FINITE GROUPS WITH MAXIMAL NORMALIZERS

BY

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Beginning with Dedekind's famous determination of the groups in which each subgroup is normal, there are several results in the literature dealing with groups having many normal or "nearly normal" subgroups (for finite groups see e.g. [6] and the references given there). One of the possibilities for defining "nearly normal" is to require that the subgroup in question has a "big" normalizer. Thus, B. H. Neumann determined in [8] all infinite groups, the normalizers of all of whose subgroups have finite indices. In this work we are interested in finite groups, in which the normalizer of each (non-normal) subgroup is maximal. The main result is that there exists only one semi-simple such group, namely $PSL(2, 13)$. An analogous result states that if the centralizer of each (non-central) element is maximal, the group is solvable. Lemma 7, dealing with a special class of p-groups, may also have some independent interest.

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1. To begin with, we formulate more precisely our main result.

THEOREM 1.[1] *Let G be a non-solvable finite group. Suppose that for each subgroup H, satisfying* (a) *H is not subnormal and* (b) *H is either a p-group or a $\{p, q\}$-group, p and q being primes, the normalizer $N(H)$ is a maximal subgroup of G. Then $G = K \times S$, where $K \cong PSL(2, 13)$ or $K \cong SL(2, 13)$, S is abelian, and the orders of K and S are relatively prime.*

Conversely, if $G = K \times S$, K and S as above, then each non-normal subgroup of G has a maximal normalizer.

Proof. We assume at first that G is semi-simple. Then G has no subnormal solvable subgroups, so in particular each p-subgroup and each $\{p, q\}$-subgroup of G has a maximal normalizer (it will be seen from the proof, that in the semi-simple case the maximality of $N(H)$ is needed only for p-subgroups and subgroups which are extensions of p-groups by q-groups). The following lemmas, with the exception of Lemma 7, are proved under the assumption that G is semi-simple and satisfies the hypothesis of Theorem 1. It follows from this assumption that the order of G is even.

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LEMMA 1. *Let $A \neq 1$ be a p-subgroup of G, and let $N = N(A)$. Then N has a normal series $1 \lhd A \lhd K \lhd N$, in which N/K is a p-group, each subgroup of K/A is normal in N/A, and K/A is a Dedekind group of order relatively prime to p.*

Proof. If N is a p-group, choose $K = A$. If N is not a p-group, let $B/A \neq 1$ be any q-group of N/A, where $q \neq p$ and q is a prime. Then A is a normal Sylow subgroup of B, hence characteristic, hence normal in $N(B)$. Therefore $N(B) \subseteq N = N(A)$. However, $N(B)$ is a maximal subgroup, so $N(B) = N$ implying $B \lhd N$ and $B/A \lhd N/A$.

Letting B be, in turn, any of the Sylow subgroups of N/A for primes different from p, we find that N/A has a normal p-complement K/A. Then $|N:K| = |N/A:K/A|$ is a power of p. Any subgroup of K/A is generated by its Sylow subgroups, whose orders are relatively prime to p, therefore each such subgroup is normal in N/A. In particular, all subgroups of K/A are normal in K/A, so K/A is a Dedekind group.

Notice that Lemma 1 implies that the group N is solvable.

LEMMA 2. *Let A and N be as in Lemma 1. If A is not elementary abelian, then N has a normal p-complement.*

Proof. A not being elementary abelian is equivalent to $\Phi(A) \neq 1$, where $\Phi(A)$ is the Frattini subgroup of A. Suppose this is the case. Then $\Phi(A)$ is characteristic in A, so normal in N, therefore $N \subseteq N(\Phi(A))$, and $N = N(\Phi(A))$ by maximality of N. Apply the previous lemma to $\Phi(A)$. Let K be the subgroup defined there, and let T be a p-complement of K. Then $K = \Phi(A)T$ and $K \lhd N$. This implies $T^n \subseteq K$ for any $n \in N$. K is solvable, therefore all its p-complements are conjugate, so $T^n = T^k$ for some $k \in K$. Writing $k = ta$, $t \in T$, $a \in \Phi(A)$, we find $T^n = T^a$, $T^{na^{-1}} = T$, so $na^{-1} \in N_N(T)$, $n \in N_N(T)a$ and $N = N_N(T)\Phi(A)$. Since $\Phi(A) \lhd N$, a result of Gaschütz [2, Satz 5] yields $\Phi(A) \subseteq \Phi(N)$, so the factorization of N shows $N = N_N(T)$ and T is a normal p-complement of N.

