TAME ORDERS, TAME RAMIFICATION AND GALOIS COHOMOLOGY

BY

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Introduction

Let R be a commutative ring, and let G be a finite group, represented as R-automorphisms of a commutative R-algebra S. Consider the cohomology group $H^2(G, U(S))$, where $U(S)$ is the multiplicative group of units of S. To each cohomology class f, we associate a tower

$$R \subset S \subset \Delta(f, S, G)$$

of R-algebras, where $\Delta(f, S, G)$ is the crossed product algebra, or semi-linear group ring with factor set f. Classically, this tower has received considerable attention. For example, when S/R is a Galois extension of fields, or of rings, then $\Delta(f, S, G)$ is an R-central simple, or separable, algebra split by S, and defines an element of the Brauer group $B(S/R)$.

This is the case when S is an unramified extension of the integrally closed noetherian domain R, in a Galois extension L/K of their quotient fields. It is natural to consider the tower in a more general setting—for example, when the integral closure S of R in L is a tamely ramified extension of R. It is this case which is the focal point for the present investigations.

In particular, we consider the structure-forgetting functor from $\Lambda_f = \Delta(f, S, G)$-modules to S-modules. In our main theorem, we show that this functor preserves homological dimension for every cohomology class f if, and only if, the extension S/R is tamely ramified. The major corollary of this result indicates that all of the ramification in the tower $R \subset S \subset \Lambda_f$ takes place in the extension $R \subset S$.

A fortiori, a crossed product in a tamely ramified extension S/R is an order over R which is a reflexive R-module, and whose localization at every minimal prime ideal p of R is an hereditary order over R_p. Such an order is called a *tame order*. In Chapter I, we study the ideals and automorphisms of a tame order.

In Chapter II, we consider crossed product algebras and apply the theorems of Chapter I.

In Chapter III, we consider also Amitsur Cohomology, and applications to the study of the Brauer Group.

Although the main applications of these results are to integral extensions of integrally closed noetherian domains whose quotient field extension is finite

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Galois, the formal techniques applied here make it undesirable to assume such a restrictive hypothesis as a blanket condition. We make only the conventions that all rings have unit elements, that all modules and ring homomorphisms are unitary, and that all modules are finitely generated unless explicitly stated otherwise. Additional hypotheses will be stated in the text as they become necessary.

We mention here several notational conventions. If E is a right module over a ring Γ, then $\mathrm{End}_\Gamma(E)$ denotes the ring of Γ-endomorphisms of E, and E will always be considered as a left $\mathrm{End}_\Gamma(E)$-module. A similar convention will hold for left Γ-modules. The trace ideal $\tau_\Gamma(E)$ is defined to be the image of

$$\tau : E \otimes_\Gamma \mathrm{Hom}_\Gamma(E, \Gamma) \to \Gamma,$$

given by $\tau(e \otimes f) = f(e)$, and is a two-sided ideal of Γ. Its main application will be with the Morita theorems [3, Appendix]. We also recall here the notion of conductor, i.e., if $\Lambda \subset \Gamma$ are orders in the same simple algebra Σ, then the conductor $C_\Lambda(\Gamma)$ of Γ with respect to Λ is defined to be

$$C_\Lambda(\Gamma) = \{x \in \Sigma;\ \Gamma x \subset \Lambda\} = \mathrm{Hom}^l_\Lambda(\Gamma, \Lambda).$$

(For a functor F of Λ-modules, we will use the notations $F^l(*)$, $F^r(*)$, i.e., left and right, whenever there is ambiguity.)

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I. Tame orders

1. Orders over a tame order. In this section we prove an analogue of Theorem 1.7 of [12] for orders over general integrally closed noetherian domains.

Throughout chapter I, the following hypotheses will hold without any further mention. R denotes a commutative integrally closed, noetherian domain with quotient field K. All R-modules and R-algebras are finitely generated over R.

DEFINITION. Let R be an integrally closed domain. An order Λ over R will be called *tame* if

(i) Λ is reflexive as an R-module;
(ii) Λ_p is an hereditary order over R_p for every minimal prime p of R.

LEMMA 1.1. *Let R be an integrally closed domain and X the set of minimal primes of R. Then*

(i) $R = \bigcap_p R_p$;
(ii) *a torison-free R-module E is reflexive if and only if $E = \bigcap_p E_p$.*

Proof. Well known. For example, it follows directly from 3.4 of [1].

