1$, apply Corollary 1.15(i) to the circle $| z | = 1$ to get $\| x \| \leq Mr$ and let $r \to 0$.

1.17 THEOREM. *T is bounded if and only if each vector x in X has a bounded spectrum.*

Proof. If T is bounded then each local spectrum is contained in the compact set $\sigma(T)$. Conversely, if each x in X has a compact spectrum then $D(T) = X$. This follows from the closure of T and the integral representation of Corollary 1.15(i).

For a proof of the lemma which follows see [2, Theorem 7 and Corollary 2]. Note that the boundedness of T is not required but merely the representation $x = (2\pi i)^{-1} \int_C R(z)x\,dz$ if the spectrum of x consists of a single point.

1.18 LEMMA. *$\sigma(x) = \{\alpha\}$ if and only if $(T - \alpha I)^n x = 0$.*

1.19 THEOREM (The Unique Representation Property). *If $\Delta_1, \Delta_2, \cdots, \Delta_m$ are pairwise disjoint subsets of the real line and if x_j and y_j are vectors with spectra in Δ_j for each j and if $\sum_{j=1}^m x_j = \sum_{j=1}^m y_j$ then $x_j = y_j$ for each $j = 1, 2, \cdots, m$.*

Proof. The vector $x_1 - y_1 = \sum_{j=2}^m (y_j - x_j)$, by 1.2 (ii), has spectrum in the set $\Delta_1 \cap (\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_m)$ which is empty. By Lemma 1.16 we get $x_1 = y_1$. A similar argument for any index j shows that $x_j = y_j$.

1.20 COROLLARY. *If $\sigma(x)$ consists of the distinct real numbers $a_1, a_2, \cdots, a_m$ then x can be written uniquely as a sum $\sum_{j=1}^m x_j$ with $\sigma(x_j) = \{a_j\}$ for each $j = 1, \cdots, m$.*

Proof. The uniqueness follows from 1.19. For the existence of such a representation, let C_j be a (small) circle with center a_j, $j = 1, \cdots, m$ and with C_k exterior to C_j for $j \neq k$. By Corollary 1.15(i) we have

$$x = (2\pi i)^{-1} \int_C \hat{x}(z)\,dz = \sum_{j=1}^m (2\pi i)^{-1} \int_{C_j} \hat{x}(z)\,dz = \sum_{j=1}^m x_j$$

where $x_j = (2\pi i)^{-1} \int_{C_j} \hat{x}(z)\,dz$. To show that $\sigma(x_j) = \{a_j\}$, define

$$w(\mu) = 2\pi i)^{-1} \int_{C_j} \hat{x}(z)(\mu - z)^{-1}\,dz.$$

Then w is analytic for μ outside C_j and a direct computation using the closure of T shows that $(\mu I - T)w(\mu) = x_j$. Hence $\sigma(x_j)$ is interior to any sufficiently small circle having a_j as center, so that $\sigma(x_j) \subset \{a_j\}$. If $\sigma(x_j) = \emptyset$ then $x_j = 0$ and $a_j \notin \sigma(x)$, contrary to the hypothesis. Hence $\sigma(x_j) = \{a_j\}$ and the proof is complete.

The next theorem, a generalization of 1.13, might serve as the basis of a (germinal) operational calculus which has much in common with an operational calculus due to W. G. Bade (see [1]). As this is not needed in the sequel it will not be developed here.

1.21 THEOREM. *If $p(z)$ is a polynomial of degree m and x is in $D(T^m)$ then*

$$\lim_r (2\pi i)^{-1} \int_{C(-r,r)} p(z)[1 - (z/r)^2]^n R(z)x \, dz$$

exists and equals $p(T)x$.

Proof. If $p(z) = 1$, the result is merely 1.13. Using this for x in $D(T)$ and Tx in place of x and the fact that $R(z)Tx = TR(z)x = zR(z)x - x$, we get

$$Tx = \lim_r (2\pi i)^{-1} \int_{C(-r,r)} [1 - (z/r)^2]^n zR(z)x \, dz.$$

The assertion follows for $T^k x$ by induction and then, by linearity, for $p(T)x$.

This section concludes with a proof of the Lorch Approximation in the case where $R(z)$ satisfies the first order growth (G_1). A proof for bounded T can be found in [2, Theorem 9]. The proof given here uses the following lemma whose proof is the observation that if x has compact spectrum, then in the Laurent expansion for $\hat{x}(z)$ in any annulus between $\sigma(x)$ and infinity the coefficients of the positive powers must be zero by 1.15(iii).

