

# HOMOTOPY GROUPS OF THE SPACE OF HOMEOMORPHISMS ON A 2-MANIFOLD

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## 1. Introduction

This is the final paper in a series of papers concerning the homotopy groups of the space of homeomorphisms on a 2-manifold. If  $M$  is a compact 2-manifold with boundary  $M'$ , and  $K$  is a closed subset, denote by  $H(M, K)$  the space of homeomorphisms of  $M$  onto itself leaving  $K$  pointwise fixed and by  $H_0(M, K)$  its identity component. Kneser proved [14] that the space of rigid motions on  $S^2$  is a deformation retract of  $H_0(S^2)$ . Thus  $\pi_n H_0(S^2) = \pi_n(P^3)$  for each  $n$ ,  $\pi_n H_0(S^2) = \pi_n(S^3)$  for  $n > 1$ , and  $\pi_n H_0(S^2) = \pi_n(S^2)$  for  $n > 2$ . In particular  $\pi_1 H_0(S^2) = Z_2$  and  $\pi_n H_0(S^2) = 0$ . If  $M$  is a disc with holes or a Moebius strip,  $H_0(M, M')$  is homotopically trivial ([6], [8] and [12]). In fact Alexander's classic result [1] that the space of homeomorphisms of an  $n$ -cell onto itself leaving the boundary pointwise fixed is contractible and locally contractible is a most important tool in the study of these problems. If  $M$  is a torus,  $\pi_i H_0(M) = \pi_i(M)$  for each  $i$ , and if  $M$  is a torus with the interiors of a finite number of disjoint discs removed,  $H_0(M, M')$  is homotopically trivial [11]. For real projective space,  $\pi_i H_0(P^2) = \pi_i(P^2)$  for  $i > 2$ ,  $\pi_2 H_0(P^2) = 0$ ,  $\pi_1 H_0(P^2) = Z_2$ ,  $\pi_1 H_0(P^2, x) = Z$ , where  $x \in P^2$  and  $\pi_i H_0(P^2, x) = 0$  for  $i > 1$  (see [12]). For the Klein bottle  $K$ ,  $\pi_i H_0(K) = 0$  for  $i > 1$ ,  $\pi_1 H_0(K) = Z$  and  $\pi_i H_0(K, x) = 0$  for each  $i$  [12]. In this present paper, it is shown that  $H_0(M)$  is homotopically trivial for all compact 2-manifolds (without boundary) of genus greater than 1, if orientable, and greater than 2, if non-orientable; and that, if  $M$  is a compact 2-manifold with non-empty boundary,  $H_0(M, M')$  is homotopically trivial.

Further related results may be found in McCarty's paper [16], where he proves among other things, that

$$\pi_1 H_0(S^2, x) = \pi_1 H_0(S^2, x \cup y) = Z$$

and

$$\pi_i H_0(S^2, x) = \pi_i H_0(S^2, x \cup y) = 0 \quad \text{for } i > 1$$

and that if  $K$  is a finite subset of  $S^2$  with more than two points  $H_0(S^2, K)$  is homotopically trivial. Quintas proves in [18] that if  $M$  is an orientable compact manifold with two or more handles or is non-orientable with three or more cross-caps and  $M_k$  is the manifold obtained from  $M$  by deleting  $k$  points,  $\pi_n H_0(M) = \pi_n H_0(M_k)$  for each  $n$ . It thus follows from McCarty's work that in this case  $\pi_n H_0(M) = \pi_n H_0(M, x)$ .

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Received February 15, 1965.

<sup>1</sup> Presented to the American Mathematical Society, April 20, 1964. Research supported in part by the National Science Foundation.

The interested reader is referred also to [6] and [15], where local homotopy properties of these spaces are studied and to Fisher's paper [4], where further historical remarks may be found.

This paper is constructed in the following way. In Section 2, the relation between  $H_0(M)$  and  $H_0(M, x)$  is studied. In Section 3 is found the main result for the double torus. In order not to disturb the continuity of the argument, certain parts of the proof are gathered together as lemmas in Section 4. Finally, the results for surfaces of higher genus and for non-orientable surfaces are to be found in Section 5. The proofs there are merely outlined, for the main ideas for the arguments are in Sections 3 and 4 and can be carried over without essential change.

### 2. Relation between $H_0(M)$ and $H_0(M, x)$

McCarty proves in [16] that if  $M$  is a compact 2-manifold and  $x$  is a point of  $M$ , then  $H(M)$  is a fiber space over  $M$  with fiber  $H(M, x)$ , where the projection map  $p$  is such that  $p(f) = f(x)$ . If  $M$  is neither  $S^2$  nor  $P^2$ ,  $\pi_n(M) = 0$  for  $n > 1$ , so it follows from the properties of the homotopy sequence of this fibering that  $\pi_n H(M) = \pi_n H(M, x)$  for each  $n > 1$ . Part of this homotopy sequence is

$$\cdots \rightarrow \pi_2(M, x) \xrightarrow{d_2} \pi_1 H(M, x) \xrightarrow{i_1} \pi_1 H(M) \xrightarrow{p_1} \pi_1(M, x) \rightarrow \cdots .$$

McCarty observes that  $p_1(\pi_1 H(M))$  is central in  $\pi_1(M, x)$  (Remarks 5.24, p. 302 of [16]). Using a result of Baumslag [2, Theorem 3] Quintas notes in [18] that if  $M$  is orientable and has two or more handles, then  $\pi_1(M)$  has trivial center. It thus follows that  $\pi_1 H(M, x) = \pi_1 H(M)$ . This is, of course, false if  $M$  is a 2-sphere, torus, Klein bottle or  $P^2$ . Quintas argument goes through if  $M$  is non-orientable and has three or more cross caps. Griffiths, in [5], proves that, in this case,  $\pi_1(M)$  has trivial center (see p. 10, Theorem 4.4). Thus, for all compact manifolds,  $M$ , with two or more handles or three or more cross caps,  $\pi_n H(M) = \pi_n H(M, x)$  for each  $n$ .