COROLLARY 1.2. *If Λ is a tame order, then $\Lambda = \bigcap_p \Lambda_p$. If A, B are R-reflexive ideals of Λ, then $A = B \Leftrightarrow A_p = B_p$ for all minimal primes p.*

DEFINITION. Two ideals of a tame order Λ are *quasi-equal* if $A^{**} = B^{**}$, i.e., if $A_p = B_p$ for all minimal primes p. The relation of quasi-equality is obviously an equivalence relation on the set of left, or right, or two-sided fractionary ideals of Λ [11]. An R-reflexive two-sided ideal A such that A is quasi-equal to A^2 will be called *quasi-idempotent.*

LEMMA 1.3. *Let R be any commutative ring and $M \subset R$ a multiplicative set. Let $F \subset E$ be R-modules, not necessarily finitely generated. Then $F_M = E_M$ if and only if for each $x \in E$, $\mathrm{Ann}_R(F + Rx/F)$ and M have a non-empty intersection.*

Proof. Clear by standard techniques.

COROLLARY 1.4. *Let Λ be an R-algebra which is reflexive as an R-module. Suppose that, for each minimal prime p, we are given a left ideal $I(p)$ of Λ_p such that $I(p) = \Lambda_p$ for all but a finite number of p. Then $I = \bigcap_p I(p)$ is a left ideal of Λ and I is R-reflexive as a module.*

Proof. Suppose first that $I(p) \subset \Lambda_p$ for all p. Then by 1.2, $I \subset \Lambda$. Consider the exact sequence

$$0 \to I \to \Lambda \to \textstyle\sum_q (\Lambda_q/I(q)).$$

Localizing at p, we obtain the exact sequence

$$0 \to I_p \to \Lambda_p \to (\textstyle\sum_q \Lambda_q/I(q))_p.$$

$(\Lambda_q/I(q))_p = (\Lambda_q)_p/I(q)_p$, and $\mathrm{Ann}_R(\Lambda_q/I(q)) = q^n R_q \cap R = q^{(n)}$, the nth symbolic power of the prime ideal q. If $p \neq q$, then $q^{(n)} \not\subset p$, i.e., $q^{(n)} \cap (R - p) \neq \varphi$. By Lemma 1.3, $(\Lambda_q)_p = I(q)_p$ if $q \neq p$. Thus

$$(\textstyle\sum_q \Lambda_q/I(q))_p = \Lambda_p/I(p) \quad \text{and} \quad I(p) = I_p.$$

So I is reflexive as an R-module.

In general let $0 \neq x \in K$ such that $xI \subset \Lambda$. Then $xI(p) \subset \Lambda_p$ for all p, and $xI(p) = \Lambda_p$ for almost all p, since x is a unit of R_p for almost all p. Then $xI = \bigcap_p xI(p)$ is a reflexive R-module and an ideal of Λ by the foregoing, and so is I.

LEMMA 1.5. *If E, F are Λ-modules and F is R-reflexive, then* $\mathrm{Hom}_\Lambda(E, F)$ *is R-reflexive.*

Proof. $\mathrm{Hom}_\Lambda(E, F) = \mathrm{Hom}_\Lambda(E, F^{**}) \cong \mathrm{Hom}_R(F^* \otimes_\Lambda E, R)$ by [7, II.5.2].

COROLLARY 1.6. *If Λ is an R-reflexive algebra, then a Λ-reflexive module is R-reflexive.*

LEMMA 1.7. *Let Λ be any ring and P a (finitely generated) left Λ-projective module. Then the trace ideal $\tau_\Lambda(P)$ is an idempotent two-sided ideal.*

Proof. This follows easily from the appendix of [3].

THEOREM 1.8. *Let Λ be a tame order over a domain R. Then every R-reflexive order containing Λ is again tame, and there exists a one-to-one correspondence between R-reflexive orders containing Λ, and quasi-idempotent two-sided ideals of Λ, through the conductor.*

Proof. *If* $\Lambda \subset \Gamma$, then $\Lambda_p \subset \Gamma_p$ for all minimal primes $p \subset R$, so by [12, Corollary 1.4], Γ_p is hereditary for all p. Hence Γ is tame if Γ is R-reflexive.