LEMMA 3. *A Sylow 2-subgroup of G is either elementary abelian or maximal.*

Proof. Let P be a Sylow 2-subgroup of G, and $N = N(P)$. Suppose that P is not elementary abelian. Let T be the normal 2-complement of N, which exists by Lemma 2. Lemma 1, for $A = P$, shows that T is a Dedekind group. Also, $N = T \times P$. Suppose $T \neq 1$, and let A be any non-identity p-subgroup of T. Then $A \lhd N$, so $N = N(A)$ by maximality. Another application of Lemma 1 (for the given A), shows that P is also a Dedekind group. Hence, for any subgroup $Q \neq 1$ of P, $N = N(Q)$. Since N has a normal 2-complement, the Frobenius theorem [4, Th. 14.4.7] shows that G has a normal 2-complement. But then G is solvable, a contradiction.

Therefore $T = 1$, and $P = N$ is a maximal sub-group.

2. We treat first the case that P is abelian, P denoting always a Sylow 2-subgroup of G. The order of P is at least 4. If it is equal to 4, P is elementary abelian. Applying Lemma 1 for A a subgroup of order 2 in P, we find that $N(A) = C(A)$ has a 2-complement which is a Dedekind group of odd order, hence abelian. Therefore, remembering that G has no normal solvable subgroup, so, in particular, no normal subgroup of odd order, we see that the main result of [3] implies that G is $PSL(2, q)$, $PGL(2, q)$ or A_7. The order of $PGL(2, q)$ and A_7 is divisible by 8, so $PSL(2, q)$ (q odd) remains as the only possibility for G when the order of P is 4.

LEMMA 4. *Let P be elementary abelian of order at least* 8. *Let* $1 \neq a \in P$, *and let* $M = C(a)$. *Then for each* $1 \neq g \in M$, $C(g) \subseteq M$.

Proof. Denote $A = \langle a \rangle$; then $M = N(A)$, so M is maximal and its structure is determined by Lemma 1. Let H be any subgroup of odd order of M; then Lemma 1 implies $AH \lhd M$. But $AH = A \times H$ and H is characteristic in AH, so $H \lhd M$ and $M = N(H)$. If g is any element of odd order in M, then, taking $H = \langle g \rangle$, we have $C(g) = C(H) \subseteq N(H) = M$, so our assertion holds for all elements of M of odd order, hence also for all elements that are not 2-elements.

M has a 2-complement, L say. The preceding paragraph shows $L \lhd M$. If also $P \lhd M$, then $M = N(P)$, and a theorem of Burnside [4, Th. 14.3.1] shows that G has a normal 2-complement, a contradiction. Hence it is not true that $P \lhd M$ (so $P \neq M$).

Let H be a subgroup of M of odd prime order. Then we have seen above that $M = N(H)$. Let $Q = C_P(H)$. The same argument as above, with H replacing A, yields that each subgroup of Q is normal in M. Therefore, $P \neq Q$, and P/Q, which is the automorphism group induced on H by P, is non-trivial. This automorphism group is elementary abelian, because P is elementary abelian, and cyclic, because H has a prime order. So $|P:Q| = 2$.

Let $1 \neq g \in Q$, and let $A = \langle g \rangle$. We have remarked that $M = N(A)$, so again $C(g) = C(A) \subseteq N(A) = M$.

Now let $g \in P - Q$, and suppose $C(g) \not\subseteq M$. Then $P \neq C(g)$ (but of course $P \subseteq C(g)$, P being abelian). Let h be an element of odd prime order of $C(g)$, and let $A = \langle h \rangle$. Repeating for $C(g)$ the arguments used for $M = C(a)$, we find $C(g) = N(A)$. If $h \in M$, we also have $M = N(A)$, hence $M = C(g)$, a contradiction. Therefore $h \notin M$.

