A CHARACTERIZATION OF THE SUZUKI GROUPS

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1. Introduction and Notation

Let G be a finite group. Suppose p is a prime and P is a Sylow p-subgroup of G. We say that G is *p-core-free* if G has no normal p'-subgroup except the identity subgroup. Consider the following conditions:

(a) *Two elements of P are conjugate in G if and only if they are conjugate in the normalizer of P.*

(b) *The centralizer of every element of order p in P has a normal p-complement.*

(c) *Every pair of elements of order p in G generates a p-solvable subgroup of G.*

(d) *P is not an elementary Abelian 2-group whose non-identity elements are all conjugate in G.*

The object of this paper is to obtain the following characterization of Suzuki's simple groups:

MAIN THEOREM. *Let p be a prime, and let P be a Sylow p-subgroup of a finite p-core-free group G. Suppose P is not a normal subgroup of G. Then G satisfies* (a), (b), (c), *and* (d) *if and only if $p = 2$ and G is a Suzuki group.*

The consequences of this result are different for $p = 2$ and for p odd. Suppose, first, that $p = 2$. Then every finite group G satisfies (c). If P is an Abelian group with two generators but is not elementary Abelian, then G satisfies (a), (b), and (d) (Lemma 2.1(ii)). Thus we obtain a theorem of Brauer [3, pages 317–320], which asserts that P is normal if G is 2-core-free. Brauer's proof uses his method of "columns"; we shall require Brauer's method and also Feit's construction [5] of exceptional characters and Suzuki's characterization [11], [12] of certain doubly transitive permutation groups.

Suppose $p = 2$, G is 2-core-free, and G satisfies (a) and (b) but not (d). Assume that G does not have a normal Sylow 2-subgroup.In this case the main theorem gives no information. However, the possible groups have been determined. By a theorem of Gorenstein [7] that uses several recent deep results, G must be a group of one of the following types:

(i) a subgroup of $P\Gamma L(2, q)$ that contains $PSL(2, q)$ as a subgroup of odd index, for some odd prime power q such that $q \equiv \pm 3$, mod 8;
(ii) $PSL(2, 2^n)$, for some integer $n \geq 2$.

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Conversely, the groups of types (i) and (ii) all satisfy (a) and (b) but not (d).

Unfortunately, when p is odd, it seems impossible to verify (c) in most situations. However, we obtain the following result:

COROLLARY. *Let p be a prime and let G be a finite group with a cyclic Sylow p-sub-group. Then G is a p-solvable group if and only if every pair of elements of order p in G generates a p-solvable group.*

These results originated in a problem proposed by J. Thompson. We are indebted to E. Shult for permission to quote his results before publication. We also thank the National Science Foundation for its partial support during the preparation of this paper.

Most of our notation is standard. All groups considered in this paper are finite. Suppose G is a finite group. Denote the order of G by $|G|$. The *exponent* of G is the smallest positive integer r such that $x^r = 1$ for all $x \in G$. If H is a subset of G, we write $H \subseteq G$ (respectively, $H \subset G$) to indicate that H is a subgroup (respectively, a proper subgroup) of G. Denote the identity element and identity subgroup of G by 1. If G is isomorphic to a group G_1, we write $G \cong G_1$.

Suppose $H \subseteq G$. We say that H is an *elementary Abelian* group if H is Abelian and the exponent of H is either one or some prime. In this case G is said to act *irreducibly* on H if H is a minimal normal subgroup of G.

Suppose $H, K \subseteq G$. Let $H^{\#}$ be the set of all non-identity elements of H. The normalizer and the centralizer of H in K are denoted by $N_K(H)$ and $C_K(H)$. When there is no danger of confusion, we will also write $N(H)$ and $C(H)$ for $N_G(H)$ and $C_G(H)$. If H is generated by a single element x, let $C_K(x) = C_K(H)$ and $C(x) = C(H)$. Let $[H, K]$ be the subgroup of G generated by the commutators $h^{-1}k^{-1}hk$, where $h \in H$ and $k \in K$. For $x, y \in G$, let $y^x = x^{-1}yx$ and $K^x = x^{-1}Kx$. We say that K is *weakly closed* in H with respect to G if $K \subseteq H$ and if $K^x = K$ whenever $x \in G$ and $K^x \subseteq H$.

Let $Z(G)$ be the center of G and let G' be $[G, G]$, the commutator subgroup of G. Suppose H is a normal subgroup of G. We say that G is a *Frobenius group* with *Frobenius kernel* H if $1 \subset H \subset G$ and if $C(x) \subseteq H$ for every $x \in H^{\#}$. In this case, a subgroup Q of G that satisfies $HQ = G$ and $H \cap Q = 1$ is called a *complement* of H in G. It is well known that Q must exist [6, (25.2), page 150].

Suppose p is a prime. Then G is called a *p'-group* if p does not divide $|G|$. We say that G is *p-solvable* if every composition factor of G is a p-group or a p'-group. If $H \subseteq G$ and H is a p'-group, then H is said to be a *p'-subgroup* of G. It is easy to see that the product of any number of normal p'-subgroups of G is itself a normal p'-subgroup of G. Hence G has a unique maximal normal p'-subgroup $K_p(G)$, called the *p-regular core* of G. Note that G is p-core-free if $K_p(G) = 1$.

An element x of G is called a *p'-element* if p does not divide the order of x. Suppose $x \in G$. Then there exist unique elements $y, z \in G$ such that $yz = zy = x$,

z is a p'-element, and the order of y is either one or a positive power of p. We call y the *p-part* of x. If $G/K_p(G)$ is a p-group, G is said to have a *normal p-complement* (namely, $K_p(G)$). Clearly, this occurs if and only if the p'-elements of G form a subgroup of G.

Let q be a power of a prime. Let S be the set of elements of the Galois field $GF(q)$ of order q, together with the symbol ∞. We let $P\Gamma L(2, q)$ be the group of permutations of S described by

$$x \to (ax^\tau + b)/(cx^\tau + d) \qquad (x \in S), \tag{2.1}$$

where $a, b, c, d \in GF(q)$ and τ is an automorphism of $GF(q)$, and $ad - bc \neq 0$. Let $PSL(2, q)$ be the subgroup of $P\Gamma L(2, q)$ that consists of those permutations of the form (2.1) for which τ is the identity automorphism and $ad - bc = 1$. If $q = 2^{2n+1}$ for some positive integer n, let $G(q)$ be the group defined on page 133 of [12]. The groups $G(q)$ are called the *Suzuki groups.*

2. Some Consequences of (a), (b), and (c)

LEMMA 2.1. *Let P be a Sylow p-subgroup of a finite group G. Suppose W is a subgroup of $Z(P)$ that is weakly closed in P with respect to G.*

(i) *Assume that whenever two elements of P are conjugate in G, they are conjugate in P. Then G has a normal p-complement.*

(ii) *Two elements of P are conjugate in G if and only if they are conjugate in $N(W)$.*

(iii) *If G satisfies* (b) *and $W \neq 1$, then G satisfies* (a).

Proof. (i) This is a theorem of Frobenius that was generalized by Brauer in Theorem 3 of [1].

(ii) Let $x, y \in P$ and $g \in G$. Suppose $x^g = y$. Then $W \subseteq C(y)$ and $W^g \subseteq C(x)^g = C(y)$. Let Q be a Sylow p-subgroup of $C(y)$ that contains W, and take $h \in C(y)$ such that $(W^g)^h \subseteq Q$. Take $k \in G$ such that $Q^k \subseteq P$. Then

$$W^k \subseteq Q^k \subseteq P \quad \text{and} \quad (W^{gh})^k \subseteq Q^k \subseteq P.$$

Since W is weakly closed in P, $W = W^k = W^{ghk}$. Thus $gh \in N(W)$ and $x^{gh} = (x^g)^h = y^h = y$.

(iii) Suppose G satisfies (b) and $W \neq 1$. Since W is weakly closed in P, $N(P) \subseteq N(W)$. Suppose $x, y \in P$ and x and y are conjugate in G. By (ii), there exists $g \in N(W)$ such that $x^g = y$. Since P is a Sylow p-subgroup of $C(W)$ and since $C(W)^g = C(W)$, there exists $h \in C(W)$ such that $(P^g)^h = P$. By (b), $C(W)$ has a normal p-complement. Take $j \in P$ and $k \in K_p(C(W))$ such that $kj = h$. Then

$$P^{gk} = (P^{gh})^{j^{-1}} = P^{j^{-1}} = P.$$

Thus $gk \in N(P)$. Since

$$y^{-1}x^{gk} \equiv y^{-1}x^g \equiv 1 \mod K_p(C(W)),$$

$y^{-1}x^{gk} \in P \cap K_p(C(W)) = 1$. Thus $x^{gk} = y$.