Now $C_\Lambda(\Gamma) = \mathrm{Hom}^l_\Lambda(\Gamma, \Lambda)$ is an R-reflexive two-sided ideal of Λ. $C_\Lambda(\Gamma)_p = C_{\Lambda_p}(\Gamma_p)$ is idempotent for all p, and $(C_\Lambda(\Gamma)^2)_p = C_{\Lambda_p}(\Gamma_p)^2$, i.e., $(A^2)_p = (A_p)^2$ for any ideal A since they are both the image in Λ_p of $(A \otimes_\Lambda A)_p = A_p \otimes_{\Lambda_p} A_p$. Hence by Theorem 1.7 of [12], $C_\Lambda(\Gamma)$ is quasi-idempotent. By localization, it is obvious that $\mathrm{End}^r_\Lambda(C_\Lambda(\Gamma)) = \Gamma$.

Conversely, if A is a quasi-idempotent ideal of Λ, then $\mathrm{End}^r_\Lambda(A)$ is a tame order containing Λ, and, finally, $C_\Lambda(\mathrm{End}^r_\Lambda(A)) = A$ by localization.

COROLLARY 1.9. *If Λ is a tame order over R, then there are only a finite number of R-reflexive orders containing Λ. In fact, there are exactly $e = \sum_p (e(p))$ tame orders containing Λ (including Λ), where $e(p)$ is the number of (necessarily hereditary) orders containing Λ_p for each minimal prime p of R.*

Proof. It follows from the conductor that Λ_p is a maximal order over R_p for all but a finite number of p. By [12, Theorem 1.7], the number $e(p)$ is finite. Since a quasi-idempotent ideal is R-reflexive by definition, we have the first statement and the inequality $e \leq \sum_p(e(p))$.

For each minimal prime p, let $I(p)$ be an idempotent two-sided ideal of Λ_p. Let $I = \bigcap_p I(p)$. Since Λ_p is maximal for almost all p, we must have $I(p) = \Lambda_p$ for almost all p. It follows from 1.4 that I is a two-sided R-reflexive ideal of Λ, with $I_p = I(p)$. It is now trivial that I is quasi-idempotent, i.e., $(I^2)_p = (I_p)^2 = I_p$ for all p. The equality $e = \sum_p (e(p))$ now follows.

If A is a quasi-idempotent ideal of a tame order Λ, then $\tau^l_\Lambda(A) = A$ by Lemma 1.5 of [12] and localization. On the other hand, by Lemma 1.7, $\tau_\Lambda(A)^{**}$ is quasi-idempotent for every left ideal A of Λ. The following simple proposition will clarify further the relation between A and $\tau_\Lambda(A)^{**}$.

PROPOSITION 1.10. *Let A be an R-reflexive left ideal of the tame order Λ. Let $\Lambda \subset \Gamma$ be the tame order corresponding to $\tau_\Lambda(A)^{**}$ as in Theorem 1.8. Then Γ is the unique largest order of Σ such that A is a left Γ-module. Moreover, $(\tau_\Gamma(A))^{**} = \Gamma$.*

Proof. It suffices to prove this statement when Λ is an h-order. We have $\Gamma = \mathrm{End}^r_\Lambda(\tau_\Lambda(A))$. Also, $A = \tau_\Lambda(A)\cdot A$, hence

$$\gamma A = \gamma\tau_\Lambda(A)\cdot A \subseteq \tau_\Lambda(A)\cdot A = A \qquad \text{for } \gamma \in \Gamma,$$

so A is a left Γ-module. If $x \in \Sigma$ such that $xA \subset A$, then $xAA^{-1} \subset AA^{-1}$, i.e., $x\tau_\Lambda(A) \subset \tau_\Lambda(A)$, so $x \in \Gamma$ by definition. Finally, by 1.8, $\tau_\Gamma(A) = \Gamma$, since $\tau_\Gamma(A)$ is idempotent and $\mathrm{End}^r_\Gamma(\tau_\Gamma(A)) = \Gamma$ by the first assertion of this proposition.

From this proposition, we can see that the theory of R-reflexive ideals of a tame order Λ has two separate facets: The first is the determination of all tame orders containing Λ over R, e.g., Theorem 1.8. The second is the study of those ideals which admit Λ as left order, i.e., $\Gamma = \Lambda$ in Proposition 1.10. We begin this study in the next section.

2. Quasi-equality classes. We consider in this section the set of quasi-equality classes of certain two-sided ideals of a tame order. It is clear that every quasi-equality class has a unique representative which is a reflexive R-module, and we will deal with these representatives. We consider here a generalization of a group formulated by O. Goldman for maximal orders in [11].

The conventions stated in I.1 remain in force throughout this section.