Let $R = C_P(A)$. As for Q, we prove $|P:R| = 2$. Since the order of P is at least 8, $Q \cap R \neq 1$. Let $1 \neq b \in Q \cap R$. Then $h \in C(b)$ and $b \in Q$, so $C(b) \subseteq M$ and $h \in M$, a contradiction.

We have shown that the assertion of the lemma holds for all elements of P, hence for all 2-elements of M, which ends the proof.

According to Suzuki [12, Th. 1] if a finite group G has a subgroup M, of even order and containing the centralizer of each of its non-identity elements, then

G is either a Frobenius group or a so-called (ZT) group, and in the second case M is either the Sylow 2-subgroup or its normalizer. Since the kernel of a Frobenius group is a normal nilpotent subgroup, our G, being semi-simple, is not a Frobenius group. The second alternative is also impossible, since we have seen in the proof of Lemma 4, that M is neither the Sylow 2-subgroup nor its normalizer.

Hence, if P is abelian, its order must be 4 and $G \cong PSL(2, q)$ for some odd q.

3. We now assume that P is a maximal subgroup. The following lemmas, excepting Lemma 7, are proved under this additional hypothesis.

LEMMA 5. *Let A be a 2-subgroup of G. If A is not elementary abelian, $N(A)$ is a 2-group.*

Proof. By contradiction. Let A be 2-subgroup which is not elementary abelian, whose normalizer is not a 2-group, and which is maximal relative to these two properties. Let T be the normal 2-complement of $N(A)$, which exists by Lemma 2. Then $T \neq 1$, and $T \lhd N(A)$, $A \lhd N(A)$ and $T \cap A = 1$ imply $T \subseteq C(A)$.

Let P be a Sylow 2-subgroup of G containing A, and z a central involution of P. Then $z \in N(A)$. Maximality of P implies $P = C(z)$. Hence $T \not\subseteq C(z)$, and therefore $z \notin A$. Denote $B = \langle A, z \rangle$, then B contains A properly, so maximality of A implies that $N(B)$ is a 2-group.

Since z centralizes A and $z^2 = 1$, $B^2 = A^2 \neq 1$, the last since A is not elementary. B^2 is characteristic in B, so normal in $N(B)$. $N(B)$ being maximal, we get $N(B^2) = N(B)$. However, the same reasoning shows that $N(A^2) = N(A)$, hence $N(B) = N(A)$, where $N(B)$ is a 2-group and $N(A)$ is not. This is the desired contradiction.

LEMMA 6. *Let P be a Sylow 2-subgroup of G, and let A be a subgroup of P which is not elementary abelian. Then $A \lhd P$.*

Proof. By the previous lemma, $N(A) = N$ is a 2-group, which must be a Sylow 2-subgroup, if it is to be maximal. Suppose $N \neq P$, and let B be a maximal intersection of two Sylow subgroups containing A. With A, B is not elementary abelian, so $N(B)$ is also a 2-group. This is well known to be impossible for a maximal Sylow intersection (e.g. [13, Th. 7, p. 138]). Hence $N = P$.

We digress now to deal with p-groups having the property proved for P in Lemma 6.

LEMMA 7. *Let Q be a p-group. Suppose that each subgroup of Q which is not elementary abelian is normal. Then at least one of the following holds:* (a) *Q has exponent p;* (b) *Q has class ≤ 2; or* (c) *Q is the dihedral group of order* 16.

Proof. We suppose Q has exponent larger than p, and prove that either (b) or (c) holds. Let $a \in Q$ have order p^2, let $A = \langle a \rangle$ and $B = \langle a^p \rangle$. Then $A \lhd Q$, so also $B \lhd Q$. Let $D = \langle d \rangle$ be any subgroup of order p not contained in A. Then DA has order p^3 and $DA \lhd Q$. If p is odd, then DB is the set of all elements of order $\leq p$ in DA and is characteristic in DA, so $DB \lhd Q$. This shows that in Q/B all subgroups of order p, and hence all elementary abelian subgroups, are normal. The normality of the subgroups which are not elementary abelian in Q/B is inherited from Q, so Q/B is a Dedekind group of odd order, therefore abelian, and Q has class ≤ 2.