LEMMA 2.2. *Let P be a Sylow p-subgroup of a finite group G. Assume G satisfies* (a) *and* (b). *Let $N = N(P)$. Suppose $x \in P^{\#}$. Then*

$$C(x) = C_P(x)K_p(C(x)) \quad \textit{and} \quad C_N(x) = C_P(x)K_p(N).$$

Proof. Let Q be an arbitrary Sylow p-subgroup of $C(x)$. Take $g \in G$ such that $Q^g \subseteq P$. Then $x^g \in Q^g \subseteq P$. By (a), there exists $n \in N$ such that $x = (x^g)^n = x^{gn}$. Now, Q^{gn} is a Sylow p-subgroup of $C(x^{gn})$, that is, of $C(x)$. Since $Q^{gn} \subseteq P$, $Q^{gn} = C_P(x)$. As some power of x has order p, $C(x)$ has a normal p-complement by (b). Therefore, $C(x) = C_P(x)K_p(C(x))$.

Let $M = C_N(x)$ and $K = K_p(M)$. Since $M \subseteq C(x)$, M has a normal p-complement. Now

$$[P, K_p(N)] \subseteq P \cap K_p(N) = 1,$$

so $K_p(N) \subseteq K$. Conversely, since K normalizes $C(x)$ and P,

$$[K, Z(P)] \subseteq [K, C(x) \cap P] \subseteq K \cap (C(x) \cap P) = 1.$$

Thus $K \subseteq C(Z(P))$, so that by (b),

$$K \subseteq K_p(C(Z(P))) \subseteq K_p(N(Z(P))).$$

Thus

$$K \subseteq K_p(N(Z(P))) \cap N \subseteq K_p(N).$$

This completes the proof of Lemma 2.2.

LEMMA 2.3. *Let H be a Frobenius group with Frobenius kernel K. Suppose Q is a complement of K in H and M is a normal subgroup of H. Let $q = |Q|$. Then*

(i) *if $|H/K|$ is even, then K is Abelian;*

(ii) *$M \subseteq K$ or $K \subseteq M$;*

(iii) *if $1 \subset M \subset K$, then MQ and H/M are Frobenius groups, so q divides $|M| - 1$ and $|K/M| - 1$.*

Proof. Part (i) is a well-known result of Burnside [Theorem 25.9, page 156, 6]. Parts (ii) and (iii) follow from Theorem 25.3, page 152 of [6].

PROPOSITION 2.1. *Let p be a prime and let P be a Sylow p-subgroup of a p-core-free finite group G. Assume that G satisfies* (a) *and* (b) *and that P is not a normal subgroup of G. Let $N = N(P)$. Then*

(i) *$N/K_p(N)$ is a Frobenius group with Frobenius kernel $PK_p(N)/K_p(N)$;*

(ii) *G possesses a unique minimal normal subgroup M, and M is a simple group that contains P and satisfies* (a) *and* (b).

Proof. (i) Let $K = K_p(N)$. Since P is not a normal subgroup of G, $P \neq 1$. Suppose $N = PK$. By Lemma 2.2, $K \subseteq C(x)$ for every $x \in P$. Hence $N = PC(P)$. By (a), two elements of P are conjugate in G if and only if they are conjugate in P. By Lemma 2.1(i), G has a normal p-complement. Since G is p-core-free, $G = P$, contrary to hypothesis. Thus $N \neq PK$.

Suppose $x \in N$, $y \in P^{\#}$, and the cosets of x and y in N/K commute with each other. Then

$$x^{-1}y^{-1}xy \in K \quad \text{and} \quad x^{-1}y^{-1}xy = (x^{-1}yx)^{-1}y \in P.$$

Hence $xy = yx$. By Lemma 2.2, $x \in C_N(y) \subseteq PK$. Thus N/K is a Frobenius group with Frobenius kernel PK/K.

(ii) We prove this part by induction on $|G|$. Let M be a minimal normal subgroup of G. Since G is p-core-free, p divides $|M|$. Thus $M \cap P$, which is a Sylow p-subgroup of M, is a non-identity normal subgroup of P. By a well known result of P. Hall [15, page 144], $M \cap Z(P) \neq 1$. Let $W = M \cap Z(P)$. Clearly, W is a normal subgroup of N so $N \subseteq N(W)$. By (a), W is a weakly closed subgroup of P with respect to G. Therefore, W is a weakly closed subgroup of P with respect to M. Since M obviously satisfies (b), M satisfies (a) by Lemma 2.1(iii).

Let $g \in N(W)$. Since P is a Sylow p-subgroup of $C(W)$ and $C(W)$ is a normal subgroup of $N(W)$, there exists $c \in C(W)$ such that $(P^g)^c = P$. Thus $g = (gc)c^{-1} \in NC(W)$. Since g is arbitrary, $N(W) = NC(W)$. By (b), $C(W) = PK_p(C(W))$, so

$$N(W) = NK_p(C(W)). \tag{2.1}$$

Suppose M is a p-group. Then $W \subseteq M \subseteq P$. Since N normalizes W, G normalizes W, by (a). Therefore, $K_p(C(W)) \subseteq K_p(G) = 1$. By (2.1), $G = N$, contrary to hypothesis. Thus M is not a p-group. Since M is a minimal normal subgroup of G, $M \cap P$ is not a normal subgroup of M.

Suppose $P \not\subseteq M$. Obviously, MP satisfies (b). As above with M, we see that MP satisfies (a). Since $M \cap P$ is not normal in M, M does not normalize P. Moreover,

$$K_p(MP) \subseteq K_p(M) \subseteq K_p(G) = 1,$$

because MP/M is a p-group. By induction hypothesis, $MP = G$.

In this case, since $N \cap M$ is a normal subgroup of N, $(N \cap M)K/K$ is a normal subgroup of N/K. By (i) and Lemma 2.3,

$$PK/K \subseteq (N \cap M)K/K \quad \text{or} \quad (N \cap M)K/K \subseteq PK/K.$$

Since $M \cap P$ is a proper subgroup of P and is a Sylow p-subgroup of M, $P \not\subseteq (N \cap M)K$. Therefore,

$$N \cap M \subseteq (N \cap M)K \subseteq PK.$$

Since G/M is a p-group, $N/(N \cap M)$ is a p-group. Hence

$$N = P(N \cap M) \subseteq PPK = PK.$$

But this violates (i). Therefore, $P \subseteq M$. If G has a minimal normal subgroup M_1 different from M, we obtain similarly $P \subseteq M_1$. But then $P \subseteq M \cap M_1 = 1$, a contradiction.

Since M is a minimal normal subgroup of G, M is a direct product of isomorphic simple groups [8, page 131], say, $M = S_1 \times S_2 \times \cdots \times S_n$. As p divides $|M|$ and M is not a p-group, $S_1, S_2, \cdots$, and S_n are not p-solvable. Let x be an element of order p in S_1. By (b), $C(x)$ is p-solvable. Therefore, $n = 1$. This completes the proof of Proposition 2.1.

PROPOSITION 2.2. *Let p be a prime, and let P be a Sylow p-subgroup of a finite group G that satisfies* (a) *and* (c). *Suppose $x, y \in G$, P_0 is a normal subgroup of $N(P)$ contained in P, and x and y are conjugate to elements of P_0. Assume that x is conjugate to an element of $Z(P)$. Let H be the subgroup of G generated by x and y, and let H_p be a Sylow p-subgroup of H. Then*

(i) $H = H_p K_p(H)$;
(ii) *H_p is an Abelian group and is conjugate to a subgroup of P_0;*
(iii) *$H \cap P \subseteq P_0$; and*
(iv) *if xy^{-1} is a p'-element, x is conjugate to y.*

Proof. Let $K = K_p(H)$. Clearly, we may assume that $x \in H_p$. Take $g \in G$ such that $(H_p)^g \subseteq P$. By (a), x^g is conjugate in $N(P)$ to an element of $Z(P)$. Consequently, $x^g \in Z(P)$. Hence $x^g \in Z((H_p)^g)$, and $x \in Z(H_p)$. Let M/K be the largest normal p-subgroup of H/K. Then $M \cap H_p$ is a Sylow p-subgroup of M. Therefore, $M = (M \cap H_p)K$, so

$$xz \equiv zx \quad \text{modulo } K, \tag{2.2}$$

for all $z \in M$. By Lemma 1.2.3 of Hall and Higman [9], $x \in M$. Thus H is generated by M and y, and H/M is a p-group. By the definition of M, $H = M$. Thus $H = H_p K$. This proves (i). Since H is generated by x and y, H/K is Abelian by (2.2).

Take $h \in H$ such that $y^h \in H_p$. Since H_p is Abelian and $H = H_p K$, x and y^h generate H_p, and

$$y^h \equiv y \quad \text{modulo } K. \tag{2.3}$$

Therefore, x^g and y^{hg} generate $(H_p)^g$. Since P_0 is a normal subgroup of $N(P)$, x^g and y^{hg} lie in P_0, by (a). Therefore, $(H_p)^g \subseteq P_0$. This completes the proof of (ii).