### 3. The double torus

Let  $M$  be a double torus. This section is devoted to the proof that  $H_0(M)$  is homotopically trivial. Since it is the injection map of  $H(M, P)$  into  $H(M)$  that induces the isomorphism of  $\pi_k H(M, P)$  onto  $\pi_k H(M)$ , it will suffice to prove that every map of  $S^k$  into  $H_0(M, P)$  is homotopic to 0 in  $H_0(M)$ . For  $i = 1, 2$  let  $M_i$  denote a torus from which has been removed the interior of a disc. Sew these tori together along their boundaries to obtain  $M$  and denote the now common boundary of  $M_1$  and  $M_2$  by  $C$ . Let  $a_i$  and  $b_i$  be simple closed curves in  $M_i$  meeting only at a point  $P$  of  $C$ , one being a meridian, the other a longitude of  $M_i$  and let  $a_i^*, b_i^*$  be elements of  $\pi_1(M_i, P)$  determined by homeomorphisms of  $S^1$  onto  $a_i$  and  $b_i$ . The  $a_i^*$  and  $b_i^*$  may also be considered as generators of  $\pi_1(M, P)$  (hereafter, the  $P$  is dropped), where

$$a_1^* b_1^* a_1^{*-1} b_1^{*-1} = a_2^* b_2^* a_2^{*-1} b_2^{*-1} = c^*, \quad 2$$

the class of a homeomorphism of  $S^1$  onto  $C$ . Finally, let  $\omega$  be a mapping of  $M$  onto the union of two tori  $T_1$  and  $T_2$  with common point  $c$  such that  $\omega(C) = c$ , and  $\omega(M_i - C)$  is a homeomorphism onto  $T_i - C$ . Denote the images of  $a_i^*$  and  $b_i^*$  in  $\pi_1(T_1 \cup T_2)$  under the induced homomorphism by  $a_i^*$  and  $b_i^*$  also. Then  $\pi_1(T_1 \cup T_2)$  is the free product of  $\pi_1(T_1)$  and  $\pi_1(T_2)$ ,  $a_i^*$  and  $b_i^*$  generating the abelian rroup  $\pi_1(T_i)$ .

Suppose that  $f$  is a mapping of  $S^k$  into  $H_0(M, P)$ . Let  $\mathfrak{f}$  be an arc on  $f(y)(C)$  that lies, except for its endpoints, which lie on  $C$ , in  $M - C$  and which does not, when joined to either component of  $C - \mathfrak{f}$ , bound a disc in  $M$ . Call such an arc an *arc of type  $\alpha$* . The mapping  $\omega$  associates  $\mathfrak{f}$  with a simple closed curve in  $T_1$  or  $T_2$  and thus, the direction on  $\mathfrak{f}$  being derived from that on  $[f(y)]^{-1}(\mathfrak{f})$ ,  $\omega$  associates  $\mathfrak{f}$  with an element  $W(\mathfrak{f})$  of either  $\pi_1(T_1)$  or  $\pi_1(T_2)$ . Suppose that  $\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_n$  are the consecutive arcs of type  $\alpha$  in  $f(y)(C)$ . Then  $W(\mathfrak{f}_i) \neq 1$ . However, in  $\pi_1(T_1 \cup T_2)$ ,

$$W(\mathfrak{f}_1)W(\mathfrak{f}_2) \cdots W(\mathfrak{f}_n) = 1$$

(since  $f(y)$  is in the identity component  $H_0(M)$ , of  $H(M)$ ). This implies that for some  $j, \text{ mod } n$ ,  $\mathfrak{f}_j$  and  $\mathfrak{f}_{j+1}$  lie in the same one of  $M_1$  and  $M_2$  and the arc  $t_j$  in  $f(x)(C)$  from the last point of  $\mathfrak{f}_j$  to the first point of  $\mathfrak{f}_{j+1}$  is such that  $\omega(t_j)$  is a union of simple closed curves each bounding a disc in  $T_1 \cup T_2$ .