1.22 LEMMA. *If* $\sigma(x) \subset [a, b] \subset$ Interior (C) *and* $a \leq t \leq b$, *then*

$$\int_C (z - t)^{-m} R(z)x \, dz = 0 \qquad \text{for } m = 1, 2, 3, \cdots .$$

1.23. THEOREM (The Lorch Approximation for (G_1)). *Let* $R(z)$ *satisfy* (G_1) *and let* $\varepsilon > 0$ *be given. If* J *is any closed interval of length less than* 2ε *and* k *is any point in* J, *then*

$$\| (T - kI)x \| < \varepsilon(2K + 1) \| x \|$$

for every x *with spectrum in* J.

Proof. Let $J = [a, b]$ with $b - a < 2\varepsilon$ and let $c = \frac{1}{2}(a + b)$. Choose s and t such that $s < a < b < t$ and so that $\frac{1}{2}(s + t) = c$. Define the operator $S = S(s, t)$ by

$$S = (2\pi i)^{-1} \int_C \frac{(z - s)(z - t)}{z - c} R(z) \, dz,$$

where $C = C(s, t)$ is the circle with center c and radius $r = \frac{1}{2}(t - s)$. An estimate on norms using (G_1) shows that on C, the integrand for S is bounded by $2K$ so that $\| S \| \leq 2Kr$. Let $\sigma(x) \subset J$. Then from 1.22 (with $m = 1$), 1.15(i) and the identity

$$(z - a)(z - b)/(z - c) = z - c + (ab - c^2)/(z - c)$$

it follows that

$$(T - cI)x = (2\pi i)^{-1} \int_C (z - c)R(z)x \, dz$$

$$= (2\pi i)^{-1} \int_C \frac{(z - a)(z - b)}{z - c} R(z)x \, dz.$$

Thus

$$(T - cI)x - Sx = (2\pi i)^{-1}\int_C \frac{(z - a)(z - b) - (z - s)(z - t)}{z - c} R(z)x\,dz$$

$$= (ab - c^2)(2\pi i)^{-1}\int_C (z - c)^{-1}R(z)x\,dz = 0,$$

again by 1.22. Hence $Sx = (T - cI)x$ and so $\| (T - cI)x \| \leq 2Kr \| x \|$. Since this holds for each $r = \frac{1}{2}(t - s)$ greater than $\frac{1}{2}(b - a)$, it follows that

$$\| (T - cI)x \| \leq 2K\varepsilon \| x \|.$$

If k is in J, the triangle inequality now yields

$$\| (T - kI)x \| \leq 2K\varepsilon \| x \| + \tfrac{1}{2}(b - a) < (2K + 1)\varepsilon \| x \|, \text{ Q.E.D.}$$

Section 2. Assumption of no point spectrum

2.0. Throughout this section we shall assume that T lacks a point spectrum. Thus, T is closed, densely defined, has a real spectrum containing no eigenvalues and $R(z)$ satisfies (G_n).

2.1 *Remark.* A non-zero vector cannot have a discrete spectrum. For, by 1.16, 1.18, 1.20 and 2.0 it follows that if x has a finite discrete spectrum then $x = 0$. Hence, if x has an infinite discrete spectrum then each approximating vector $I_r x$ to x is zero and $x = 0$.

2.2 LEMMA. *Let $J_1, J_2, \cdots, J_m$ be closed intervals in $(-\infty, \infty)$ which pairwise are disjoint or have at most a common endpoint. If $\sum_{j=1}^m x_j = \sum_{j=1}^m y_j$ with $\sigma(x_j)$, $\sigma(y_j) \subset J_j$ for each j then $x_j = y_j$.*

Proof. This follows from 2.1 and 1.19.

We now consider the case where $\sigma(x)$ is contained in an interval.