Take $k \in H$ such that $(H \cap P)^k \subseteq H_p$. Then $(H \cap P)^{kg} \subseteq P_0$, and

$$H \cap P \subseteq (P_0)^{g^{-1}k^{-1}} \cap P.$$

By (a), $H \cap P \subseteq P_0$.

Finally, suppose xy^{-1} is a p'-element. Then $xy^{-1} \in K$. By (2.3), $x(y^h)^{-1} \in K \cap H_p = 1$. Thus $x = y^h$. This completes the proof of Proposition 2.2.

PROPOSITION 2.3. *Let p be a prime, and let P be a Sylow p-subgroup of a finite group G that satisfies* (a) *and* (c). *Assume that P is not an elementary Abelian group on which $N(P)$ acts irreducibly. Then there exists a normal subgroup P_0*

of $N(P)$ such that $P_0 \subseteq Z(P)$ and $1 \subset P_0 \subset P$. If $v \in P_0$ and $g, h \in G$ and the the p-part of $v^g(v^h)^{-1}$ lies in P, then the p-part of $v^g(v^h)^{-1}$ lies in P_0.

Proof. If P is not an elementary Abelian group let P_0 be the set of those elements of $Z(P)$ that satisfy $x^p = 1$. If P is an elementary Abelian group let P_0 be any proper non-identity subgroup of P that is normalized by $N(P)$. By Proposition 2.2(iii), P_0 has the desired property.

3. Characters of groups satisfying (a) and (b)

The basic tool for our proof of the main theorem is the theory of group characters, including the theory of blocks as developed by Brauer [2], [3]. Throughout this section we assume that G is a finite group that satisfies (a) and (b) for some prime p that divides $|G|$ and for some Sylow p-subgroup P of G. We define a mapping $\theta \to \theta^G$ of the generalized characters of P into the ring of generalized characters of G; this mapping corresponds to the concept of the "column" of θ as defined bv Brauer (see Remark 3.1). Then we use a technique of Feit to define a related mapping that yields "exceptional" characters of G under certain conditions on P.

Let $N = N(P)$ and $q = |N/PK_p(N)|$. We assume that G does not have a normal p-complement. By Lemmas 2.1(i) and 2.2, $q > 1$.

Let $\zeta_0, \zeta_1, \cdots$ be the irreducible complex characters of P. Assume ζ_0 is the principal character of G and ζ_1 is a one-dimensional character of P. Let $z_i = \zeta_i(1)$ for $i = 0, 1, \cdots$. Let T be a complete set of coset representatives of $PK_p(N)$ in N. We assume that $1 \in T$. Let χ_0 be the principal character of G. For each irreducible complex character χ of G, we denote the kernel of χ by Ker χ.

Suppose H is a subgroup of G. Let $B_0(H)$ be the principal p-block of H; we regard $B_0(H)$ as a collection of irreducible complex characters (but not modular characters) of G. We define

$$\lambda(H) = \textstyle\sum_\chi \chi(1)^2 \qquad (\chi \in B_0(H)).$$

Let θ and η be generalized characters of H. As usual, we define the inner product of θ and η by

$$(\theta, \eta)_H = (1/|H|) \textstyle\sum_{h \in H} \theta(h)\eta(h^{-1})$$

and the norm of θ by $\|\theta\|^2 = (\theta, \theta)_H$. For every subgroup K of H, we denote the restriction of θ to K by $\theta|_K$.

Suppose θ is a generalized character of P. Let θ^* be the generalized character of G induced by θ and let θ^G be the generalized character of G given by

$$\theta^G(g) = \textstyle\sum_{\chi \in B_0(G)} (\theta^*, \chi)_G \chi(g) \qquad \text{for} \quad g \in G.$$

For $n \in N$, let θ^n be the generalized character of P given by $\theta^n(x) = \theta(x^n)$, $x \in P$. If $n = ht$ for $h \in PK_p(N)$ and $t \in T$, then $\theta^n = \theta^t$, by Lemma 2.2.

Remark 3.1. Let θ and η be any generalized characters of P. As is well known,

$$(\theta^G, \eta^G)_G = \sum (\theta^G, \chi)_G(\eta^G, \chi)_G,$$

where χ ranges over all the irreducible complex characters of G. Since $(\theta^G, \chi)_G = 0$ whenever $\chi \notin B_0(G)$, we obtain

$$(\theta^G, \eta^G)_G = \sum\nolimits_{\chi \in B_0(G)} (\theta^G, \chi)_G(\eta^G, \chi)_G = \sum\nolimits_{\chi \in B_0(G)} (\theta, \chi|_P)_P(\eta, \chi|_P)_P$$

by the Frobenius reciprocity theorem. Thus $(\theta^G, \eta^G)_G$ coincides with the inner product of columns $(\mathfrak{a}(\theta), \mathfrak{a}(\eta))$ as defined by Brauer [2, pages 165–6].

Remark 3.2. Suppose $K_p(N) = 1$. Let $\pi = \{p\}$, $H = N$, and $D = P^{\#}$. By (a) and Lemma 2.2, G satisfies the conditions of Dade for lifting certain characters of N ((i) and (ii), page 373, of [14]). Let θ be a generalized character of P, and let $\tilde{\theta}$ be the generalized character of N induced by θ. Suppose $\theta(1) = 0$. It can be shown by Lemma 1 and Theorems 6 and 7 of [14] that θ^G is the generalized character of G that corresponds to $\tilde{\theta}$ under Dade's mapping.

Our first lemma does not require any of the properties of P and G.

LEMMA 3.1. *Let θ be a character of a subgroup L of G. Suppose H is a normal subgroup of L, $L = PH$, and $P \cap H = 1$. Suppose $(\theta|_P, \zeta_0)_P = 0$, and $\theta(xy) = \theta(x)$ for all $x \in P^{\#}$ and $y \in H$. Then H is contained in the kernel of θ.*

Proof. Let $\theta|_P = \sum c_i\zeta_i$. Then $c_0 = 0$, $c_i \geqq 0$ for all $i \geqq 1$, and

$$\theta(1) = \sum c_i z_i. \tag{3.1}$$

For $i = 0, 1, 2, \cdots$, let ζ_{i0} be the irreducible character of L given by $\zeta_{i0}(xy) = \zeta_i(x)$ for $x \in P$ and $y \in H$. Since $(\theta|_P, \zeta_0)_P = 0$, $(\theta, \zeta_{00})_L = 0$. Hence for each i,

$$\begin{aligned}(\theta, \zeta_{i0})_L &= (\theta, \zeta_{i0})_L - z_i(\theta, \zeta_{00})_L = (\theta, \zeta_{i0} - z_i\zeta_{00})_L \\ &= (1/|L|)\sum\nolimits_{x,y} \theta(xy)(\zeta_{i0}(y^{-1}x^{-1}) - z_i) \qquad (x \in P,\ y \in H) \\ &= (|H|/|L|)\sum\nolimits_x \theta(x)(\zeta_i(x^{-1}) - z_i) \qquad (x \in P) \\ &= (\theta|_P, \zeta_i - z_i\zeta_0)_P = c_i - z_ic_0 = c_i.\end{aligned}$$

Therefore, $\theta = \sum c_i\zeta_{i0} + \nu$, where $\nu = 0$ or ν is a character of L. By (3.1), $\nu(1) = 0$. Consequently, $\nu = 0$. This completes the proof of Lemma 3.1.

By Proposition 1, page 310, of [3], every subgroup H of G that has a normal p-complement satisfies $\lambda(H) = |H/K_p(H)|$. Let $x \in P^{\#}$. By Lemma 2.2 $C(x) = C_P(x)K_p(C(x))$ and $C_N(x) = C_P(x)K_p(N)$. Therefore,

$$\lambda(C(x)) = |C_P(x)| \tag{3.2}$$

and

$$\lambda(C(x))/|C_N(x)| = |C_P(x)|/|C_P(x)||K_p(N)| = 1/|K_p(N)|. \tag{3.3}$$

Assume θ and η are generalized characters of P. Since η is a class function on P and since $PK_p(N) = PC(P)$,

$$\sum_{n \in N} (\theta, \eta^n)_P = |PK_p(N)| \sum_{t \in T} (\theta, \eta^t)_P . \tag{3.4}$$

By (3.3), (3.4), and a result of Brauer (Lemma 3, page 315, of [3]), we obtain:

LEMMA 3.2. *Let θ and η be generalized characters of P such that $\theta(1) = 0$. Then*

$$(\theta^G, \eta^G)_G = \sum_{t \in T} (\theta, \eta^t)_P . \tag{3.5}$$

By Lemma 2.2, $PK_p(N) = P \times K_p(N)$. As in the proof of Proposition 2.1, we see that $N/K_p(N)$ is a Frobenius group with Frobenius kernel $PK_p(N)/K_p(N)$. Consequently, Lemma 2.2 of [5] yields:

LEMMA 3.3. *Let ζ_i be an irreducible complex character of P. Let $t \in T$. Then $\zeta_i^t = \zeta_i$ if and only if $t = 1$ or $i = 0$.*

Let $\mathcal{S}$ be a set of non-principal irreducible complex characters of P with the following property: for every non-principal irreducible complex character ζ of P, there exists one and only one element ζ_i of $\mathcal{S}$ such that $(\zeta_i)^t = \zeta$ for some $t \in T$. We assume that $\zeta_1 \in \mathcal{S}$. Let $\mathcal{I}(\mathcal{S})$ be the set of all integral linear combinations of elements of $\mathcal{S}$, and let $\mathcal{I}_0(\mathcal{S})$ be the set of all $\theta \in \mathcal{I}(\mathcal{S})$ such that $\theta(1) = 0$.