Let  $n(y)$  denote the number of arcs of type  $\alpha$  in  $f(y)(C)$  and let  $N = \max n(y), y \in S^k$ . It is clear that the set  $K$  of all  $y$  in  $S^k$  such that  $n(y) = N$  is closed, for if  $\mathfrak{f}_i \subset f(y_i)(C)$  and is an arc of type  $\alpha$  and  $\{y_i\}$  converges to  $y$ , then some subsequence of  $\{\mathfrak{f}_i\}$  converges to an arc in  $f(y)(C)$  that contains an arc of type  $\alpha$ . Also, if  $h_i$  is an arc in  $f(y_i)(C), y_i \in K$ , that does not lie in an arc of type  $\alpha$ , but does lie, except for its endpoints, which lie in  $C$ , in  $M_1 - C$  or  $M_2 - C$ , then  $\{h_i\}$  does not converge to a set containing an arc of type  $\alpha$ , since then,  $n(y) > n(y_i)$  for some  $i$ . It is seen then, that for each component  $L$  of  $K$ , if  $x, y \in L$  and the arcs of type  $\alpha$  in  $f(x)(C)$  and  $f(y)(C)$  are  $\mathfrak{f}_{1x}, \dots, \mathfrak{f}_{Nx}$  and  $\mathfrak{f}_{1y}, \dots, \mathfrak{f}_{Ny}$ , then  $W(\mathfrak{f}_{ix}) = W(\mathfrak{f}_{iy})$ . (It is necessary to note that the convergence of  $\{f(y_i)(C)\}$  to  $f(y)(C)$  is equicontinuous.)

Suppose  $N > 0$ . For  $x \in K$ , let  $T_x$  denote the union of the arcs of type  $\alpha$  in  $f(x)(C)$ . Add onto the closure of each component  $l$  of  $M_1 \cap [f(x)(C) - T_x]$  the discs in  $M_1$  bounded by parts of  $l$  and parts of  $C$ . Call the resulting set  $D_l$ . If  $l$  has two limit points on the same arc  $\mathfrak{f}$  of type  $\alpha, \mathfrak{f} \cup l = f(x)(C)$ . Since  $W(\mathfrak{f}) \neq 1$  and  $f(x)$  is isotopic to the identity, this is impossible. Also, if  $l$  has limit points on two arcs,  $\mathfrak{f}_1$  and  $\mathfrak{f}_2$ , of type  $\alpha$ , then one of the components of  $f(x)(C) - (\mathfrak{f}_1 \cup \mathfrak{f}_2)$  lies entirely in  $M_1$ . For  $x \in K$ , consider the closure of the union of these  $D_l$ . The components of this and the points of  $M_1$  not in it yield an upper semicontinuous decomposition  $G_x$  of  $M_1$ . No non-degenerate element separates  $M_1$  or  $C$ , lies in  $M_1 - C$  or contains  $C$  (since  $N > 0$ ). Thus the decomposition space is  $M_1$  and the non-degenerate elements correspond to boundary points. If  $x \in S^k - K$ , let  $G_x$  be the decomposition of  $M_1$  into its own points.

In  $M_1 \times S^k$ , the collection  $\{g \times x\}$ , for  $g \in G_x, x \in S^k$ , is an upper semicontinuous decomposition  $G$  of  $M_1 \times S^k$ . It follows from Lemma 4.3 that the hyperspace  $X$  of this decomposition is  $M_1 \times S^k$ . Let  $A$  denote an annulus in  $M_1$  bounded by  $C$  and another simple closed curve,  $C'$ . There is a homeomorphic mapping of  $A \times S^k$  into the hyperspace of  $G$  such that  $A \times x$  goes into  $G_x \times x$  and the only non-degenerate elements in the image are images of points of  $C \times S^k$ . This yields a homeomorphism  $\eta$  of  $A \times S^k$  into  $M_1 \times S^k$  such that for no  $x$  does  $\eta(C' \times x)$  meet a non-degenerate element of  $G_x \times x$  (Corollary to Lemma 4.2). Let  $A(x)$  denote the annulus  $\pi\eta(A \times x)$ , where  $\pi$  is the projection map of  $M_1 \times S^k$  into  $M_1$ .

Remove from  $A(x) \times x$  the non-degenerate elements of  $G_x \times x$ . Remove the components of  $(f(x)(C) \cap A(x)) \times x$  that intersect  $C' \times x$  but not  $C \times x$  and remove the discs bounded by parts of these components and parts of  $C' \times x$ . Some components of  $(f(x)(C) \cap A(x)) \times x$  have both endpoints on  $C' \times x$ . Let  $s$  be such a component and let  $a$  and  $b$  denote the first and last points of  $s$  on  $C \times x$ . If  $a = b$ , let  $p_{sx} \times x = a$ . If  $a \neq b$ , the portion of  $s$  between  $a$  and  $b$  can be isotopically deformed in  $A(x) \times x$  into an arc on  $C \times x$  with endpoints  $a$  and  $b$  under an isotopy leaving  $a$  and  $b$  fixed. Denote this portion of  $C \times x$  by  $p_{sx} \times x$ . Remove all of  $s$  and the discs bounded by portions of it and of  $C' \times x$  or  $C \times x$  except the points of  $f(x)(C) \times x$  on  $p_{sx} \times x$ . Remove  $C' \times x$  and all of  $C \times x$  except the points of the various sets  $p_{sx} \times x$ . Call the resulting set  $A'(x) \times x$ . The only points of  $A'(x) \times x$  that are not interior points of  $A(x) \times x$  are on the sets  $p_{sx} \times x$ .