Definition, Notation and Discussion. If a and b are finite real numbers let (a, b) denote the open interval $a < t < b$ and $[a, b]$ its closure. Let

$$M(a, b) = \{x : \sigma(x) \subset [a, b]\}, \qquad M(-\infty, a) = \{x : \sigma(x) \subset (-\infty, a]\} \text{ and}$$

$$M(b, \infty) = \{x : \sigma(x) \subset [b, \infty)\}.$$

Let $L(a, b)$ denote the closure of the range and $N(a, b)$ the nullspace of the operator $K(a, b)$ defined in 1.7. The *span* (denoted by $\vee \cdots$) of any collection of manifolds is the smallest closed linear manifold containing each of them. Two closed linear manifolds M and N are called *quasi-complementary* if $M \vee N = X$ and $M \cap N = (0)$. This is equivalent to the existence of a closed densely defined idempotent E which has M as range and N as nullspace [10, Lemma 10]. E will be called a *closed projection* or, simply, a *projection.*

E. R. Lorch in [9] showed that $L(a, b)$ and $N(a, b)$ are orthogonal complements in the case where T is a self-adjoint operator in a Hilbert space X. In this section we show that $M(a, b)$ and $N(a, b)$ are quasi-complements if X is reflexive and in Section 3 we prove that these manifolds are quasi-complements for an arbitrary space X in the case where T has a purely continuous spectrum.

2.3 LEMMA. *The vector x is in $N(a, b)$ if and only if $(a, b) \subset \rho(x)$.*

Proof. Suppose first that $K(a, b)x = 0$. Let $C(a, b)$ be an admissible contour and let μ be inside $C(a, b)$ and outside $[a, b]$. If $R(\mu)$ is applied to $0 = K(a, b)x$ (written as an integral over $C(a, b)$) the resolvent equation 0.3 yields the result

$$R(\mu)x = (\mu - a)^{-n}(\mu - b)^{-n}(2\pi i)^{-1} \int_{C(a,b)} (z - a)^n(z - b)^n R(z)x \, dz.$$

The right-hand side of this equation represents an analytic function of μ for μ inside $C(a, b)$, and an application of $\mu I - T$ to this expression yields x. This follows by an argument similar to that in the proof of 1.9. This expression is thus an analytic extension of $R(z)x$ into the interior of $C(a, b)$ so that (a, b) is in $\rho(x)$.

Conversely, suppose that $(a, b) \subset \rho(x)$. For real s and t with $a < s < t < b$, let $C(a, s)$, $C(s, t)$, $C(t, b)$ and $C(a, b)$ be admissible contours no two of which meet off the real axis and such that the first three are in $C(a, b)$. If $F(z)$ is defined as

$$(z - a)^n(z - s)^n(z - t)^n(z - b)^n R(z)x,$$

then

$$(2\pi i)^{-1} \int_{C(a,b)} F(z) \, dz = (2\pi i)^{-1} \cdot \left\{ \int_{C(a,s)} F(z) \, dz + \int_{C(s,t)} F(z) \, dz + \int_{C(t,b)} F(z) \, dz \right\}.$$

The analyticity of $\hat{x}(z)$ inside and on $C(s, t)$ shows that the integral over this contour is zero so that by 1.11(i) this equation can be written as

$$(T - sI)^n(T - tI)^n K(a, b)x = (T - tI)^n(T - bI)^n K(a, s)x + (T - aI)^n(T - sI)^n K(t, b)x.$$

Letting $s \to a+$ and using the closure of $(T - tI)^n(T - bI)^n$, by 0.4, and the fact that $K(a, s)x \to 0$, by 1.12, we get

$$(T - aI)^n(T - tI)^n K(a, b)x = (T - aI)^{2n} K(t, b)x.$$

The invertibility of $T - aI$ by 2.0 now yields $(T - aI)^n K(t, b)x = (T - tI)^n K(a, b)x$. Letting $t \to b-$ and using the closure of $(T - aI)^n$ and the fact that $K(t, b)x \to 0$ we get $0 = (T - bI)^n K(a, b)x$. Hence $K(a, b)x = 0$ by the invertibility of $T - bI$, Q.E.D.

2.4 LEMMA. *A necessary and sufficient condition for*

$$(*) \qquad (T - aI)^n(T - bI)^n x = K(a, b)x$$

is that x be in $N(c, d)$ for each interval (c, d) disjoint from (a, b).

Proof. Let x satisfy $(*)$. Apply $K(c, d)$ to this equation for any interval (c, d) which is disjoint from (a, b) and use 1.8(ii) and 2.0 to get $K(c, d)x = 0$.