By Lemma 3.3,

$$(\theta, \eta^t)_P = 0 \qquad \text{if} \quad t \in T, t \neq 1, \quad \text{and} \quad \theta, \eta \in \mathcal{I}(\mathcal{S}). \tag{3.6}$$

Hence by (3.5),

$$(\theta^G, \eta^G)_G = (\theta, \eta)_P \qquad \text{if} \quad \theta, \eta \in \mathcal{I}_0(\mathcal{S}). \tag{3.7}$$

If $\theta \in \mathcal{I}_0(\mathcal{S})$, then $\theta^*(x) = 0$ for every p'-element x in G. By a theorem of Brauer and Nesbitt [14, Lemma 1], $\theta^G(x) = 0$ for every p'-element x in G. In particular,

$$\theta^G(1) = 0 \qquad \text{if} \quad \theta \in \mathcal{I}_0(\mathcal{S}). \tag{3.8}$$

By (3.7), the mapping given by $\theta \rightarrow \theta^G$ is a (linear) isometry of $\mathcal{I}_0(\mathcal{S})$ into the ring of generalized characters of G. We say that $\mathcal{S}$ is *coherent* if $\mathcal{I}_0(\mathcal{S}) \neq \{0\}$ and if there exists a linear isometry τ of $\mathcal{I}(\mathcal{S})$ into the ring of generalized characters of G with the property that $\theta^\tau = \theta^G$ for all $\theta \in \mathcal{I}_0(\mathcal{S})$. Using a method introduced by Feit in [5], we obtain the following four lemmas:

LEMMA 3.4. *Let $q = |N/PK_p(N)|$. Suppose that P is Abelian and $|P| > q + 1$ or that P is a non-Abelian and $|P/P'| > 4q^2$. Then $\mathcal{S}$ is coherent.*

Proof. Let $y_1, \cdots, y_k$ be the distinct degrees of the elements of $\mathcal{S}$, arranged in strictly ascending order. Then $y_1 = 1$. For $i = 1, \cdots, k$, let $\mathcal{S}_i$ be the elements of degree y_i in $\mathcal{S}$, and let n_i be the number of elements of $\mathcal{S}_i$. By Lemma 3.3,

$$|P/P'| = 1 + n_1 q.$$

By our hypothesis, $n_1 \geq 2$. By Theorem 31.2, page 183, of [6], $\mathcal{S}$ is coherent unless there exists an integer m such that $1 < m \leq k$ and

$$\sum_{i=1}^{m-1} n_i(y_i)^2 \leq 2y_m. \tag{3.9}$$

Assume m exists. Since $m > 1$, P is not Abelian. By Lemma 2.1 of [5] and by (3.9) and Lemma 3.3,

$$(y_m)^2 \leq 1 + q\sum_{i=1}^{m-1} n_i(y_i)^2 \leq 1 + 2qy_m.$$

Therefore, $y_m(y_m - 2q) \leq 1$. Since $y_m > 1$, $y_m \leq 2q$. By Lemma 2.3(iii), q divides $|P/P'| - 1$. Since y_m is a power of p and $q > 1$, we have $y_m < 2q$. But by (3.9),

$$\begin{aligned} y_m \geq \tfrac{1}{2}n_1(y_1)^2 = \tfrac{1}{2}n_1 &= (1/2q)(|P/P'| - 1) \\ &\geq (1/2q)4q^2 = 2q. \end{aligned}$$

This contradiction completes the proof of Lemma 3.4.

LEMMA 3.5. *Suppose $x \in P^\#$ and y is a p'-element of G. Then*

$$\sum \chi(x)\chi(x^{-1}) = |C_P(x)| \quad \textit{and} \quad \sum \chi(x)\chi(y^{-1}) = 0,$$

where χ ranges over all the elements of $B_0(G)$.

Proof. By Proposition 2, page 310, of [3],

$$\sum \chi(x)\chi(x^{-1}) = \lambda(C(x)).$$

Thus the first equation follows from (3.2). The second equation is a special case of equation (85.21), page 609, of [4].

Let I be the set of subscripts i such that $\zeta_i \in \mathcal{S}$. Suppose $\mathcal{S}$ is coherent and $i \in I$. Then $\|\zeta_i^\tau\|^2 = \|\zeta_i\|^2 = 1$. Thus there exists $\varepsilon_i = \pm 1$ such that $\varepsilon_i\zeta_i^\tau$ is an irreducible character of G. Let $\chi_i = \varepsilon_i\zeta_i^\tau$. We call the characters χ_i, $i \in I$, the *exceptional* characters of G. Let $\varepsilon = \varepsilon_1$. Define

$$\zeta_i' = \sum_{t \in T}\zeta_i^t \qquad \text{for all } i \in I.$$

Denote the character of the regular representation of P by ρ.

LEMMA 3.6. *Suppose $\mathcal{S}$ is coherent. Then $\varepsilon_i = \varepsilon$* and *$\chi_i \in B_0(G)$ for all $i \in I$. Moreover,*

$$z_j\zeta_i^G - z_i\zeta_j^G = \varepsilon(z_j\chi_i - z_i\chi_j) \tag{3.10}$$

for all $i, j \in I$. There exist integers a and c such that

$$\chi_i|_P = \varepsilon\zeta_i' + z_i c\zeta_0 + z_i a\rho \qquad \textit{for all } i \in I. \tag{3.11}$$

$B_0(G)$ must contain at least one non-principal, non-exceptional character. If

χ_i is a non-exceptional character in $B_0(G)$, there exist integers a_i and c_i such that

$$\chi_i|_P = c_i\zeta_0 + a_i\rho. \tag{3.12}$$

Proof. Let $i, j \in I$. Then $z_j\zeta_i(1) - z_i\zeta_j(1) = 0$. Hence

$$z_j\zeta_i^G - z_i\zeta_j^G = (z_j\zeta_i - z_i\zeta_j)^G$$
$$= (z_j\zeta_i - z_i\zeta_j)^\tau = z_j\zeta_i^\tau - z_i\zeta_j^\tau = \varepsilon_i z_j\chi_i - \varepsilon_j z_i\chi_j.$$

By (3.8)

$$0 = \varepsilon_i z_j\chi_i(1) - \varepsilon_j z_i\chi_j(1).$$

Taking $j = 1$, we see that $\varepsilon_i = \varepsilon_1 = \varepsilon$. Hence $\varepsilon_i = \varepsilon_j = \varepsilon$ for all $i, j \in I$. Thus we obtain (3.10). Clearly, $\chi_i \in B_0(G)$ for all $i \in I$.

Since χ_1 is a class function on G,

$$(\chi_1|_P, \zeta_i^t)_P = (\chi_1|_P, \zeta_i)_P \qquad \text{for all} \quad i \in I \quad \text{and} \quad t \in T^\cdot$$

Therefore, there exist integers c_0 and c_i, $i \in I$, such that

$$\chi_1|_P = c_0\zeta_0 + \textstyle\sum_{i\in I} c_i\zeta_i'. \tag{3.13}$$

Take $i \in I$ different from 1. Then

$$c_i - z_ic_1 = (\chi_1|_P, \zeta_i)_P - z_i(\chi_1|_P, \zeta_1)_P = (\chi_1|_P, \zeta_i - z_i\zeta_1)_P$$
$$= (\chi_1, (\zeta_i - z_i\zeta_1)^*)_G = (\chi_1, (\zeta_i - z_i\zeta_1)^G)_G.$$

Therefore, by (3.10),

$$c_i - z_ic_1 = \varepsilon(\chi_1, \chi_i - z_i\chi_1)_G = -z_i\varepsilon. \tag{3.14}$$

Let $a = c_1 - \varepsilon$ and $c = c_0 - a$. By (3.13) and (3.14),

$$\chi_1|_P = (c + a)\zeta_0 + \varepsilon\zeta_1' + \textstyle\sum_{i\in I} z_i a\zeta_i' = c\zeta_0 + \varepsilon\zeta_1' + a\rho. \tag{3.15}$$