Now let $p = 2$. Suppose Q has a cyclic subgroup of order 4, A_1, such that $A_1 \neq A$, and let $B_1 = A_1^2$. Then $A_1 \lhd Q$, $B_1 \lhd Q$. As above, we get $DA \lhd Q$, $DA_1 \lhd Q$, implying $DA \cap DA_1 \lhd Q$. The group DA has order 8, has elements of order 4, and more than one element of order 2. Hence DA is either the direct product $D \times A$ or the dihedral group of order 8. In the first case $D \times B$ is again the totality of elements of order ≤ 2 in $D \times A_1$ and $DB \lhd Q$. In the second case, DA does not contain A_1, as the dihedral group has only one cyclic subgroup of order 4. Therefore $DA \cap DA_1 \neq DA$, and $DA \cap DA_1$ is either D or DB.

We may suppose that D can be chosen to be nonnormal, otherwise Q is a Dedekind group and has class at most 2. Then $D \neq DA \cap DA_1$, so $DA \cap DA_1 = DB$, and also $DA \cap DA_1 = DB_1$.

B and B_1, being normal subgroups or order 2, are in the center of Q. If $B \neq B_1$, then $DB = DB_1 = BB_1$ is also central, implying $D \lhd Q$. We may suppose, then, that $B = B_1$. Since A_1 can be taken to be any cyclic subgroup of order 4 different from A_1 this means that all elements of order 4 in Q have the same square a^2. Also, we have obtained $DB \lhd Q$, so, as for the case that p is odd, we see that Q/B is a Dedekind group.

If Q/B is not abelian, it has a quaternion subgroup R/B, say. R does not have elements of order 8, since no group of order 16 having an element of order 8 can be homomorphic to the quaternion group. Since all elements of order 4 or less in Q have their squares in B, R/B must be elementary abelian, a contradiction. This means that Q/B is abelian, so again B has class 2 at most.

There remains the case in which A_1 cannot be found, namely: Q has only one cyclic subgroup of order 4. But G. A. Miller has proved [7, p. 129] that such a Q is either cyclic or dihedral. In the second case the assumption on normality of non-elementary subgroups shows that the order of Q is 16 at most, so either (b) or (c) holds also in this case.

Now resume the proof of Theorem 1 (for the semi-simple case). If P is maximal, the two last lemmas imply that P is either of class two at most, or the dihedral group of order 16 (recall that groups of exponent 2 are abelian). If P has class two (or is abelian) then G is solvable (e.g. [5]). Hence P is the dihedral group of order 16. Once again we deduce from [3] that G is A_7,

$PGL(2, q)$, or $PSL(2, q)$. This time A_7 is ruled out because the order of its Sylow 2-subgroup is too small, not too big.

Suppose $G \cong PGL(2, q)$. Let N be the subgroup of index 2 of G such that $N \cong PSL(2, q)$. $P \cap N$ is a Sylow 2-subgroup of N, and it has order 8. Dickson's list of subgroups of $PSL(2, q)$ [1, pp. 285–286] shows that $P \cap N$ is contained in a dihedral subgroup D, of order $q + 1$ or order $q - 1$, of N. Then $Z = Z(P \cap N) = Z(D)$, so $D \subseteq N(Z)$. But we also have $Z(P \cap N) = Z(P)$, so $P \subseteq N(Z)$ and maximality of P implies $P = N(Z)$, hence $D \subseteq P \cap N$ and $D = P \cap N$. This yields $q = 7$ or $q = 9$. Both $PSL(2, 7)$ and $PSL(2, 9)$ have subgroups isomorphic to S_4, which of course can be taken to contain $P \cap N$. Let T be such a subgroup of N, and let A be the normal 4-group of T. Then $N_G(A) \supset T$, since T is not maximal in G. However, T is maximal in N, and N is simple, so $T = N_N(A) = N \cap N_G(A)$, therefore $| N_G(A) : T | = 2$ and $| N_G(A) | = 48$. This shows that $N_G(A)$ contains properly a Sylow 2-subgroup of G_1 of order 16, contrary to the maximality of P.

So, in the event that P is maximal, as well as in the abelian case (see beginning of Section 2) we find that we must have $G \cong PSL(2, q)$, for some odd q. Referring again to the list of subgroups of $PSL(2, q)$, it turns out that the only possibility is $G \cong PSL(2, 13)$.