PROPOSITION 2.1. *Let A be a two-sided fractionary ideal of a tame order Λ. The following are equivalent:*

(i) $\tau^l_\Lambda(A)^{**} = \Lambda$;
(ii) $\mathrm{End}^r_\Lambda(A) = \Lambda$;
(iii) $\tau^r_\Lambda(A)^{**} = \Lambda$;
(iv) $\mathrm{End}^l_\Lambda(A) = \Lambda$;

Proof. By localization, it suffices to prove the equivalence for an h-order Λ. By the same arguments as in Proposition 1.10, it is clear that

$$\mathrm{End}^r_\Lambda(\tau^l_\Lambda(A)) = \mathrm{End}^r_\Lambda(A) \quad \text{and} \quad \mathrm{End}^l_\Lambda(\tau^r_\Lambda(A)) = \mathrm{End}^l_\Lambda(A),$$

so (i) $\Leftrightarrow$ (ii), (iii) $\Leftrightarrow$ (iv) are clear. By symmetry, it is enough to show (i) $\Rightarrow$ (iv). By Theorem A.5 of [3], Λ and $\mathrm{End}^l_\Lambda(A)$ have the same number of maximal two-sided ideals. But $\Lambda \subset \mathrm{End}^l_\Lambda(A)$, so equality holds, cf. Theorem 4.3 of [12].

DEFINITION. A two-sided R-reflexive fractionary ideal A satisfying one, hence all, of the conditions of Proposition 2.1 will be called a *divisor* of Λ. It is clear that A is a divisor if, and only if, A is R-reflexive and A_p is invertible at every minimal prime p of R. For the rest of this section, the word "ideal" will mean a two-sided fractionary divisor of Λ.

We proceed as in [11]. If A, B are ideals, we define their product by

$A \cdot B = (AB)^{**}$. This may be looked at as the product of the quasi-equality classes of A and B.

LEMMA 2.2. *Let A, B, C be ideals.*

(i) *$A \cdot B$ is an ideal.*
(ii) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.
(iii) $A \cdot B = B \cdot A$.

Proof. It suffices to show this when A, B, C are divisors of an h-order Λ. However, by Theorem 6.1 of [12], the divisors of an h-order are simply the invertible ideals, and they form a cyclic group generated by the radical.

THEOREM 2.3. *Let $D(\Lambda)$ denote the set of ideals (or of quasi-equality classes of two-sided ideals) of the tame order Λ. Then $D(\Lambda)$ is an abelian group. In fact, $D(\Lambda)$ is the free abelian group generated by the ideals $r(\Lambda_p) \cap \Lambda$, for all minimal primes p, where $r(\quad)$ denotes the Jacobson radical of the ring.*

Proof. If $A \in D(\Lambda)$, then $\mathrm{Hom}^l_\Lambda(A, \Lambda) \in D(\Lambda)$. In the h-order Λ_p, we know that $\mathrm{Hom}^l_\Lambda(A_p, \Lambda_p)$ is again invertible. Moreover, $A \cdot \mathrm{Hom}^l_\Lambda(A, \Lambda) = \tau^l_\Lambda(A)^{**} = \Lambda$ by definition of A. Hence $D(\Lambda)$ is a group. Define $\alpha : D(\Lambda) \to \sum_p D(\Lambda_p)$ by $(\alpha(A))_p = A_p$. By Lemma 1.2, α is one-to-one. By definition of the multiplication in $D(\Lambda)$, α is a homomorphism. Finally, if $(A(p)) \in \sum_p D(\Lambda_p)$, then by 1.4, $A = \bigcap_p A(p)$ is a finitely generated R-reflexive two-sided Λ-module in Σ, and is thus an element of $D(\Lambda)$ with $\alpha(A) = (A(p))$. Therefore, α is onto and an isomorphism.

COROLLARY 2.4. *For $A \in D(\Lambda)$, $\mathrm{Hom}^r_\Lambda(A, \Lambda) = \mathrm{Hom}^l_\Lambda(A, \Lambda)$.*

Proof. By uniqueness of the inverse in a group and Proposition 2.1, i.e., $A \cdot \mathrm{Hom}^l_\Lambda(A, \Lambda) = \mathrm{Hom}^r_\Lambda(A, \Lambda) \cdot A = \Lambda$.