Let  $C(x) \times x$  be a simple closed curve in  $A'(x) \times x$  that bounds, together with  $C' \times x$ , an annulus in  $A(x) \times x$ . The curve  $C(x) \times x$  meets  $C \times x$  in the arcs  $p_{sx} \times x$  described above. Note that if  $C$  were replaced by  $C(x)$ , (in the sense that arcs of type  $\alpha$  on  $f(x)(C)$  are to be considered as arcs  $\mathfrak{f}$  in  $f(x)(C)$  that lie except for their endpoints in  $M - C_x$  and that do not, when joined to either component of  $C_x - \mathfrak{f}$ , bound a disc in  $M$ ) the number of arcs of type  $\alpha$  in  $f(x)(C)$  will not increase and will decrease if  $x \in K$  and if, as a result of the fact that  $W(\mathfrak{f}_1)W(\mathfrak{f}_2) \cdots W(\mathfrak{f}_n) = 1$ , the consecutive arcs  $\mathfrak{f}_j$  and  $\mathfrak{f}_{j+1}$  that lie in the same one of  $M_1$  and  $M_2$  lie, in fact, in  $M_2$ . The only way this number can increase is for  $C(x) \times x$  to meet a component of  $(f(x)(C) \cap A(x)) \times x$  that doesn't meet  $C \times x$  and this has been avoided by construction. If the  $\mathfrak{f}_j$  and  $\mathfrak{f}_{j+1}$  mentioned above lie in  $M_2$ , the arc  $t_j$  (defined earlier in this section) on  $f(x)(C)$  between  $\mathfrak{f}_j$  and  $\mathfrak{f}_{j+1}$  has the property that either it lies in  $M_2$  or each piece of it in  $M_1$  bounds, together with a piece of  $C$ , a disc in  $M_1$ . The construction prevents  $C(x)$  from meeting such pieces; thus  $\mathfrak{f}_j \cup \mathfrak{f}_{j+1} \cup t_j$ , which is an arc, lies in one of the components of  $M - C(x)$  and the number of arcs of type  $\alpha$  is reduced.

Now it follows from Lemma 4.4 that there is a mapping  $\phi$  of  $C \times S^k$  into  $M_1$  such that for each  $x, \phi|C \times x$  is a homeomorphism into  $A'(x)$ . (The map  $\phi$  is the composition of  $\phi^*$  of Lemma 4.4 and the projection of  $M_1 \times S^k$  into  $M_1$ .) The simple closed curve  $\phi(C \times x)$  has the properties described for

$C(x)$  above. Lemma 4.5 implies that there is a mapping  $\Phi$  (the composition of  $\Phi^*$  of Lemma 4.5 with the projection on  $M_1$ ) of  $M_1 \times S^k \times I$  into  $M_1$  such that  $\Phi|_{M_1 \times x \times t}$  is a homeomorphism,  $\Phi(y, x, 0) \in \phi(C \times x)$  for each  $y$  in  $C$ , and  $\Phi(y, x, 1) = y$  for each  $y$  in  $M_1$ . Thus there is a mapping  $\psi$  of  $S^k \times I$  into  $H(M)$  such that

$$\psi(x, t)(M_1) \subset M_1, \psi(x, 0)(C) = \phi(C \times x), \text{ and } \psi(x, 1) = i.$$

Let

$$\psi^*(x, t) = \psi(x, 1 - t)[\psi(x, 0)]^{-1}f(x).$$

Then  $\psi^*(x, 1) = f(x)$  and  $\psi^*(x, 0) = [\psi(x, 0)]^{-1}f(x)$ . Since  $[\psi(x, 0)]^{-1}$  takes  $\phi(C \times x)$  into  $C$ , it is seen that the intersection of  $\psi^*(x, 0)(C)$  with  $C$  is precisely like that of  $f(x)(C)$  with  $\phi(C \times x)$ .

If for some  $x_1$  in  $K$ , there is a  $j$  such that  $\mathfrak{k}_j$  and  $\mathfrak{k}_{j+1}$  (this notation being that described earlier in this paper) lie in  $M_2$ , then this is true for all  $x$  in the component  $L$  of  $K$  containing  $x_1$ . (Note that for each  $x$  in  $K$ , there is a  $j$  such that  $\mathfrak{k}_j$  and  $\mathfrak{k}_{j+1}$  lie in the same one of  $M_1$  and  $M_2$ .) Therefore, if  $K$  contains such an  $x_1$ ,  $\psi^*(x, 0)(C)$  has fewer arcs of type  $\alpha$  than does  $f(x)(C)$ , for each  $x$  in  $L$ , and, in any case, for no  $x$  in  $S^k$  does  $\psi^*(x, 0)(C)$  have more arcs of type  $\alpha$  than does  $f(x)(C)$ . If  $M_2$  is considered in place of  $M_1$  in the above arguments, it is seen that  $\psi^*(x, 1)$  is homotopic in  $H(M)$  to a mapping  $f^*$  of  $S^k$  into  $H(M)$  such that the maximum number of arcs of type  $\alpha$  on the curves  $f^*(x)(C)$  is less than that on the curves  $f(x)(C)$ .

In the above argument,  $\psi^*(x, 0)(P)$  may not be constant, but for each  $x$ , it does lie in  $M_2$ . The map  $\rho$  of  $S^k$  into  $M_2$  such that  $\rho(x) = \psi^*(x, 0)(P)$  is homotopic to  $0$ , so it follows from the theorems of [9] that  $\psi^*(x, 0)$  is homotopic in  $H(M, C)$  to a mapping  $\psi^{**}$  such that  $\psi^{**}(x)(P)$  is constant for  $x$  in  $S^k$ . Consequently  $f^*(x)(P)$  may be presumed to be constant also.