Take $j \in I$ different from 1. Let ζ be an irreducible complex character of P. By (3.10),

$$(\chi_j|_P - z_j\chi_1|_P, \zeta)_P = (\chi_j - z_j\chi_1, \zeta^*)_G$$
$$= (\chi_j - z_j\chi_1, \zeta^G)_G = \varepsilon((\zeta_j - z_j\zeta_1)^G, \zeta^G)_G.$$

Hence, by (3.5),

$$(\chi_j|_P - z_j\chi_1|_P, \zeta)_P = \varepsilon\textstyle\sum_{t\in T}(\zeta_j - z_j\zeta_1, \zeta^t)_P. \tag{3.16}$$

Since χ_j and χ_1 are class functions on G, there exist integers d_0 and d_i, $i \in I$, such that

$$\chi_j|_P - z_j\chi_1|_P = d_0\zeta_0 + \textstyle\sum_{i\in I} d_i\zeta_i'.$$

By (3.16), $d_0 = 0$ and $d_i = 0$ if $i \in I$ and $i \neq 1, j$. Also, $d_1 = -\varepsilon z_j$ and $d_j = \varepsilon$. Thus

$$\chi_j|_P - z_j\chi_1|_P = \varepsilon\zeta_j' - \varepsilon z_j\zeta_1'. \tag{3.17}$$

By (3.15) and (3.17), we obtain (3.11).

Assume for the moment that χ_0 is the only non-exceptional character in $B_0(G)$. Take $x \in P^{\#}$. Equation (3.11) yields

$$\chi_i(1) = \varepsilon q z_i + z_i c + z_i a |P| = z_i \chi_1(1) \qquad \text{for } i \in I.$$

By Lemma 3.5,

$$0 = 1 + \textstyle\sum_{i \in I} \chi_i(x)\chi_i(1) = 1 + \chi_1(1) \sum z_i \chi_i(x).$$

Since all the quantities in the above equation are algebraic integers, $\chi_1(1) = 1$. Thus $G' \subseteq \operatorname{Ker} \chi_1$. Since $q > 1$, $P \subseteq N'$ by Lemma 2.3. Therefore $\chi_1|_P = \zeta_0$. If I contains more than one subscript, (3.11) yields $a = \epsilon = 0$, which is false. Therefore, I $= \{1\}$. By Lemma 3.5,

$$0 = 1 + \chi_1(x)\chi_1(1) = 1 + 1,$$

a contradiction. Thus χ_0 is not the only non-exceptional character in $B_0(G)$.

Let χ_i be a non-exceptional character in $B_0(G)$. There exist integers c_{i0} and c_{ij}, $j \in I$, such that

$$\chi_i|_P = c_{i0}\zeta_0 + \textstyle\sum_{j \in I} c_{ij}\zeta_j'.$$

For $j \in I$,

$$\begin{aligned} &c_{ij} - z_j c_{i1} \\ &\quad = (\chi_i|_P, \zeta_j - z_j\zeta_1)_P = (\chi_i, (\zeta_j - z_j\zeta_1)^G)_G = \varepsilon(\chi_i, \chi_j - z_j\chi_1)_G = 0. \end{aligned}$$

Therefore

$$\chi_i|_P = c_{i0}\zeta_0 + \textstyle\sum_{j \in I} c_{i1} z_j \zeta_j' = (c_{i0} - c_{i1})\zeta_0 + c_{i1}\rho.$$

Thus we obtain (3.12).

LEMMA 3.7. *Suppose* $\mathcal{S}$ *is coherent. Assume* P *is not an elementary Abelian group on which* $N(P)$ *acts irreducibly. Then there exists a non-negative integer* a *such that*

$$\chi_i|_P = \varepsilon\zeta_i' + z_i a\rho \qquad \textit{for all } i \in I. \tag{3.18}$$

Proof. By our hypothesis, there exists a normal subgroup P_1 of N such that $1 \subset P_1 \subset P$. By Lemma 2.3 (iii), q divides $|P_1| - 1$ and $|P/P_1| - 1$. Therefore,

$$|P| \geq (q+1)^2 = q^2 + 2q + 1. \tag{3.19}$$

By Lemma 3.6, there exist integers a and c such that

$$\chi_i|_P = \varepsilon\zeta_i' + z_i c\zeta_0 + z_i a\rho \qquad \text{for all } i \in I. \tag{3.11}$$

Let $\mu = \zeta_1 - \zeta_0$. By Lemma 3.2,

$$\begin{aligned} \|\mu^G\|^2 &= \textstyle\sum_{t \in T} (\mu, \mu^t)_P = (\zeta_1 - \zeta_0, \sum_{t \in T}(\zeta_1^t - \zeta_0))_P \\ &= (\zeta_1 - \zeta_0, \zeta_1' - q\zeta_0)_P = 1 + q. \end{aligned} \tag{3.20}$$

Take $i \in I$. Then

$$(\mu^G, \chi_i)_G = (\mu, \chi_i|_P)_P = (\zeta_1, \chi_i|_P)_P - (\zeta_0, \chi_i|_P)_P .$$

Hence by (3.11),

$$(\mu^G, \chi_1)_G = \varepsilon - c \quad \text{and} \quad (\mu^G, \chi_i)_G = -cz_i \qquad \text{if } i \in I \text{ and } i \neq 1.$$

Moreover,

$$(3.21) \qquad (\mu^G, \chi_0)_G = (\mu, \chi_0|_P)_P = (\zeta_1 - \zeta_0, \zeta_0)_P = -1.$$

Therefore, (3.20) yields

$$1 + q \geq (-1)^2 + (\varepsilon - c)^2 + \sum_{1 \neq i \in I} z_i^2 c^2 = 2 - 2\varepsilon c + c^2 \sum_{i \in I} z_i^2.$$

By Lemma 3.3, $\sum_{i \in I} z_i^2 = (1/q)(|P| - 1)$. Consequently, by (3.19),

$$\begin{aligned} 1 + q &\geq 2 - 2\varepsilon c + (c^2/q)(|P| - 1) \geq 2 - 2\varepsilon c + (q + 2)c^2 \\ &\geq 2 + c^2(q + 2) - 2|c| = 2 + |c|(|c|(q + 2) - 2). \end{aligned}$$

Since c is an integer, c must be zero. Thus (3.18) coincides with (3.11). Since $a = (\chi_1|_P, \zeta_0)_P$, $a \geq 0$.

LEMMA 3.8. *Suppose $x \in G$ and $\chi \in B_0(G)$. Let π be the p-part of x. Then $\chi(x) = \chi(\pi)$ if $\pi \neq 1$.*

Proof. This is Corollary 5, page 159, of [2].

4. A commutator condition

Suppose that G satisfies (a), (b), and (c) and that P is not an elementary Abelian group on which $N(P)$ acts irreducibly. By Proposition 2.3, G satisfies the following condition:

(c′) *P contains a non-identity element v and a proper subgroup P_0 with the property that whenever $g, h \in G$ and the p-part of $v^g(v^h)^{-1}$ lies in P, the p-part of $v^g(v^h)^{-1}$ lies in P_0.*

Note that if P is Abelian, then $|P| > 1 + q$.

This section and the next one are devoted to the proof of the following result:

THEOREM 4.1. *Let p be a prime, and let P be a Sylow p-subgroup of a finite p-core-free group G. Suppose that G satisfies* (a), (b), and (c′) *and that P is not a normal subgroup of G. Let $N = N(P)$ and $q = |N/PK_p(N)|$. Then:*

(i) *P normalizes no p′-subgroup of G except 1;*
(ii) *P is a non-Abelian group and $|P/P'| < 4q^2$;*
(iii) *$q > 1$ and q is odd;*
(iv) *$Z(P)$ is not a cyclic group;*
(v) *the centralizer of every non-identity element of P is contained in P;*

and

(vi) *every pair of distinct Sylow p-subgroups of G intersects in the identity group.*

In this section we use the results of Section 3 to prove (i) and (ii). Throughout this section we assume that G satisfies the hypothesis of Theorem 4.1, and thus that G does not have a normal p-complement. We also assume that P_0 is a maximal subgroup of P. By Sylow's Theorem, P_0 is a normal subgroup of P. Therefore, $|P/P_0| = p$. We assume that $P_0 = \operatorname{Ker} \zeta_1$.