Remark. This could be demonstrated directly from $\mathrm{End}^l_\Lambda(A) = \mathrm{End}^r_\Lambda(A) = \Lambda$, using the isomorphism $\psi : \mathrm{Hom}_\Lambda(E, \Lambda) \to \mathrm{Hom}_\Gamma(E, \Gamma)$ defined just before Theorem A.2 of [3], where E is any Λ-module and $\Gamma = \mathrm{End}_\Lambda(E)$.

3. Invertible ideals.

DEFINITION. A divisor A of the tame order Λ will be called *invertible* if A is left and right Λ-projective. The following proposition justifies this terminology.

PROPOSITION 3.1. *For a divisor A of Λ, A is left Λ-projective $\Leftrightarrow \tau^r_\Lambda(A) = \Lambda$. Hence A is invertible $\Leftrightarrow AA^{-1} = A^{-1}A = \Lambda$.*

Proof. See Theorem A.2(g) of [3]. This shows that $\tau^r_\Lambda(A) = \Lambda \Rightarrow A$ is left Λ-projective. Moreover, since the map

$$\psi : \mathrm{Hom}^r_\Lambda(A, \Lambda) \to \mathrm{Hom}^l_\Lambda(A, \Lambda)$$

is an isomorphism (see the note following 2.4), we have that A is left Λ-pro-

jective $\Rightarrow \tau_\Lambda^r(A) = \Lambda$ by the same theorem. The second assertion follows trivially by symmetry.

PROPOSITION 3.2. *Let A, B be invertible. Then A^{-1} and $A \cdot B = AB$ are invertible. Hence, the invertible ideals form a subgroup of $D(\Lambda)$, and multiplication of invertible ideals in $D(\Lambda)$ corresponds to the ordinary multiplication of two-sided ideals.*

Proof. Since A is right Λ-projective, we have an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_\Lambda B \arrow[r] \arrow[d, "\alpha"'] & A \arrow[r] \arrow[d, "\beta"'] & A \otimes_\Lambda \Lambda/B \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & AB \arrow[r] & A \arrow[r] & A/AB \arrow[r] & 0
\end{tikzcd}$$

where all the maps are canonically defined. Since β is the identity and α is onto, we have by [7, III.3.3] that α is an isomorphism. Now by [7, II.5.2],

$$\mathrm{Hom}_\Lambda^l(A \otimes_\Lambda B, E) \approx \mathrm{Hom}_\Lambda^l(B, Hom_\Lambda^l(A, E))$$

for all left Λ-module E. Passing to derived functors, we see that $A \otimes_\Lambda B$ is left Λ-projective. Similarly, $A \otimes_\Lambda B$ is right Λ-projective. In particular, $AB \approx A \otimes_\Lambda B$ is already R-reflexive and is forced to be the same as $A \cdot B$.

Let $D'(\Lambda)$ denote the group of invertible divisors of Λ. Recall that $U(T)$ denotes the group of invertible elements of a ring T. Let $x \in U(K)$. Then Λx is a well-defined element of $D'(\Lambda)$, and

$$\Lambda x = \Lambda y \Leftrightarrow \Lambda x y^{-1} = \Lambda \Leftrightarrow x y^{-1} \in U(\Lambda) \Leftrightarrow x y^{-1} \in U(R).$$

Hence $D'(\Lambda)$ contains a subgroup canonically isomorphic to $U(K)/U(R)$. This subgroup can be identified as all those elements of $D'(\Lambda)$ which are isomorphic with Λ as modules over the enveloping algebra $\Lambda^e = \Lambda \otimes_R \Lambda^\circ$ of Λ. In fact, if A, B are two-sided Λ-modules in Σ, then

$$\mathrm{Hom}_{\Lambda^e}(A, B) \supset \mathrm{Hom}_\Sigma e(\Sigma, \Sigma) = K,$$

so that

$$\mathrm{Hom}_{\Lambda^e}(A, B) = \{x \in K; xA \subset B\}.$$

The set $C(\Lambda)$ of Λ^e-isomorphism classes of invertible divisors is clearly a group, under the multiplication $A \cdot B = A \otimes_\Lambda B$. Then we have

LEMMA 3.3. $0 \to U(K)/U(R) \to D'(\Lambda) \to C(\Lambda) \to 0$ *is an exact sequence of abelian groups.*

4. Automorphisms. Let R be any commutative ring, and Λ any R-central algebra, not necessarily finitely generated as an R-module. Denote by $U(\Lambda)$ the group of units of Λ, and by $\mathrm{Aut}(\Lambda)$ the group of R-algebra auto-

morphisms of Λ. There is a natural map $U(\Lambda) \to \mathrm{Aut}(\Lambda)$ which associates to a unit $u \in \Lambda$ the inner automorphism $x \to uxu^{-1}$ determined by u. The group $O(\Lambda)$, called the *group of outer automorphisms of* Λ, is defined by the exact sequence

$$0 \to U(R) \to U(\Lambda) \to \mathrm{Aut}(\Lambda) \to O(\Lambda) \to 0.$$

In this section, we will compute $O(\Lambda)$ for a tame order Λ over the integrally closed noetherian domain R. This is essentially a generalization of [15]. The conventions stated in I.1 remain in effect throughout this section.