This process is now repeated until  $f$  is homotopic in  $H$  to a mapping  $f_1$  for which no  $f_1(x)(C)$  has arcs of type  $\alpha$  and  $f_1(x)(P)$  is constant. The technique for constructing the annuli  $A(x)$  can now be applied to yield mappings  $\zeta_1$  and  $\zeta_2$  of  $C \times S^k$  into  $M_1$  and  $M_2$  respectively such that for each  $x$ ,  $\zeta_i|_{C \times x}$  is a homeomorphism, and  $f_1(x)(C)$  lies in the annulus  $B(x)$  bounded by  $\zeta_1(C, x)$  and  $\zeta_2(C, x)$ . It thus follows from Lemma 4.6 that there is a mapping  $\gamma$  of  $C \times S^k \times I$  into  $M$  such that  $\gamma|_{C \times x \times t}$  is a homeomorphism,  $\gamma(C \times x \times t) \subset B(x)$ ,  $\gamma(y, x, 0) = f_1(x)(y)$  for each point  $y$  in  $C$  and  $\gamma(y, x, 1) = y$ . The method used for obtaining  $\psi$  can now be repeated to yield a mapping  $\Psi$  of  $S^k \times I$  into  $H(M)$  such that for  $y \in C$ ,  $\Psi(x, t)(y) = \gamma(y, x, t)$ ,  $\Psi(x, 0) = f_1(x)$  and  $\Psi(x, 1) = i$ . Thus  $f$  is homotopic to the identity and  $\pi_k(H(M)) = 0$ .

### 4. Some lemmas

In this section, the notations introduced in the arguments of Section 3 will be used without further explanation.

LEMMA 4.1. *If  $q$  is a point of  $\text{int } M_1$  and  $J$  is a simple closed curve in  $\text{int } M_1$  that contains  $q$  and bounds, together with  $C$ , an annulus, then the identity component of  $H_{J,q}$ , the space of homeomorphisms of  $J$  into  $\text{int } M_1$  that leave  $q$  fixed and are extendable to elements of  $H(M_1)$  is homotopically trivial.*

*Proof.* It follows from Corollary 3 to Theorem 3.1 of [9] that  $H(M_1, q)$  is a fiber space with base space  $H_{J,q}$  and fiber  $H(M_1, J)$ . The identity components of  $H(M_1, q)$  and  $H(M_1, J)$  are homotopically trivial (Theorem 3 of [11]). Consequently, the properties of the exact homotopy sequence of this fibering imply that  $\pi_i(\pi_i(H_{J,q})) = 0$  for  $i \geq 2$ .

To see that  $\pi_1(H_{J,q}) = 0$ , Let  $\lambda$  be a mapping of  $S^1$  into  $H_{J,q}$ . Let  $C'$  be a simple closed curve in  $\text{int } M_1$  such that for each  $x$ ,  $\lambda(x)(J)$  and  $C'$  bound an annulus in  $M_1$  and let  $A$  be the annulus bounded by  $C$  and  $C'$ . It follows from Theorem 3, Corollary 4 of [6] that there is a homeomorphism  $\Lambda$  of  $A \times S^1$  into  $M_1 \times S^1$  such that  $\Lambda(y, x) \in M_1 \times x$ ,  $\Lambda(y, x) = (\lambda(x)(y), x)$  if  $y \in J$ , and  $\Lambda(C', x) = (C', x)$ . Thus there is constructed a map  $\tau$  of  $S^1 \times I$  into  $H_J$ , the space of homeomorphisms of  $J$  into  $M_1$  that extend to all of  $M_1$ , such that  $\tau(x, 0) = \lambda(x)$  and  $\tau(x, 1)(J) \subset C'$ . The map  $\sigma^*$  of  $S^1 \times I$  into  $M_1$  such that  $\sigma^*(x, t) = \tau(x, t)(q)$  demonstrates the fact that the map  $\sigma$  of  $S^1$  into  $C'$  such that  $\sigma(x) = \sigma^*(x, 1)(q)$  is homotopic to 0 in  $M_1$  and thus in  $C'$ . The techniques of [8] or [9] now show that it could be assumed of  $\tau$  that  $\tau(x, 1)(q)$  is constant and thus that  $\lambda$  is homotopic to 0 in  $H_J$ . It follows from the technique of Theorem 3.1 of [9] that  $H_J$  is a fiber space with base space  $\text{int } M_1$  and fiber  $H_{J,q}$ . The properties of this exact sequence demonstrate that the injection map of  $\pi_1(H_{J,q})$  into  $\pi_1(H_J)$  is 1-1. Consequently  $\lambda$  is homotopic to 0 in  $H_{J,q}$ .