Let $\mu = \zeta_1 - \zeta_0$. Then $\mu(x) = 0$ for all $x \in P_0$. By a result of Brauer (Lemma 4, page 316, of [3], with an obvious change in the statement and proof), condition (c$'$) yields:

LEMMA 4.1. *Let χ range over the characters of $B_0(G)$. Then*

$$\sum (\mu^G, \chi)_G |\chi(v)|^2/\chi(1) = 0.$$

By Lemma 3.5,

$$\sum |\chi(v)|^2 = |C_P(v)| \leq |P| \qquad (\chi \in B_0(G)). \tag{4.1}$$

Since $\chi_0 \in B_0(G)$, the Frobenius reciprocity theorem yields

$$(\mu^G, \chi_0)_G = (\mu, \zeta_0)_P = (\zeta_1 - \zeta_0, \zeta_0)_P = -1.$$

Thus, by Lemma 4.1 and (4.1),

$$\sum (\mu^G, \chi)_G |\chi(v)|^2/\chi(1) = 1 \tag{4.2}$$

and

$$\sum |\chi(v)|^2 \leq |P| - 1, \tag{4.3}$$

where in both equations we sum over the non-principal characters in $B_0(G)$. Therefore, there exists a non-principal character ψ in $B_0(G)$ such that

$$(\mu^G, \psi)_G \geq 1 \quad \text{and} \quad \psi(1)/(\mu^G, \psi)_G \leq |P| - 1. \tag{4.4}$$

Let $\psi|_P = c_0\zeta_0 + \sum_{i \in I} c_i \zeta_i'$. By (3.20) and (3.7),

$$\|\mu^G\|^2 = 1 + q \tag{4.5}$$

and, if $i \in I$ and $i \neq 1$,

$$\|\zeta_i^G - z_i \zeta_1^G\|^2 = \|\zeta_i - z_i\zeta_1\|^2 = 1 + z_i^2.$$

Thus

$$c_i - z_i c_1 = (\zeta_i - z_i \zeta_1, \psi|_P)_P = (\zeta_i^G - z_i\zeta_1^G, \psi)_G \geq -z_i.$$

Hence

$$\begin{aligned} \psi(1) &= c_0 + \textstyle\sum_{i \in I} c_i \zeta_i'(1) = c_0 + q \sum_{i \in I} c_i z_i \\ &\geq c_0 + qc_1 + q \textstyle\sum_{i \neq i \in I} (c_1 z_1 - z_i) z_i = c_0 + q + (c_1 - 1) q \sum_{i \in I} z_i^2 \\ &= c_0 + q + (c_1 - 1)(|P| - 1). \end{aligned} \tag{4.6}$$

Suppose $c_0 \geq 1$. Since

$$\begin{aligned} c_1 &= (\zeta_1, \psi|_P)_P = (\zeta_1 - \zeta_0, \psi|_P)_P + (\zeta_0, \psi|_P)_P = (\mu, \psi|_P)_P + c_0 \\ &\geqq (\mu, \psi|_P)_P + 1 = (\mu^G, \psi)_G + 1, \end{aligned}$$

(4.6) yields

$$\psi(1) > (\mu^G, \psi)_G(|P| - 1),$$

contrary to (4.4). Thus

$$(4.7) \qquad (\zeta_0, \psi|_P) = c_0 = 0.$$

We claim that G satisfies (i). Let H be a p'-subgroup of G that is normalized by P. We wish to show that $H = 1$.

Let $L = PH$. Take $x \in P^{\#}$ and $y \in H$, and let π be the p-part of xy. Then π is conjugate in L to an element of P. Since $L = PH = HP$, there exists $h \in H$ such that $\pi^h \in P$. Then $x^{-1}\pi^h \in P$ and $x \equiv \pi \equiv \pi^h$ modulo H. Hence $x^{-1}\pi^h = 1$. Therefore, $\psi(\pi) = \psi(x)$. By Lemma 3.8, $\psi(xy) = \psi(\pi)$. Thus

$$(4.8) \qquad \psi(xy) = \psi(x) \qquad \text{for } x \in P^{\#} \text{ and } y \in H.$$

Since $c_0 = 0$, $P \nsubseteq \operatorname{Ker} \psi$. As $\operatorname{Ker} \psi$ is a normal subgroup of G, $\operatorname{Ker} \psi = 1$ by Proposition 2.1. Let $\theta = \psi|_L$. By Lemma 3.1, $H \subseteq \operatorname{Ker} \theta = 1$. This completes the proof of part (i) of Theorem 4.1.

In the remainder of this section, we will assume that G violates (ii) and will aim for a contradiction. Let P_1 be the subgroup of P generated by those elements of P that occur as the p-part of the commutator $v^g(v^h)^{-1}$ for some $g, h \in G$. Since

$$(v^g(v^h)^{-1})^x = v^{gx}(v^{hx})^{-1} \qquad \text{for all } g,\ h,\ x \in G,$$

P_1 is a normal subgroup of N. Clearly, $P_1 \subseteq P_0 \subset P$. Since $N/K_p(N)$ is a Frobenius group,

$$(4.9) \qquad P_1 \neq 1.$$

By Lemma 2.3 (iii), $N/P_1K_p(N)$ is a Frobenius group with Frobenius kernel $P/P_1K_p(N)$. But $v^g \equiv v \bmod P_1$, for all $g \in N$. Thus

$$(4.10) \qquad v \in P_1.$$

From (3.19), $|P| \geq (q+1)^2 > q + 1$. Since G violates (ii), $\mathcal{S}$ is coherent by Lemma 3.4. By Lemma 3.7, there exist $\varepsilon = \pm 1$ and an integer a such that

$$(4.11) \qquad \chi_i|_P = \varepsilon\zeta_i' + az_i\rho \qquad \text{for all } i \in I.$$

Therefore,

$$(4.12) \qquad (\mu, \chi_1|_P)_P = (\zeta_1 - \zeta_0, \chi_1|_P) = \varepsilon$$

and, if $i \in I$ and $i \neq 1$,

(4.13) $$(\mu, \chi_i|_P)_P = 0.$$

By (4.2)

(4.14) $$1 = \varepsilon|\chi_1(v)|^2/\chi_1(1) + \sum(\mu^G, \chi)_G|\chi(v)|^2/\chi(1),$$

where χ ranges over the non-principal, non-exceptional characters in $B_0(G)$.

By (4.10),

$$v \in P_1 = (P_1)^t \subseteq (P_0)^t = \mathrm{Ker}\,(\zeta_1)^t \quad \text{for every } t \in T.$$

Hence $v \in \mathrm{Ker}\,\zeta_1'$. By (4.11),

(4.15) $$\chi_1(v) = \varepsilon q \neq 0.$$

Since $|P| > q + 1$, I contains at least two subscripts. By (4.11), $P \not\subseteq \mathrm{Ker}\,\chi_1$. By Proposition 2.1, $\mathrm{Ker}\,\chi_1 = 1$. Since $v \in \mathrm{Ker}\,\zeta_1'$, $a \geq 1$. Thus, by (4.11) and (4.12),

(4.16) $$\text{either} \quad \varepsilon = -1 \quad \text{or} \quad \chi_1(1) \geq q + |P| > |P| - 1.$$

By (4.14), (4.15), (4.16), and (4.3), there exists a non-principal, non-exceptional character ψ in $B_0(G)$ that satisfies the equation

(4.17) $$(\mu^G, \psi)_G \geq 1 \quad \text{and} \quad \psi(1)/(\mu^G, \psi)_G < |P| - 1.$$

By Lemma 3.6, there exist integers d and e such that $\psi|_P = d\zeta_0 + e\rho$. By (4.17),

$$1 \leq (\mu^G, \psi)_G = (\mu, \psi|_P)_P = (\zeta_1, \psi|_P)_P - (\zeta_0, \psi|_P)_P = -d.$$

Since $d + e = (\zeta_0, \psi|_P) \geq 0$, we obtain

$$\psi(1) = d + e|P| \geq d - d|P| = (-d)(|P| - 1) = (\mu^G, \psi)_G(|P| - 1),$$

contrary to (4.17). This contradiction completes the proof of statement (ii) of Theorem 4.1.

5. Proof of Theorem 4.1

LEMMA 5.1. *Let p be a prime, and let X and Y be subgroups of a finite group. Suppose X is a non-cyclic group of exponent p, Y is a p'-group, X normalizes Y, and $C_Y(x) = 1$ for all $x \in X^\#$. Then $Y = 1$.*

Proof. This is a theorem of Burnside, generalized by Wielandt in (3.3), page 149, of [13].

Let G be a finite group that satisfies the hypothesis of Theorem 4.1. By the previous section, G satisfies the following conditions:

(i) *P normalizes no p'-subgroup of G except 1; and*
(ii) *P is a non-Abelian group and $|P/P'| < 4q^2$.*

By Proposition 2.1 and Lemma 2.3 (i), $q > 1$ and q is odd. Thus G satisfies (iii).

Suppose that G violates (iv), i.e., that $Z(P)$ is cyclic. Let Z be the subgroup of order p in $Z(P)$. By Lemma 2.3 (iii), q divides $p - 1$. Consequently, p is odd. Since q is odd, $2q$ divides $p - 1$. Let $|P/P'| = p^n$. Then

$$(1 + 2q)^n \leq p^n = |P/P'| < 4q^2.$$

Thus $n = 1$. Therefore, P/P' is cyclic. By the Burnside Basis Theorem [8, page 176], P is generated by one element. But this contradicts (ii). Hence $Z(P)$ is not cyclic.