4. We now let G be a group satisfying the assumptions of Theorem 1, which is not semi-simple. Let S be the maximal normal solvable subgroup of G. Let A/S be any p- or $\{p, q\}$-subgroup of G/S. Then A is solvable, therefore A has some $\{p, q\}$-subgroup A_1 which maps onto A/S. $N(A_1)$ is maximal, by assumption, hence so is $N(A/S)$. G/S satisfies, then, the assumptions of Theorem 1 and is semi-simple, so by what has been proven already, $G/S \cong PSL(2, 13)$.

Let A be any p-subgroup of G which is not subnormal. Then the structure of $N(A)$ is still given by Lemma 1, since in the proof of this result the semi-simplicity of G is needed only to conclude that A is not subnormal. In particular, $N(A)$ is solvable. As $N(A)$ is maximal, $N(A)S = G$ or $N(A)S = N(A)$. In the first case G would be solvable. Hence $N(A)S = N(A)$ or $S \subseteq N(A)$.

If also $A \subseteq S$, this means $A \lhd S$, and since $S \lhd G$, $A \lhd\lhd G$, a contradiction. All subgroups of S are, then, subnormal in G and also in S. This makes S nilpotent.

Next let A be a $\{p, q\}$-subgroup of G, which is not subnormal. Then $A \not\subseteq S$, so $AS/S \neq 1$. Again $N(A)$ is maximal, and now $N(A)S = G$ would imply $AS/S \lhd G/S$, contrary to the simplicity of G/S. So once again $S \subseteq N(A)$.

Let p be any prime divisor of $| S |$, and let S^p be the p-complement of S, and A any p-subgroup of G. If $A \not\subseteq S$, then, by what we have just seen, $A \lhd AS^p$. Since also $S^p \lhd G$, we obtain $S^p \subseteq C(A)$. The same conclusion is true if $A \subseteq S$, as S is nilpotent. Denoting by $0^p(G)$ the subgroup generated by all p'-elements of G, it follows that $S_p \subseteq C(0^p(G))$, where S_p is the Sylow p-subgroup of S.

Let K be a minimal non-solvable subgroup of G. Then KS and KS/S are not solvable. But each proper subgroup of $G/S \cong PSL(2, 13)$ is solvable, so $KS = G$. Therefore $K/K \cap S \cong KS/S \cong PSL(2, 13)$.

The choice of K implies $K = K'$. Therefore $K = 0^p(K) \subseteq 0^p(G)$ for all primes p. Hence the result of the previous paragraph implies $S \subseteq C(K)$, so of course also $K \subseteq C(S)$ and $K \cap S \subseteq Z(G)$. Now it follows from a result of Schur [9, IX, p. 119] that $K \cong PSL(2, 13)$ or $K \cong SL(2, 13)$.

First, suppose $K \cong PSL(2, 13)$. Then K is simple, so $K \cap S = 1$ and $G = KS = K \times S$.

Fix some proper non-identity p-subgroup A of K. Let T be any q-subgroup of S. Then $N(T \times A) = N_S(T) \times N_K(A)$ is maximal in G. Since $N_K(A) \neq K$, we must have $N_S(T) = S$. This means that S is a Dedekind group.

Suppose $p \mid (|K|, |S|)$, p being a prime number. First let $p > 2$. Choose an element $a \in S$ and $b \in K$ of order p. Then

$$N(\langle ab \rangle) \subset S \times N_K(\langle b \rangle),$$

and $N(\langle ab \rangle)$ is not maximal.

So suppose $p = 2$. Let a be as above, and let b_1, b_2, b_3 be the non-identity elements of a Sylow 2-subgroup of K. Let $c \in K$ normalize this Sylow 2-subgroup, such that c permutes the b_i's cyclically. As above, c does not normalize the subgroup $\{1, ab_1, ab_2, b_3\}$, so this subgroup does not have a maximal normalizer.

Thus we must have $(|K|, |S|) = 1$. S is now a Dedekind group of odd order, hence abelian.

