MULTIPLICITY OF SOLUTIONS IN FRAME MAPPINGS, II

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A recent paper [11][2] established some homology results for the problem of mapping an orthogonal k-tuple on S^{n-1} into the Euclidean space R. These results were obtained under restrictive hypotheses on n and k. That paper indicated certain possibilities of generalization. In part the present brief note indicates these in more detail. Since the arguments are largely extensions of these anterior results, it seems worthwhile to preface our remarks with the proofs of some of the assertions in [11] together with their indicated generalizations. In the interests of succinctness it will be assumed the reader has available [11] and notation, terminology, numbering of results and bibliography references are consistent with that article.

The computations in [11] involve the coefficient group J_p for all cohomology rings. However, the definition of the index in [4] or [10] uses J_p for odd dimensions and J for even dimensions in computing the element $\mathbf{I}(j)$ of the j^{th} cohomology group [4, p. 331]. Since 2 is the only possible torsion coefficient entering in the various cohomology rings used the Universal Coefficient theorem is the assurance that the height of $\mathbf{I}(j)$ can be calculated by using J_p, p an odd prime throughout. For clarity we shall presently refer to the index as the J_p index.

Lemma 1 states: *If $n = 2m + 1$ is a prime, and if Δ_i is the i^{th} symmetric function in the arguments $1^2, \cdots, m^2$, then $\Delta_i = 0 \bmod n$ for $1 < i < m$.*

The elementary proof consists in the combination of the observation that $1^2, 2^2, \cdots, m^2$ are quadratic residues [12, p. 270] with respect to the prime $n = 2m + 1$ and Euler's criterion [12, p. 274]

$$x^m \equiv 1 \mod n$$

whence

$$(x^m - 1) \equiv \prod_{i=1}^{i=m} (x - i^2) \mod n$$

which implies the lemma.

The method in [11] involves the determination of the cohomology ring of

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[2] The following slips in [11] are noted: page 171, line -3; n should be m in $B^{4i}_{SO(n)}$; page 172: The book reference should be to page 332; page 173, lines 17 and 18: the footnote reference is to Corollary 5 not 3; the exposition on page 175 is clearer if one introduces $T = T'' \times T'$ Then $G^m = (G \cap T'') \times T'$; in the second line of equation (3.3), t_i not b_i and the i range is $e < i$. Equation (3.4) should be

$$\rho^*(S, SO(n))\rho^*(J_k \times T', S) = \rho^*(J_k \times T', G^m)\rho^*(G^m, T)\rho^*(T, SO(n));$$

In the next line too, S and $J_k \times T'$ should be transposed, and below, $i = e$ not k'; page 176: transpose the sentences "The ideal . . ." and "Since $p_{4i}\, i$ . . ."; line 8: insert ", $\cdots$, u_{4m-1})" after u_{4r+3}; line 14: the relation is mod I; line 16: H not B; line 19: Add Λ before (a).

$SO(n)/A$ for some subgroups A, which in turn involves the homomorphism of cohomology rings of classifying spaces [2]. These homomorphisms depend on how the subgroups are imbedded. Thus a crucial homomorphism is that indicated by $\rho^*(C, G)$ where $C \approx J_k$ and $G \approx (J_k)^n$. A natural conjecture might be $b_j \to b$, in line with [2, p. 310]. However, as stated in [11], the imbedding of C in G is such that the correct correspondence is

$$\text{(a)} \qquad b_j \to jb.$$

We present the details.

If the matrix representation of the generator of C is

$$\begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ & & 0 & \cdot & & \\ & & & \ddots & & \\ & & & & \cdot & 1 \\ 0 & & & & & 0 \\ 1 & 0 & \cdots & & & \end{pmatrix}$$

then, since the characteristic roots are the k^{th} roots of unity, this is orthogonal equivalent to the matrix

$$a = \begin{pmatrix} A & & & & \\ & A^2 & & 0 & \\ & & \ddots & & \\ & 0 & & A^m & \\ & & & & 1 \end{pmatrix}$$

where $m = [n/2]$ and

$$A = \begin{pmatrix} \cos 2\pi/k & -\sin 2\pi/k \\ \sin 2\pi/k & \cos 2\pi/k \end{pmatrix}.$$

The multiplicative representative of the j^{th} generator of $(J_k)^m$ is

$$a_j = \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & & & A_j & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

where $A_j = A$ is in the position of A^j in a. The inclusion map of a into the direct product of the a_j's is therefore obtained by mapping a into

$$\begin{pmatrix} A_1 & & & & \\ & A_2^2 & & 0 & \\ & & \ddots & & \\ & 0 & & A_m^m & \\ & & & & 1 \end{pmatrix}.$$