Take $x \in Z(P)^{\#}$. Then $P \subseteq C(x)$ and, by (b), $C(x)$ has a normal p-complement. By (ii), this complement is the identity group. Thus

$$C(x) = P \qquad \text{for } z \in Z(P)^{\#}. \tag{5.1}$$

Since $Z(P)$ is a non-cyclic Abelian p-group, $Z(P)$ contains an elementary Abelian subgroup X of order p^2. Let $y \in P^{\#}$. Then $X \subseteq C_P(y)$. By (5.1) and Lemma 5.1, $K_p(C(y)) = 1$. Lemma 2.2 yields $C(y) \subseteq P$.

Let R be a Sylow p-subgroup of G such that $P \cap R \neq 1$. Take $y \in (P \cap R)^{\#}$. Then $Z(R) \subseteq C(y) \subseteq P$. Since $Z(R)$ is conjugate to $Z(P)$ in G, $Z(R) = Z(P)$ by (a). Hence

$$R \subseteq C(Z(R)) = C(Z(P)) \subseteq P.$$

Thus $R = P$. This completes the proof of Theorem 4.1.

COROLLARY 5.1. *Let P be a Sylow 2-subgroup of a finite 2-core-free group G. Suppose that G satisfies* (a) *and* (b) *and that P is not an elementary Abelian group on which $N(P)$ acts irreducibly. Assume that P is not a normal subgroup of G. Then G is a Suzuki group.*

Conversely, every Suzuki group satisfies the above conditions.

Proof. Assume that G satisfies the above conditions. Since every pair of elements of order two generates a dihedral group, G satisfies (c). By Proposition 2.3, G satisfies (c′). By Theorem 4.1, G satisfies the following conditions:

(CIT) The centralizer of every element of order 2 is a 2-group.

(TI) Every pair of distinct Sylow 2-subgroups of G intersects in the identity subgroup.

By Theorem I.5, page 434, of [11], G is a (ZT)-group, in the notation of Suzuki. However, Suzuki has proved [12] that the only (ZT)-groups are the Suzuki groups and the groups $PSL(2, 2^n)$, $n \geqq 2$. For $n \geqq 2$, the normalizer of a Sylow 2-subgroup of $PSL(2, 2^n)$ acts irreducibly on the Sylow 2-subgroups. Conversely, every (ZT)-group satisfies (TI) and (CIT), which imply conditions (a) and (b).

Remark 5.1. Suppose p is an odd prime. There are many examples of finite simple groups that satisfy (a) and (b) and do not have elementary Abelian Sylow p-subgroups. However, we do not know whether there are any examples in which the Sylow p-subgroups are non-Abelian.

6. The Abelian case

In this section we retain our previous notation and results and work with the group ring of G over the integers. Suppose $C_1, \cdots, C_k$ are the distinct conjugate classes of G. For $i = 1, \cdots, k$, let h_i be the number of elements of C_i and let K_i be the *class sum* of C_i, that is, the sum of the elements of C_i in the group ring. For each i, choose a representative g_i of C_i. It is well known [8, page 277] that for each K_i and K_j there exist non-negative integers c_{ijs} $(s = 1, \cdots, k)$ such that

$$K_i K_j = \sum_{s=1}^{k} c_{ijs} K_s \tag{6.1}$$

and

$$h_i h_j \chi(g_i)\chi(g_j)/\chi(1) = \sum_{s=1}^{k} c_{ijs} h_s \chi(g_s) \tag{6.2}$$

for every irreducible character χ of G. Moreover, by Lemma 3.5,

$$\sum_{\chi} \chi(x)\chi(x^{-1}) = |C_P(x)| \quad \text{and} \quad \sum_{\chi} \chi(x)\chi(y^{-1}) = 0 \quad (\chi \in B_0(G)) \tag{6.3}$$

if $x \in P^{\#}$ and y is a p'-element of G. We say that an element of G is a *p-singular* element if it is not a p'-element.

THEOREM 6.1. *Let P be a Sylow p-subgroup of a finite simple, non-Abelian group G that satisfies* (a) *and* (b). *Suppose P is Abelian and $|P| > q + 1$. Let x and y be p-singular elements of G. Then there exist g, $h \in G$ such that $x^g y^h$ is a p'-element of G.*

Proof. Assume the conclusion is false. Let K_i and K_j be the class sums of the conjugate classes of x and y respectively. Take c_{ijs} as in (6.1) for $s = 1, \cdots, k$. By hypothesis, $c_{ijs} = 0$ *if* g_s is a p'-element. Let χ be any character in $B_0(G)$ such that χ is constant on the p-singular elements of G. Then $c_{ijs}\chi(g_s) = c_{ijs}\chi(x)$ for $s = 1, \cdots, k$. By (6.2),

$$h_i h_j \chi(x)^2/\chi(1) = h_i h_j \chi(x)\chi(y)/\chi(1) = \sum_{s=1}^{k} c_{ijs} h_s \chi(x). \tag{6.4}$$

Substituting $\chi = \chi_0$ in (6.4), we obtain

$$h_i h_j = \sum_{s=1}^{k} c_{ijs} h_s. \tag{6.5}$$

The set $\mathcal{S}$ of characters is coherent, by Lemma 3.4. By Lemma 3.6, there exists a non-principal, non-exceptional character χ in $B_0(G)$, and χ is constant on the elements of $P^{\#}$. From Lemma 3.8, χ is constant on the p-singular elements of G. By (6.4) and (6.5),

$$\chi(x)^2/\chi(1) = \chi(x). \tag{6.6}$$

Since $\chi \in B_0(G)$ and p divides $|G|$, χ does not have defect zero. Therefore, χ is non-zero on some p-singular element of G. Hence $\chi(x) \neq 0$. By (6.6), $\chi(x) = \chi(1)$. But then $x \in \operatorname{Ker} \chi$, which is impossible because G is simple.

THEOREM 6.2. *Let p be a prime, and let P be a Sylow p-subgroup of a finite simple non-Abelian group G that satisfies* (a), (b), *and* (c). *Suppose P is a non-identity elementary Abelian group and all the elements of $P^{\#}$ are conjugate in G. Then $p = 2$.*

Proof. Take $\pi \in P^{\#}$. Let K_i be the class sum of the conjugate class of π in G. Then K_i is the sum of all the elements of order p in G. For $s = 1, \cdots, k$, take c_{iis} as in (6.1) and let $c_s = c_{iis}$. Unlike the situation in Theorem 6.1, we may have $c_s \neq 0$ when g_s is a p'-element. However, since any two distinct subgroups of order p intersect in the identity group, $K_i K_i$ may be written as a sum of products of the form

$$(x + x^2 + \cdots + x^{p-1})(y + y^2 + \cdots + y^{p-1}),$$

where x and y are elements of order p.

Let I be the set of integers s such that $1 \leq s \leq k$ and g_s is p-singular. Suppose x and y are elements of order p in G and H is the subgroup of G that they generate. By Proposition 2.2, $H/K_p(H)$ is an Abelian p-group. Therefore, for every power x^i of x, $x^i y^j$ is p-singular for at least p-2 values of j between 1 and $p - 1$. Hence

$$\sum_{s \in I} c_s h_s \geq \frac{p-2}{p-1} \sum_{s=1}^{k} c_s h_s .$$

Let $h = h_i$. Since K_i is a sum of h elements, (6.1) and the above equa tion yield

$$(6.7) \qquad \textstyle\sum_{s \in I} c_s h_s \geq h^2(p-2)/(p-1).$$

Let

$$r = (1/h^2) \textstyle\sum_{s \in I} c_s h_s .$$

Suppose $\chi \in B_0(G)$. By Lemma 3.8 and our hypothesis, χ is constant on the p-singular elements of G. By (6.2),

$$(6.8) \qquad h^2 \chi(\pi)^2/\chi(1) = \textstyle\sum_{1 \leq s \leq k, s \notin I} c_s h_s \chi(g_s) + h^2 r \chi(\pi).$$

Taking $\chi = \chi_0$ in (6.8), we obtain

$$(6.9) \qquad \textstyle\sum_{1 \leq s \leq k, s \notin I} c_s h_s = (1 - r)h^2.$$

By (6.7), (6.8), and (6.9),

$$(6.10) \qquad r \geq (p-2)/(p-1).$$

Now, π^{-1} is conjugate to π in G. Multiply both sides of (6.8) by $\chi(\pi)$

and sum over all $\chi \in B_0(G)$. By (6.3),

$$h^2 \sum_\chi (\chi(\pi)^3/\chi(1)) = h^2 r \sum_\chi \chi(\pi)^2. \tag{6.11}$$