Conversely, suppose x satisfies the condition. Consider any approximating vector $I_r x$ to x with $r > \max(|a|, |b|)$. Let $C_1 = C(-r, a)$, $C_2 = C(a,b)$ and $C_3 = C(b, r)$ be admissible contours in $C(-r, r)$. Then

$$\begin{aligned}(T - aI)^n(T - bI)^n I_r x &= (2\pi i)^{-1} \int_{C(-r,r)} (z - a)^n(z - b)^n [1 - (z/r)^2]^n R(z)x\,dz \\ &= \textstyle\sum_{j=1}^{3} (2\pi i)^{-1} \int_{C_j} (z - a)^n(z - b)^n [1 - (z/r)^2]^n R(z)x\,dz \\ &= p(T)K(-r, a)x + (I - T^2/r^2)^n K(a, b)x + q(T)K(b, r)x\end{aligned}$$

where $p(T)$ and $q(T)$ are polynomials in T by 1.11(i). By the assumption, $K(-r, a)x = K(b, r)x = 0$ so that

$$(T - aI)^n(T - bI)^n I_r x = (I - T^2/r^2)^n K(a, b)x.$$

Letting $r \to \infty$ and using the closure of $(T - aI)^n(T - bI)^n$ we get $(T - aI)^n(T - bI)^n x = K(a, b)x$, Q.E.D.

2.5 THEOREM. *$M(a, b)$ consists precisely of those vectors which satisfy equation $(*)$ of Lemma 2.4.*

Proof. x is in $M(a, b)$ if and only if $\rho(x) \supset (-\infty, a) \cup (b, \infty)$. The result now follows from 2.3 and 2.4.

2.6 COROLLARY. *$M(a, b)$ is a closed linear manifold which is invariant under T.*

Proof. This follows from 1.4 and 1.6, because any vector which satisfies $(*)$ of 2.4 is in the domain of T.

These results imply the existence of (non-trivial) invariant subspaces for T. For a different proof in the case of a bounded operator see [2, Theorem 6].

2.7 THEOREM. *If $X \neq (0)$ then T has non-trivial invariant subspaces.*

Proof. We assume that T has no eigenvalues, otherwise the eigenspaces of T meet the requirements. If T is an unbounded operator, then since $X \neq (0)$ and X is the span of the manifolds $M(-r, r)$, $r > 0$ by 1.13, it follows that at least one of the manifolds $M(a, b)$ is different from X and (0). If T is bounded we may assume that $\sigma(T)$ has only one component; for, otherwise, there are invariant spaces of the form $(2\pi i)^{-1} \int_C R(z)X\,dz$ where C is a contour about one component which excludes the others. The problem thus is reduced to the case where $\sigma(T) = [a, b]$, a bounded interval. A straight-forward argument then shows that for any c between a and b at least one of $M(a, c)$ and $M(c, b)$ is different from X and (0).

2.8 *Remark.* "One-sided" improper integrals. Suppose x is such that $(s, t) \subset \rho(x)$. Then for any a and b with $s < a < t < b$ the integral

$$Y(a, b; f) = \int_{C(a,b)} f(z)(z - b)^n R(z)x\, dz$$

exists for any entire function f and satisfies:

(i) $Y(a, b; f)$ is independent of a provided that $s < a < t$ and
(ii) $\sigma(Y(a, b; f)) \subset [t, b]$.

Similarly, integrals of the form $\int_{C(a,b)} f(z)(z - a)^n R(z)x\, dz$ exist in the case where $a < s < b < t$ and have analogous properties (i′) and (ii′).

2.9 Lemma. $N(a, b) = M(-\infty, a) \vee M(b, \infty)$.

Proof. If x is in $N(a, b)$, then $(a, b) \subset \rho(x)$ by 2.3. Consider any approximating vector $I_r x$ to x with $r > \max(|a|, |b|)$. Choose any s and t with $a < s < t < b$ and admissible contours $C_1 = C(-r, s)$, $C_2 = C(s, t)$ and $C_3 = C(t, r)$ which are in $C(-r, r)$. Then $I_r x = x_1 + x_2 + x_3$, where

$$x_j = (2\pi i)^{-1} \int_{C_j} [1 - (z/r)^2]^n R(z)x\, dz.$$

Since $\hat{x}(z)$ is analytic inside and on C_2, $x_2 = 0$. By 2.8(ii) and (ii′), $\sigma(x_1) \subset [-r, a]$ and $\sigma(x_2) \subset [b, r]$ so that $I_r x$ is in $M(-\infty, a) + M(b, \infty)$. Letting $r \to \infty$, we get x is in $M(-\infty, a) \vee M(b, \infty)$ so that