These techniques also prove the

COROLLARY. *The identity component of  $H_J$  has the property that its homotopy groups of dimension greater than 1 vanish and its fundamental group is infinite cyclic.*

LEMMA 4.2. *Let  $X$  be the hyperspace of the decomposition  $G$  of  $M_1 \times S^k$  described in §3 and let  $z$  be the mapping of  $X$  onto  $S^k$  taking  $G_x \times x$  (the elements of  $G$  in  $M_1 \times x$ ) onto  $x$ . Then there is a map  $h$  of  $S^k$  into  $X$  such that for each  $x$ ,  $h(x) \in \text{int } z^{-1}(x)$ . (Note that the elements of  $\text{int } z^{-1}(x)$  are degenerate and thus are essentially points of  $M_1 \times x$ .)*

*Proof.* Each set  $\text{int } z^{-1}(x)$  is connected and locally connected. Also,  $z^{-1}(x)$  is homeomorphic to  $M_1$  and  $z$  is completely regular in the sense of [3] (see also [6]). The space  $X$  is complete. Thus  $\bigcup \text{int } z^{-1}(x)$  is complete. Also, the collection  $\{\text{int } z^{-1}(x)\}$  is lower semicontinuous. Michael's selection theorem [17] now implies the existence of  $h$  if  $k = 1$ . The proof now proceeds by induction on  $k$ —assume the map  $h$  exists for  $k = n$  and assume  $k = n + 1$ .

The sphere  $S^{n+1}$  can be regarded as  $S^n \times I$ ,  $S^n \times 0$  and  $S^n \times 1$  being reduced to points. Thus, each point  $x$  of  $S^{n+1}$  can be expressed as  $(y, t)$ ,  $y \in S^n$ ,

$t \in I$ . In the remainder of this proof, this notation for the points of  $S^{n+1}$  (the image of  $z$ ) will be used. Each set  $\text{int } z^{-1}(y, t)$  is  $LC^k$  for each  $k$ . The induction hypothesis yields for each  $t$  in  $I$  a map  $h_t$  of  $S^n \times t$  (this being regarded as a point if  $t = 0$  or  $1$ ) into  $\bigcup \text{int } z^{-1}(y, t)$  such that  $h_t(y, t) \in z^{-1}(y, t)$ . It follows from Michael's selection theorems that there are, for each  $t$ , an open set  $U_t$  containing  $S^n \times t$  and a map  $h_t^*$  of  $U_t$  into  $\bigcup \text{int } z^{-1}(y, t')$  that extends  $h_t$  and is such that  $h_t^*(y, t') \in \text{int } z^{-1}(y, t')$ . Thus there are numbers  $0 = t_0 < t_1 < \dots < t_m = 1$  and maps  $h_i$  of  $S^n \times [t_{i-1}, t_i]$  into  $X$  such that  $h_i(y, t) \in z^{-1}(y, t)$ .

In order to get  $h_{i-1}$  and  $h_i$  to agree on  $S^n \times t_{i-1}$ , a trick is used that will be useful later on. It may be assumed that  $h_{i-1}$  is actually defined on  $S^n \times (t_{i-2}, t'_{i-1})$ , where  $t_{i-1} < t'_{i-1} < t_i$ . Let  $J$  be a simple closed curve with an orientation and give each  $\text{bdry } z^{-1}(y, t)$  the orientation induced by that on  $C \times (y, t)$  as a subset of  $M_1 \times S^{n+1}$ . Let  $p$  be a point of  $J$ . The argument for Lemma 4.1 demonstrates that the space  $H_{y,t}$  of homeomorphisms of  $J$  into  $\text{int } z^{-1}(y, t)$ ,  $t_{i-1} \leq t \leq t'_{i-1}$ , taking  $p$  into  $h_{i-1}(y, t)$  and taking  $J$  into a curve oriented in the same way as  $\text{bdry } z^{-1}(y, t)$ , containing  $h_i(y, t)$  and bounding, together with  $\text{bdry } z^{-1}(y, t)$ , an annulus, is homotopically trivial. This space is  $LC^k$  for each  $k$ , as an argument exactly like that for Theorems 5.1 and 5.2 of [7] proves, and it is topologically complete. Finally,  $\bigcup H_{y,t}$  is topologically complete and the collection  $\{H_{y,t}\}$  is equi- $LC^k$  and lower semicontinuous. Michael's selection theorems now yield a homeomorphism  $h_i^*$  of

$$J \times S^n \times [t_{i-1}, t'_{i-1}] \text{ into } z^{-1}[S^n \times [t_{i-1}, t'_{i-1}]]$$

such that  $h_i^*(J \times y \times t) \subset z^{-1}(y, t)$ ,  $h_i^*(p, y, t) = h_{i-1}(h, t)$  and  $h_i^*(J \times y \times t)$  contains  $h_i(y, t)$ . These curves can be used to construct a map  $h'_i$  of  $S^n \times [t_{i-1}, t'_{i-1}]$  into  $X$  such that  $h'_i(y, t'_{i-1}) = h_i(y, t'_{i-1})$ . This map  $h'_i$  then effects a fitting together of  $h_{i-1}$  and  $h_i$  and the process can be repeated to yield the required  $h$ .

**COROLLARY.** *There is a homeomorphism  $\eta$  of  $C \times S^k$  into  $X$  such that  $\eta(C \times x) \subset \text{int } z^{-1}(x)$ , has the same orientation as  $\text{bdry } z^{-1}(x)$  and, with  $\text{bdry } z^{-1}(x)$ , bounds an annulus, and  $\eta(P, x) = h(x)$ . As noted in the statement of the lemma,  $\eta(C \times x)$  lies, essentially, in  $M_1 \times x$  and meets no non-degenerate element of  $G_x \times x$ .*

*Proof.* This follows from the arguments in the proof of Lemma 4.2.