By (6.3) (with $y = 1$), there exists $\chi_1 \in B_0(G)$ such that $\chi_1(\pi) < 0$. For every non-principal character χ in $B_0(G)$, let $f(\chi) = \chi(\pi)/\chi(1)$. From (6.11),

$$1 + f(\chi_1)\chi_1(\pi)^2 + \sum_\chi f(\chi)\chi(\pi)^2 = r + r\chi_1(\pi)^2 + r\sum_\chi \chi(\pi)^2, \tag{6.12}$$

where we sum over all $\chi \in B_0(G)$ distinct from χ_0 and χ_1. Since π is conjugate to all of its non-identity powers, $\chi_1(\pi)$ is a (negative) integer. By (6.12),

$$\sum_\chi (f(\chi) - r)\chi(\pi)^2 = r + r\chi_1(\pi)^2 - 1 - f(\chi_1)\chi_1(\pi)^2 \geqq 2r - 1. \tag{6.13}$$

Assume p is odd. We shall obtain a contradiction. By (6.10), $r \geq \frac{1}{2}$. Hence by (6.13), there exists $\chi \in B_0(G)$ such that

$$\chi \neq \chi_0 \quad \text{and} \quad f(\chi) \geq r \geq \tfrac{1}{2}. \tag{6.14}$$

Let ρ be the character of the regular representation of P, and let $w = |P|$. Since $\chi|_P - \chi(\pi)\zeta_0$ is zero on $P^{\#}$, there exists a real number b such that

$$\chi|_P = \chi(\pi)\zeta_0 + b\rho. \tag{6.15}$$

Since $b = (\chi|_P, \zeta_0) - \chi(\pi)$, b is an integer. Since $\chi \neq \chi_0$,

$$b \geq 1. \tag{6.16}$$

By (6.14) and (6.15),

$$\tfrac{1}{2} \leq f(\chi) = \chi(\pi)/\chi(1) = (\chi(1) - bw)/\chi(1).$$

Thus

$$\chi(1) \leq 2\chi(\pi) \quad \text{and} \quad b \leq \chi(1)/2w. \tag{6.17}$$

By (6.3), $\chi(\pi) \leq w^{1/2}$. Thus by (6.17),

$$b \leq \chi(\pi)/w \leq w^{-1/2} < 1,$$

which contradicts (6.16). This completes the proof of Theorem 6.2.

7. Proof of main theorem

THEOREM 7.1. *Let p be a prime, and let P be a Sylow p-subgroup of a finite p-core-free group G. Suppose P is not a normal subgroup of G. Then G satisfies* (a), (b), (c), *and* (d) *if and only if $p = 2$ and G is a Suzuki group.*

Proof. For $p = 2$, every finite group satisfies (c). By Corollary 5.1, the Suzuki groups satisfy (a), (b), and (d). Since they have even order and are simple and non-Abelian, they are 2-core-free and do not have normal Sylow 2-subgroups.

Conversely, assume G satisfies the hypothesis of the theorem. We claim that P is not Abelian. Assume P is Abelian. Then G contains a simple, non-Abelian subgroup M that contains P and satisfies (a) and (b), by Proposition 2.1. Clearly, M satisfies (c) and (d). Thus we may assume that $G = M$. By (d) and Theorem 6.2, the elements of $P^{\#}$ are not all conjugate in G. Hence $|P| - 1 > q$. Take $x, y \in P^{\#}$ such that x and y are not conjugate in G. By Theorem 6.1, there exist $g, h \in G$ such that $x^g(y^{-1})^h$ is a p'-element. But this contradicts Proposition 2.2(iv).

Thus P is not Abelian. If $p = 2$, then G is a Suzuki group, by Corollary 5.1. Assume p is odd. Since P is non-Abelian, a theorem of Shult (Corollary 1 of [10]) guarantees that the subgroups of order p in P are not all conjugate in $N(P)$. Let Q be a subgroup of order p in $Z(P)$, and choose R to be a subgroup of order p in P such that R is not conjugate to Q in $N(P)$. By (a), Q and R are not conjugate in G.

Take $g \in G$. Let H be the subgroup of G generated by Q and R^g. By Proposition 2.2, $H/K_p(H)$ is an Abelian p-group. Let x and y be generators of Q and R respectively. Then no non-identity power of x is conjugate to y^g. By Proposition 2.2(iv),

$$x^i(y^g)^{-1} \notin K_p(H), \qquad i = 1, 2, \cdots, p-1.$$

Thus $H/K_p(H)$ is not cyclic. Since x and y^g generate H, $H/K_p(H)$ is an elementary Abelian group of order p^2. Let H_p be a Sylow p-subgroup of H. Then H_p is isomorphic to $H/K_p(H)$.

Since P is non-Abelian, G satisfies (c′) by Proposition 2.3. By Theorem 4.1,

$$C(z) \subseteq P \qquad \text{for all } z \in P^{\#}.$$

Hence $C(w) \cap K_p(H) = 1$ for all $w \in (H_p)^{\#}$. Since H_p is elementary Abelian of order p^2, $K_p(H) = 1$, by Lemma 5.1. Thus $H = H_p$. Hence $y^g \in C(x)$.

Since G is simple, the subgroup of G generated by all the elements y^g, $g \in G$, coincides with G. By the above paragraph, $G \subseteq C(x)$, that is, $x \in Z(G) = 1$. This contradiction completes the proof of Theorem 7.1.

COROLLARY 7.1. *Let P be a Sylow 2-subgroup of a finite 2-core-free group G. Suppose G satisfies* (a) *and* (b). *Then G satisfies at least one of the following conditions:*

(i) *G is a Frobenius group with Frobenius kernel P;*

(ii) *P is an elementary Abelian group whose non-identity elements are all conjugate in G;*

(iii) *G is a Suzuki group.*

Proof. Every pair of elements of order two in G generates a dihedral group, so G satisfies (c).

COROLLARY 7.2. *Let p be a prime and let G be a finite group with a cyclic Sylow p-subgroup. Then G is p-solvable if and only if every pair of elements of order p in G generates a p-solvable group.*

Proof. One part of the conclusion is obvious. For the converse part, assume that every pair of elements of order p in G generates a p-solvable group.

Let P be a Sylow p-subgroup of G, $M = K_p(G)$, $\bar{G} = G/M$, and $\bar{P} = PM/M$. Then $\bar{G}$ is a p-core-free group. We may assume that $\bar{G} \neq 1$. Since every element of order p in $\bar{P}$ is a coset that contains an element of order p in P, $\bar{G}$ satisfies (c). By taking $W = \bar{P}$ in Lemma 2.1(ii), we see that $\bar{G}$ satisfies (a). Take $x \in \bar{P}$ such that $x \neq 1$, and let $C = C_{\bar{G}}(x)$. Then $\bar{P}$ is a Sylow p-subgroup of C. Moreover, C, like $\bar{G}$, satisfies (a).

Now, the identity automorphism is the only p'-element in the automorphism group of $\bar{P}$ that leaves an element of $\bar{P}^{\#}$ fixed [15, pages 145–146]. Since $N_C(\bar{P})/C_C(\bar{P})$ is a p'-group, and since $x \in Z(N_C(\bar{P}))$,

$$N_C(\bar{P}) = C_C(\bar{P}). \tag{7.1}$$

As C satisfies (a), C has a normal p-complement, by Lemma 2.1(i). Thus $\bar{G}$ satisfies (b).

If p is odd, then $\bar{P}$ is a normal subgroup of $\bar{G}$ by Theorem 7.1; therefore, $\bar{G}$ and G are p-solvable. Assume $p = 2$. Let x be the unique element of order two in $\bar{P}$. Then $N_{\bar{G}}(\bar{P}) \subseteq C$. By (7.1), $N_{\bar{G}}(\bar{P}) = C_{\bar{G}}(\bar{P})$. Since $\bar{G}$ satisfies (a), Lemma 2.1 (i) yields that $\bar{G}$ has a normal 2-complement. Therefore, $\bar{G} = \bar{P}$. Hence G has a normal 2-complement, and G is 2-solvable.

Remark 7.1. Suppose p is an odd prime and P is a cyclic Sylow p-subgroup of a finite group G that is not p-solvable. As in the above proof, $G/K_p(G)$ satisfies (a) and (b). Assume that the elements of $P^{\#}$ are not all conjugate in G. By using Proposition 2.1, and Theorems 4.1 and 6.1, we obtain the following violations of p-solvability (for a group with a cyclic Sylow p-subgroup):

(7.2) There exist two non-conjugate p-elements whose product is a p'-element.

(7.3) If $|P| > p$, there exist two elements of order p whose product has order divisible by $|P|$.

In some recent work which we hope to publish at a later date, we generalize Corollary 7.2 by showing that G is p-solvable whenever p is odd and G satisfies (b) and (c).

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