$$N(a, b) \subset M(-\infty, a) \vee M(b, \infty).$$

For the reverse inclusion we show that $N(a, b)$ contains $M(-\infty, a)$ and $M(b, \infty)$. Let x be in $M(-\infty, a)$ and consider any approximating vector $I_r x$ with $r > |a|$. As $\sigma(I_r x) \subset [-r, a]$ by 1.11(iii), it follows from 2.3 that $K(a, b)I_r x = 0$. By the continuity of $K(a, b)$ it follows that $K(a, b)x = 0$ so that $N(a, b) \supset M(-\infty, a)$. Similarly, $N(a, b) \supset M(b, \infty)$, so that $N(a, b)$, by its closure, contains $M(-\infty, a) \vee M(b, \infty)$, Q.E.D.

2.10 Theorem. *There exists a closed idempotent which has $M(a, b)$ as its range and $N(a, b)$ as its nullspace.*

Proof. Define $Ex = (T - aI)^{-n}(T - bI)^{-n}K(a, b)x$ for each x for which this expression exists and let $D(E)$ denote the domain of E. Let x be in $D(E)$ and let $y = Ex$ so that $(T - aI)^n(T - bI)^n y = K(a, b)x$. Applying $K(a, b)$ to this and using 1.8(i) we get

$$(T - aI)^n(T - bI)^n K(a, b)y = (T - aI)^n(T - bI)^n K(a, b)x$$

so that by 2.0 $K(a, b)y = K(a, b)x$. Hence

$$(T - aI)^{-n}(T - bI)^{-n}K(a, b)y = y,$$

or $Ey = y$, i.e., $E^2x = Ex$, so that E is an idempotent on $D(E)$. $M(a, b)$ is the range of E, as $Ex = x$ is equivalent to $(^*)$ of 2.4 and $N(a, b)$ is, clearly, the nullspace of E. The closure of E follows from that of $M(a, b)$ and of the operator $(T - aI)^n(T - bI)^n$.

2.11 LEMMA. *If $a < b < c$, then $N(a, c) = N(a, b) \cap N(b, c)$.*

Proof. Let $C(a, b)$ and $C(b, c)$ be admissible contours in $C(a, c)$ and let

$$F(z) = (z - a)^n(z - b)^n(z - c)^nR(z).$$

Then

$$(2\pi i)^{-1}\int_{C(a,c)} F(z)\, dz = (2\pi i)^{-1}\int_{C(a,b)} F(z)\, dz + (2\pi i)^{-1}\int_{C(b,c)} F(z)\, dz.$$

By 1.10 this equation can be written

$$(T - bI)^nK(a, c) = (T - cI)^nK(a, b) + (T - aI)^nK(b, c).$$

The assertion follows from this identity and from 2.3 and 2.0.

Since $M(a, b) \cap N(a, b) = (0)$, the sets $M(a, b)$ and $N(a, b)$ are candidates for quasi-complements in X. However, it seems difficult to show that their span is X without making further assumptions, such as (1) X is reflexive or (2) T has a purely continuous spectrum. This section concludes with a result in case (1). Section 3 is concerned with obtaining the result in case (2).

2.12 THEOREM. *Let X be reflexive and let T and its adjoint T^* have empty point spectrum. Then $M(a, b)$ and $N(a, b)$ are quasi-complements in X. Moreover, if $\cdots a_{j-1} < a_j < a_{j+1} \cdots$ is any partition of $(-\infty, \infty)$ with $+\infty$ and $-\infty$ as the only limit points of the a_j, then $X = \bigvee_j M(a_{j-1}, a_j)$.*

Proof. Let $L^*(a, b)$ be the closure of the range and $N^*(a, b)$ be the nullspace of $K(a, b)^*$, the adjoint of $K(a, b)$. As X is reflexive, T^* has the same properties as T; viz., (i) T^* is closed and densely defined in X^*, (ii) T^* has a real (and eigenvalue-free) spectrum and (iii) its resolvent $R(z; T^*)$ satisfies (G_n), since $R(z; T^*) = R(z; T)^*$ and $R(z) = R(z; T)$ is a bounded operator. Hence since $L(a, b) \cap N(a, b) = (0)$ by 2.10 and the fact that $L(a, b) \subset M(a, b)$ it follows that $L^*(a, b) \cap N^*(a, b) = (0^*)$. Taking the orthogonal complement of this in X we get X is the span of $N(a, b)$ and $L(a, b)$ so that $X = N(a, b) \vee M(a, b)$. By 2.10 it follows that $M(a, b)$ and $N(a, b)$ are quasi-complements.