Lastly, suppose $K \cong SL(2, 13)$. As $K/K \cap S \cong PSL(2, 13)$, we must have $K \cap S = Z(K)$. $G/K \cap S$ has the same structure as G in the case $K \cong PSL(2, 13)$. Therefore $S/K \cap S$ is abelian and of odd order. Letting A be the 2-complement of S, we obtain $G = K \times A$. This completes the proof of Theorem 1.

5. In analogy with Theorem 1, one may ask which finite non-solvable groups have the property, that the centralizer of each non-central element is a maximal subgroup. It turns out that there are no such groups. This follows rather easily from the deep results in finite group theory obtained in recent years.

Let, then, G be a finite group with the property, that for each $g \in G$, either $C(g) = G$ or $C(g)$ is a maximal subgroup of G. Let S be the maximal solvable normal subgroup of G. Then G/S has the same property of centralizers characterizing G. Therefore, we are going to assume that G is semi-simple.

If G is semi-simple, then obviously $C(g) \neq G$ whenever $g \neq 1$. Let n be a natural number, and suppose $g^n \neq 1$ for some $g \in G$. Then $C(g) \subseteq C(g^n)$ and maximality of $C(g)$ implies $C(g) = C(g^n)$.

Let H be any semi-simple group satisfying: if $h \in H$ and $h^n \neq 1$, then

$C(h) = C(h^n)$. We will prove that H is a (CN) group, i.e. the centralizer of each element is nilpotent.

Indeed, it is enough to prove this for p-elements. Let, then, h be a non-identity p-element of H, for some prime p. We may suppose $C(h)$ not to be a p-group. Let $k \in C(h)$ be a q-element, where q is a prime and $q \neq p$. Then both h and k are powers of hk, therefore $C(h) = C(k) = C(hk)$. Hence $k \in Z(C(h))$. Let l be a p-element of $C(h)$. Since $C(h) = C(k)$, the same reasoning shows $C(l) = C(k) = C(h)$, and $l \in Z(C(h))$. Hence each element of prime-power order in $C(h)$ is central, and $C(h)$ is abelian.

All the semi-simple (CN) groups were determined by Suzuki [10, Th. 5, p. 468], and it is easy to check that in none of them $C(h)$ is always maximal. We shall verify this only for the so-called (ZT) groups, which appear in the formulation of Suzuki's result just mentioned, since these were not completely classified in the paper [10] (they were classified in a later paper [11], but this is not needed here). Thus, a (ZT) group, G, is, by definition, a doubly transitive group of odd degree, which is not a Frobenius group, and in which only the identity fixes three letters. Let H be the subgroup of G fixing a letter, then it follows from the above definition that H is a Frobenius group with a Sylow 2-subgroup of G as a kernel. Hence, this Sylow subgroup is not maximal in G. On the other hand, (ZT) groups are known to be (CIT) groups, i.e. the centralizer of each involution is a 2-group. Hence, centralizers of involutions in G are not maximal subgroups.

It is obvious from the above proof that the maximality of $C(g)$ needs to be required only when the order of g is $p^\alpha q^\beta$. If the order of G is assumed to be even (in particular, if we invoke the Feit-Thompson theorem), it is even enough to consider elements of order $2^\alpha q^\beta$ (and we may assume $\beta > 0$). That is because the preceding proof then shows that G is a (CIT) group, and Suzuki's result actually deals with this type of groups.

Finally, if one wishes to show only that G is a (CIT) group, it is enough to require $C(g) = C(g^n)$ for elements g of order $2p$ only. Thus, let h be an involution, and suppose that $C(h)$ is not a 2-group. Let $k \in C(h)$ have an odd prime order. Then, as before, we obtain $C(h) = C(k) = C(hk)$. If l is another involution in $C(h)$, we again obtain $C(l) = C(k) = C(h)$. Now let $a \neq 1$ be any element of $C(h)$. Then some power of a, a^n say, has prime order, so $C(a) \subseteq C(a^n) = C(h)$. As $|C(h)|$ is even, another result of Suzuki [12, Th. 1] shows that G is either a Frobenius group or a (ZT) group. If G is assumed to be semi-simple, it is not a Frobenius group. As have already been mentioned, (ZT) groups are (CIT) groups.

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