In an additive representation with α the generator of C corresponding to a and $\{\alpha_j\}$ the additive generators corresponding to a_j

$$\alpha \xrightarrow{i} \alpha_1 \oplus 2\alpha_2 \oplus \cdots \oplus m\alpha_m .$$

Let β and β_j be the dual generators for Hom (C, J_p) and Hom (G, J_p) with $k = p$ respectively. Thus

$$\beta(\alpha) = 1, \qquad \beta_r(\alpha_s) = \delta_s^r \mod p$$

and the induced homomorphism

$$\text{Hom}\,(G, J_p) \xrightarrow{i} \text{Hom}\,(C, J_p)$$

is specified as usual by

$$(i^{\#}\beta_r)\alpha = \beta_r(i\alpha) = \beta_r \oplus_{j=1}^{j=m} j\alpha_j = r.$$

Accordingly

(b) $$i^{\#}\beta_r = r\beta.$$

On referring to [11, Eq. 2.4] for B_C^* and B_G^*, it is clear that (b) implies (a), i.e.

$$\rho^*(C, G)b_i = ib.$$

We now take up some generalizations. One is that of replacing $k = p$ by $k = p^s$, $s > 1$. This generalization amounts to replacing C_p by C_{p^s} maintaining the coefficient group J_p and the auxiliary Lemmas 1 and 2 go over intact. Thus Lemma 1 remains valid with the interpretation $m = (p^s - 1)/2$

$$\Delta_i \equiv 0 \mod p \qquad\qquad 0 < i < m.$$

For this note that

$$(p^s - 1)/2 = p(p^{s-1} - 1)/2 + (p - 1)/2$$

whence mod p, the sum of the $2i$ power of the first $p(p^{s-1} - 1)/2$ integers vanishes and only the integers $1^2, 2^2, \cdots, ((p-1)/2)^2$ enter. However for these, Lemma 1 is valid.

The squared integers entering in the correspondent of (a) may be restricted to those prime to p since the coefficient j in the mapping $b_j \to jb$ is understood mod p. This implies that [11, (2.9)] is valid provided the $(p^{s-1} - 1)/2$ integers divisible by p are stricken from the product

$$\textstyle\prod_1^m (1 + (jb)^2).$$

Accordingly the right hand side of [11, (2.9)] would be

(2.9′) $$1 + Ab^{(p-1)p^{s-1}}$$

That is to say $2m$ is replaced by $(p-1)p^{s-1}$, etc. and so [11, (2.8) and (2.10)] are available, and thus eventually the index is $2(p-1)p^{s-1} - 1$.

A comment is in order here on the application of the index method. The index is derived under the assumption that only the identity of C leaves any point fixed. Accordingly the diagonal of R^{p^s} must be replaced by the *fat diagonal* ∇ defined as the diagonal of

$$\sum^p = \sum \times \cdots \times \sum,$$

the p fold product of $\sum = R^{p^{s-1}}$. Plainly $\nabla \supset \Delta$. (In his current doctoral thesis Mr. Masami Wakae has independently noted (2.9′) and has carried through the analysis of the case $n \neq k$ also and has established bounds for $\nu(R^n - \nabla)$ for $n = p^s$.)

For the case of n a composite number, a simple observation is in order.

THEOREM 7. *If the odd number $n = kq$, the J_p index, p an odd prime, of $SO(n)/C_k$ is the same as that of $SO(k)/C_k$.*

By the Kunneth theorem, since there is no odd torsion

$$H(SO(n)/C_k, J_p) \approx H(SO(n)/SO(k), J_p) \otimes H(SO(k)/C_k, J_p)$$

and the height facts are those of the factor

$$H(SO(k)/C_k, J_p).$$

The case $p = 2, n = 2^s$ is open. The difficulty here is that the action desired of $C \approx J_{2^s}$ is not that of reflections, but of a rotation of angle $2\pi/2^s$. Accordingly the plausible attempt of replacing the maximal torus T in the ρ^* homomorphisms cited in [11] by the subgroup of diagonal matrices Q according to the pattern of [13] is not available.

However, there is still an interesting class of frame problems involving $C_2 \approx J_2$ following the ideas of the index (apart from the procedures of Bourgin and of Yang cited in the bibliography of [4]). Here the equivariant mapping condition

$$ft = tf \tag{c}$$

restricts the admissible mappings. Assume below that t is the antipodal mapping $tw = -w$. Then

$$f(-w) = -f(w). \tag{d}$$

For these restricted mappings we may again apply our methods and as a notable advantage the results can be stated for the general k-tuple. Remark first that already with C_k, $k \neq 2$, the orthogonal k-tuples in [11] can be replaced by equispaced k-tuples in the sense that rotation through $2\pi/k$ about a suitable axis leaves the k-tuple unchanged (Cf[5, p. 300]). However for the antipodal mappings each diameter is unaffected. Hence there need be no relation to any other diameter in our $C_2 \approx J_2$ arguments and therefore an arbitrarily spaced k-tuple can be assigned and the results stated for the rotation equivalents of this k-tuple. An alternative argument invokes the Gram-Schmidt orthogonalization process and was given in [14] where it was also remarked that the linearly dependent k-tuple can be treated as a limiting case of linearly independent k-tuples. Specifically then as an analogue of Theorem 6 [11],