If x is in $N(a_{j-1}, a_j)$ for every j, then $x = 0$ by 2.3 and 2.1. Thus $\bigcap_j N^*(a_{j-1}, a_j) = (0^*)$ by symmetry in X^*. Taking orthogonal complements in X we get $\bigvee_j L(a_{j-1}, a_j) = X$ so that $\bigvee_j M(a_{j-1}, a_j) = X$, Q.E.D.

Section 3. Assumption that T has a purely continuous spectrum; the "resolution of the identity" for T

3.0. Throughout this section we will assume, in addition to its other properties, that T has a pure continuous spectrum.

This will guarantee that certain direct sums will be dense in X. For convenience, we use the following notation: let $D_0 = X$, $D_1 = D(T)$ and inductively for $k > 1$,

$$D_k = \{x : x, Tx, \cdots, T^{k-1}x \text{ are in } D(T)\}.$$

The lemma which follows is a direct consequence of 0.3 and induction on m. In fact, the asserted set inclusion can be replaced by an equality.

3.1 Lemma. *If z is non-real, then $R(z)D_m \subset D_{m+1}$, $m = 0, 1, 2, \cdots$.*

3.2 Lemma. *The set $(T - a_1 I)(T - a_2 I) \cdots (T - a_m I)D_m$ is dense in X for any real numbers $a_1, a_2, \cdots, a_m$.*

Proof. Introduce the "Cayley transform" operator defined by

$$V = (iI - T)(iI + T)^{-1} = (T - iI)R(-i).$$

Since V is closed and has X as domain, it is bounded. If

$$k_j = (i - a_j)(i + a_j)^{-1},$$

then a computation yields $V - k_j I = (1 + k_j)(T - a_j I)R(-i)$. By assumption T has only continuous spectrum, hence $(V - k_j)X = (T - a_j I)D_1$ is dense in X. An induction argument and 3.1 complete the proof.

3.3 Theorem. *The manifolds $M(-\infty, a)$ and $M(a, \infty)$ are quasi-complements in X, for each real a.*

Proof. By 2.1, $M(-\infty, a) \cap M(a, \infty) = (0)$. Let $D' = (T - aI)^n D_n$ and let D'' be the set of all vectors y of the form

$$y = (T - aI)^n I_r x = (2\pi i)^{-1} \int_{C(-r,r)} (z - a)^n [1 - (z/r)^2]^n R(z)x \, dz$$

for all x in X and positive r. By 3.2, D' is dense in X. By 1.21, the elements of D' are approximable by vectors in D'', so that D'' is dense in X. We will be finished if we show that each element of D'' is in $M(-\infty, a) + M(a, \infty)$. For this let $y = (T - aI)^n I_r x$ be a vector in D''. If $r \leq a$, then

$$\sigma(x) \subset [-r, r] \subset [-r, a]$$

so that y is in $M(-\infty, a)$. If $-r \geq a$, then $\sigma(x) \subset [a, r]$, so that y is in $M(a, \infty)$. Finally if $-r < a < r$, then $y = y' + y''$ where

$$y' = (2\pi i)^{-1} \int (z - a)^n [1 - (z/r)^2]^n R(z)x \, dz$$

is in $M(-\infty, a)$ and

$$y'' = (2\pi i)^{-1} \int (z - a)^n [1 - (z/r)^2]^n R(z)x \, dz$$

is in $M(a, \infty)$, the integrals being taken over admissible contours $C(-r, a)$ and $C(a, r)$, resp., in $C(-r, r)$. Hence

$$X = \mathrm{Cl}\,(D'') = M(-\infty, a) \vee M(a, \infty).$$

3.4 *Remark.* By 3.3, there exists a closed projection $E(a)$ which has $M(-\infty, a)$ as range and $M(a, \infty)$ as nullspace. We call the family

$$\{E(t) : -\infty < t < \infty\}$$

so obtained the *resolution of the identity for* T. Let $M(E)$ denote the range and $N(E)$ the nullspace of any projection E. The family satisfies:

(1) if $s < t$ then $E(s) \leq E(t)$; i.e., $M(E(s)) \subset M(E(t))$ and $N(E(s) \supset N(E(t))$,

(2) any bounded linear operator which commutes with T commutes with each projection of the family and

(3) $\lim_{t\to\infty} E(t)x = x$ and $\lim_{t\to-\infty} E(t) = 0$, for a dense set of vectors.