**LEMMA 4.3.** *The space  $X$  is homeomorphic to  $M_1 \times S^k$ .*

*Proof.* Since the only non-degenerate elements of  $X$  are in  $\bigcup \text{bdry } z^{-1}(x)$ ,  $\eta$ , defined in the above corollary, may be considered as a map into  $M_1 \times S^k$ . There is also a homeomorphism  $\eta'$  of  $C \times S^k$  into  $M_1 \times S^k$  such that  $\eta'(C \times x)$  lies in  $M_1 \times x$ , is separated from  $C \times x$  by  $\eta(C \times x)$  and bounds, together with  $\eta(C \times x)$ , an annulus. Finally, there is a curve  $C'$  in  $M_1 - C$  such that  $C \cup C'$  bounds an annulus,  $A$ , and  $A \times S^k$  does not meet  $\eta(C \times S^k)$ . As in the argu-

ment for Lemma 4.1, there are homeomorphisms  $h'$  and  $h''$  ( $\Lambda$  of that Lemma) of  $A \times S^k$  into  $M_1 \times S^k$  such that  $h'(C \times x) = C' \times x$ ,  $h'(C' \times x) = h''(C \times x) = \eta(C \times x)$  and  $h''(C' \times x) = \eta'(C' \times x)$ . From this construction, it is easily seen that there is a homeomorphism  $h_2$  of  $\text{cl}(M_1 - A) \times S^k$  into  $M_1 \times S^k$  such that  $h_2(C' \times x) = \eta(C \times x)$ . Now, back in  $X$ , the proof of Lemma 4.1 yields a homeomorphism  $h_1$  of  $A \times S^k$  into  $X$  such that  $h_1(C \times x) = \text{bdry } z^{-1}(x)$  and  $h_1(C' \times x) = \eta(C \times x)$ . The fitting together of  $h_1$  and  $h_2$  yields a homeomorphism of  $M_1 \times S^k$  onto  $X$ .

LEMMA 4.4. *There is a homeomorphism  $\phi^*$  of  $C \times S^k$  into  $M_1 \times S^k$  such that for each  $x$ ,  $\phi^*(C \times x) \subset A'(x) \times x$ , the set defined in §3.*

*Proof.* Suppose that there is a map  $\mu$  of  $S^k$  into  $M_1 \times S^k$  such that  $\mu(x) \in A'(x) \times x$ . Let  $H(x)$  denote the space of homeomorphisms  $\beta$  of  $C$  into  $A'(x) \times x$  such that  $\beta(P) = \mu(x)$ ,  $\beta(C)$  separates  $C \times x$  from  $C' \times x$  and  $\beta(C)$  has, in the obvious sense, the same direction as  $C$ . The arguments in the proof of Lemma 4.1 or in the proof of Theorem 4.1 of [9] demonstrate that  $H(x)$  is homotopically trivial. The proofs of Theorems 5.1 and 5.2 of [7] demonstrate that it is  $LC^n$ . The set  $A'(x) \times x$  is topologically complete, so  $H(x)$  is topologically complete. Also,  $H^*(x)$ , the space of homeomorphisms of  $C$  into  $A(x) \times x$  with the properties of those in  $H(x)$  is topologically complete. Let  $H^*$  and  $H$  denote  $\bigcup H^*(x)$  and  $\bigcup H(x)$  respectively.

The space  $H^*$ , as a closed subset of the space of all homeomorphisms of  $C$  into  $M_1 \times S^k$ , is seen to be topologically complete. The subspace  $H_1^*$  of  $H^*$  consisting of mappings  $\beta$  such that for some  $x$ ,

$$\beta(C) \cap (C \times x) \supset (A'(x) \cap C) \times x,$$

is closed in  $H^*$  and is thus complete. The union of the sets  $(A'(x) \cap C) \times x$  is closed in  $M_1 \times S^k$ . It is therefore the intersection of a countable sequence  $U_1, U_2, \dots$  of open sets. Let  $H_{i1}^*$  be the subset of  $H_1^*$  consisting of these maps  $\beta$  such that for some  $x$ ,  $\beta(C) \cap (C \times x) \subset U_i$ . This is open on  $H_1^*$ . Thus the sets  $H_{i1}^*$  have as their intersection the complete subspace  $H_2^*$  consisting of these mappings  $\beta$  such that for some  $x$ ,

$$\beta(C) \cap (C \times x) = (A'(x) \cap C) \times x.$$

The set  $H_3^*$  of elements  $\beta$  of  $H_2^*$  such that for some  $x$ ,  $\beta(C) \subset A'(x) \times x$  is open in  $H_2^*$ . The subset of  $H_3^*$  consisting of the elements  $\beta$  such that for some  $x$ ,  $\beta(P) = g(x)$  is closed in  $H_3^*$ . This set is  $H$  and is topologically complete.

It is evident that the collection  $\{H(x)\}$  is equi- $LC^n$  for each  $n$  and lower semicontinuous. Michael's selection theorems now yield the required homeomorphism.

That the map  $\mu$  does, in fact, exist, follows now from the arguments in the proof of Lemma 4.2.