THEOREM 8. *Let f be of type* (b) *on S^{n-1} to R^l. Then for $n = 2^s$, the set of rotation equivalents of an arbitrary k-tuple on S^{n-1} whose end points have a common image constitutes a symmetric set D' with*

$$H_{N-j}(D') \neq 0 \qquad (k-1)l \leq j \leq n-1$$

$$N = \dim SO(n)/C_2 = n(n-1)/2.$$

In particular if $n - 1 = (k-1)l$, there is a non-bounding cycle of such k-tuples of dimension $(n/2)(n/2 - 1)$.

COROLLARY 9. *Under the hypotheses above there is a rotation equivalent of an arbitrary k-tuple on S^{n-1} whose end points map into the same point if $n - 1 = (k-1)l$.*

This corollary has been found by Geraghty [14].

The proof of the theorem depends on the fact that $C \approx J_2$ viewed as an involution is a subgroup of $SO(2^s)$. According to [2, p. 3.10] the cohomology ring is

$$H\left(\frac{SO(2^s)}{C_2}, J_2\right) \approx \frac{J_2(a)}{(a^n)} \otimes V \qquad \dim a = 1$$

with (a^n) the ideal generated by a^n. Hence

$$\nu(SO(2^s)/C_2) = n - 1.$$

The rest of the proof follows the detail of that of Theorem 2.

Typical of the directness of the proof afforded by these methods is the following treatment of the generalized Borsuk-Ulam theorem [4, p. 338].

THEOREM 10. *If $f : S^n \to R^l$ and if $\{x \mid f(x) = f(tx)\} = D$ where t is an involution, then*

$$H_k(D', J_2) \neq 0, \qquad 0 \leq k \leq n - l.$$

Note $F(x) = f(x) - f(tx)$ is equivariant and

$$F : S^n - D \to R^l - 0.$$

Since $\gamma(R^l - 0) = l - 1$ and $\nu(S^n) = n$, then just as in [11, p. 173]

$$I(i, A) = 0, \qquad l \leq i.$$

Accordingly $I(m, S^n)$, $m = l, \cdots, n$, maps into $0 \in H^m(A', J_2)$ in [11, Equation (1.5)]. Hence with $P^n = S^{n\prime}$ and $D' = P^n - A$,

$$H_{n-m}(D') \neq 0, \qquad m = l, \cdots, n$$

Moreover by commutativity in Equation (1.5), the inclusion mapping of D' into P^n induces non-trivial homomorphisms for these values of m.

It is well known that the inscribed cube problem for convex bodies is generally impossible. A related problem is that of the inscribed equilateral frame: If K is a convex body in R^n is there a frame $\{\pm a^i\}$ about some origin, with $\| a^i \|$ independent of i, terminating on the boundary of K? This problem is still unsolved for $n \geq 3$, (though it seems likely that an equilateral frame with end points on the boundary of a compactum K in R^n exists if K is merely contractible and locally contractible). However, with an added symmetry hypothesis the problem becomes a corollary of Theorem 2. Specifically, K is symmetric with respect to the origin 0 if $x \in K$ implies $-x \in K$.

THEOREM 11. *Let K be a compact symmetric star convex set with respect to an interior point θ, contained in R^n, n an odd prime. Let E be the set of inscribed equilateral frames (with end points on the boundary M of K). Then E/C, $C \approx J_n$, has the homology properties of D' in Theorem 2. In particular, for $n = 3$ the representative in $SO(3)/C$ of E/C contains a nonbounding 1 cycle.*

Furthermore Corollary 5 is valid with circumscribing cubes replaced by equilateral inscribed frames.

For the proof we need merely take for $f(w)$ the length of the radius to M along the line through $w \in S^{n-1}$. (If the existence of merely one such frame is at issue, the same definition of $f(w)$ may be applied to the Kakutani or Yamabe-Yujobo [15] proofs for general n).

An x-diameter of a convex body K is a segment containing x whose end points are on the boundary of K. n equal orthogonal x-diameters constitute an x-flare.

THEOREM 12. *Let K be an arbitrary convex body (not necessarily symmetric) in R^n and let x be any inner point of K. Let E be the set of x-flares. Then the properties of E/C are those listed in Theorem 11.*

Added in Proof (See Theorem 10). No more than four points on a great circle of S^2 need have a common image in R^1. A simple example is $f(x) = x_1 + x_2 + x_3^2$ essentially suggested by C. Pucci.

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