The first assertion is obvious. The second follows from 1.3. For (3), let x have bounded spectrum, say $\sigma(x) \subset [a, b]$. Then x is in $M(-\infty, t)$ for all $t \geq b$ and x is in $M(s, \infty)$ for all $s \leq a$. Hence (3) holds (at least) on the dense set of vectors with compact spectra.

Theorem 3.3 has the following generalization.

3.5 THEOREM. *Let* $\{a_1, a_2, \cdots, a_m\}$ *be any set of real numbers with* $a_{j-1} < a_j$. *Then*

$$(0) = M(-\infty, a_1) \cap M(a_1, a_2) \cap \cdots \cap M(a_m, \infty)$$

and

$$X = M(-\infty, a_1) \vee M(a_1, a_2) \vee \cdots \vee M(a_{m-1}, a_m) \vee M(a_m, \infty).$$

Proof. Let

$$D' = (T - a_1 I)^n (T - a_2 I)^n \cdots (T - a_m I)^n D_{mn}$$

and let D'' be the set of all vectors of the form

$$y = (T - a_1 I)^n \cdots (T - a_m I)^n I_r x$$

for x in X, $r > 0$, and then proceed as in the proof of 3.3.

3.6 COROLLARY. *If* $a < b$, *then* $M(a, b)$ *and* $N(a, b)$ *are quasi-complements in* X.

Proof. $X = M(a, b) \vee (M(-\infty, a) \vee M(b, \infty)) = M(a, b) \vee N(a, b)$ by 3.5 and 2.9, while $(0) = M(a, b) \cap N(a, b)$ by 2.10.

This shows that the closed indempotent $E = E(a, b)$ introduced in (the proof of) 2.10 is a projection. The next result shows the relation of $E(a, b)$ to the spectral family of 3.4.

3.7 THEOREM. *If $a < b$, then $E(b) - E(a)$ is a densely defined idempotent which has $M(a, b)$ as its range, $M(-\infty, a) + M(b, \infty)$ as its nullspace and $E(a, b)$ as its closure.*

Proof. If P and Q are idempotents such that $P \leq Q$ (i.e., such that $M(P) \subset M(Q)$ and $N(P) \supset N(Q)$) then $Q - P$ is an idempotent which has $M(Q) \cap N(P)$ as range and $M(P) + N(Q)$ as nullspace. Hence $E(b) - E(a)$ is an idempotent with $M(-\infty, b) \cap M(a, \infty) = M(a, b)$ as range and $M(-\infty, a) + M(b, \infty)$ as nullspace. The proof of 3.6 shows that $E(b) - E(a)$ has a dense domain. Thus taking closures in the relation $E(b) - E(a) \subseteq E(a, b)$, we get $[E(b) - E(a)] = E(a, b)$.

3.8 COROLLARY. *The projections $E(t)$ are pairwise densely defined.*

3.9 *Remark.* More generally, any finite subset of $\{E(t) : -\infty < t < \infty\}$ has a common dense domain. For, suppose $t_1 < t_2 < \cdots < t_m$ and let $D = \bigcap_{j=1}^{m} D(E(t_j))$. Let x be any vector of the form $x = \sum_{j=0}^{m} x_j$ with x_0 in $M(-\infty, t_1)$, x_m in $M(t_m, \infty)$ and x_j in $M(t_j, t_{j+1})$ for $j \neq 0, m$. Then each x_j is in D and $E(t_k)x_j$ is 0 for $k < j$ and is x_j for $k \geq j$. Thus

$$M(-\infty, t_1) + M(t_1, t_2) + \cdots + M(t_m, \infty)$$

is contained in D and hence D is dense in X by 3.5.

The family $\{E(t) : -\infty < t < \infty\}$ is seen to have many of the properties of the classical resolution of the identity for a self-adjoint transformation in a Hilbert space. The results above could now be used to obtain the integral representation of such a transformation in the case of a pure continuous spectrum. The details are given in the author's dissertation.

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