LEMMA 4.5. *There is a mapping  $\Phi^*$  of  $M_1 \times S^k \times I$  into  $M_1 \times S^k$  such that*



$\Phi^* | M_1 \times S^k \times t$  is a homeomorphism,  $\Phi^*(y, x, t) \in M_1 \times x$  for each  $y, x, t$ ,  $\Phi^*(y, x, 0) \in \Phi^*(C, x)$  for each  $y$  in  $C$  and  $\Phi^*(y, x, 1) = (y, x)$  for each  $y$  in  $M_1$ .

*Proof.* As in the proof of Lemma 4.1 (although expressed differently this time) there is a homeomorphism  $\Lambda'$  (corresponding to the  $\Lambda$  of that proof) of  $C \times I \times S^k$  into  $M_1 \times S^k$  such that

$$\Lambda'(y, t, x) \in M_1 \times x, \Lambda'(y, 1, x) = (y, x), \text{ and } \Lambda'(y, \frac{1}{2}, x) \in \phi^*(C \times x).$$

The required isotopy can now be constructed by letting

$$\Phi^*(y, x, t) = (y, x)$$

if  $(y, x)$  is not in  $\Lambda'(C \times I \times S^k)$  and

$$\Phi^*(y, x, t) = (y', x)$$

if  $(y, x) = \Lambda'(y_0, s, x)$  and  $(y', x) = \Lambda'(y_0, (t + 1)s/2, x)$ .

LEMMA 4.6. *The symbols  $B, \zeta$ , and  $f_1$  having the meaning of the last paragraph of §3, there is a mapping  $\gamma$  of  $C \times S^k \times I$  into  $M$  such that  $\gamma | C \times x \times t$  is a homeomorphism,  $\gamma(C \times x \times t) \subset B(x)$ ,  $\gamma(y, x, 0) = f_1(x)(y)$  for each  $y$  in  $C$ , and  $\gamma(y, x, 1) = y$ .*

*Proof.* It follows from Corollary 4 to Theorem 3 of [6] that there is a homeomorphism  $\Gamma_1$  of  $C \times I \times S^k$  into  $M \times S^k$  such that

$$\Gamma_1(C \times t \times x) \subset B(x) \times x, \quad \Gamma_1(y, 0, x) = (f_1(x)(y), x)$$

$$\text{and } \Gamma_1(y, 1, x) \subset \zeta_1(C \times x) \times x.$$

Also, there is a homeomorphism  $\Gamma_2$  of  $C \times I \times S^k$  into  $M \times S^k$  such that

$$\Gamma_2(C \times t \times x) \subset B(x) \times x, \quad \Gamma_2(y, 0, x) = \Gamma_1(y, 1, x)$$

$$\text{and } \Gamma_2(y, 1, x) \in C \times x.$$

The projection of  $M \times S^k$  onto  $M$  now yields a map  $\gamma'$  of  $C \times S^k \times I$  into  $M$  satisfying the conditions required of  $\gamma$  except that it is only certain so far that  $\gamma'(y, x, 1) \in C$ . However, the map  $\nu^*$  of  $I \times S^k$  into  $M$  such that  $\nu^*(t, x) = \gamma'(P, x, t)$  demonstrates, since  $f_1(x)(p)$  is constant, that the map  $\nu$  of  $S^k$  into  $C$  such that  $\nu(x) = \gamma'(P, x, 1)$  is homotopic to 0 in  $M$  and hence in  $C$ . The techniques of [8] and [9] now show that  $\gamma'$  could have been constructed so that  $\gamma'(y, x, 1) = y$ . This completes the proof of the lemma.

### 5. The remaining compact 2-manifolds

The arguments in Sections 3 and 4 demonstrate that the identity component of the space of homeomorphisms of a 2-handled surface into itself is homotopically trivial. The results of Section 2 combined with the methods of [8] or [11] demonstrate that if  $M$  has 2-handles and a hole,  $H_0(M, M)$  is homotopically trivial. It is now easy to extend these results by induction to obtain the next two theorems.

**THEOREM 5.1.** *If  $M$  is a compact orientable 2-manifold with two or more handles,  $H_0(M)$  is homotopically trivial.*

**THEOREM 5.2.** *If  $M$  is a compact orientable 2-manifold with boundary and two or more handles,  $H_0(M, M')$  is homotopically trivial.*

*Proof.* If  $M$  has two handles, these theorems are proved. Suppose  $M$  has  $n$  handles,  $n > 2$ , and no boundary. Let  $C$  be a simple closed curve in  $M$  and  $M_1, M_2$  the closures of the components of  $M - C$ ,  $M_1$  having one handle,  $M_2$  having  $(n - 1)$  handles. The arguments in Sections 2 and 3 carry over without change. The set  $T_2$  is here a compact 2-manifold with  $(n - 1)$  handles not a torus, but the fact that, in Section 2,  $\pi_1(T_2)$  is abelian, was not used. Theorem 5.2 follows as was indicated above for the 2-handled case.

It was proved in [12] that if  $M$  is a Moebius strip,  $H_0(M, M)$  is homotopically trivial and if  $M$  is a Klein bottle and  $p \in M$ ,  $H_0(M, p)$  is homotopically trivial. This last fact and the techniques of [8] and [11] yield the fact that if  $M$  is a Klein bottle with holes  $H_0(M, M')$  is homotopically trivial. The techniques in the present paper now carry over without change to prove.

**THEOREM 5.3.** *If  $M$  is a non-orientable surface, with or without boundary and with three or more cross caps, then  $H_0(M, M')$  is homotopically trivial.*

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