LEMMA 4.1. *Let $t \in \Sigma$. Suppose that Λt is an invertible two-sided ideal of Λ. Then $\Lambda t = t\Lambda$ and $t \in U(\Sigma)$, hence $\alpha(\lambda) = t\lambda t^{-1}$ defines an automorphism α of Λ.*

Proof. $t\Lambda \subset \Lambda t$ because Λt is a right ideal. Since $\tau_\Lambda^r(\Lambda t) = \Lambda$ by 3.1, $1 = \sum_i (x_i(\lambda_i t))$, where $\lambda_i \in \Lambda$ and $x_i \in (\Lambda t)^{-1}$, so $(\sum_i x_i \lambda_i)t = 1$ and t is a unit of Σ. Then

$$t^{-1} \in \mathrm{Hom}_\Lambda^l(\Lambda t, \Lambda) = \mathrm{Hom}_\Lambda^r(\Lambda t, \Lambda)$$

so $t^{-1}\Lambda t \subset \Lambda$. Hence $\Lambda t \subset t\Lambda$ and we have equality. The final assertion is now trivial, since $t^{-1}\Lambda = \Lambda t^{-1}$ is also invertible.

We define now $C'(\Lambda)$ to be the set of left Λ-isomorphism classes of invertible divisors of Λ. Multiplication in $C'(\Lambda)$ is again defined by the tensor product $\otimes_\Lambda$.

PROPOSITION 4.2. *$C'(\Lambda)$ is an abelian group.*

Proof. Let A, B be left Λ-modules in Σ. Then

$$\mathrm{Hom}_\Lambda(A, B) \subset \mathrm{Hom}_\Sigma(\Sigma, \Sigma)$$

if $A \otimes_R K = B \otimes_R K = \Sigma$, so $\mathrm{Hom}_\Lambda(A, B) = \{x \in \Sigma; Ax \subset B\}$. Hence A and B are left Λ-isomorphic if, and only if there exists $t \in \Sigma$ such that $At = B$.

Now suppose that A and B are invertible divisors which are isomorphic as left Λ-modules and write $At = B$. Then $\Lambda t = A^{-1}B$ is an invertible divisor, so that t is a unit of Σ and $t\Lambda t^{-1} = \Lambda$ by 4.1. Moreover, by commutativity of $D(\Lambda)$, $At = tA$ for any invertible divisor A. Now let $A' = At$, $B' = Bs$, where A, A', B, B' are all invertible divisors. Then $A'B' = (At)(Bs) = A(tB)s = A(Bt)s = ABts$ so that multiplication in $C'(\Lambda)$ is well defined. Since the map $j : C(\Lambda) \to C'(\Lambda)$, which associates to every Λ^e-isomorphism class its left Λ-isomorphism class, is onto and preserves multiplication, the proposition follows.

Remark. It is clear by the above proof that invertible ideals are left Λ-isomorphic $\Leftrightarrow$ they are right Λ-isomorphic. Of course, this is not the same thing as Λ^e-isomorphic.

THEOREM 4.3. *Let Λ be a tame order. Then there exists an exact sequence*

$$0 \to O(\Lambda) \xrightarrow{i} C(\Lambda) \xrightarrow{j} C'(\Lambda) \to 0.$$

Proof. We have already noticed that j is clearly onto. Let α be an R-algebra automorphism of Λ. Then α is uniquely extended to

$$\alpha \otimes 1 : \Lambda \otimes_R K \to \Lambda \otimes_R K,$$

which is a K-algebra automorphism of the K-central simple algebra Σ. By the Skolem-Noether theorem, there exists $t \in U(\Sigma)$ such that $\alpha(x) = txt^{-1}$ for all $x \in \Sigma$. Then, in particular, $\Lambda = t\Lambda t^{-1}$ and $\Lambda t = t\Lambda$, so Λt is an invertible divisor of Λ. $i(\alpha)$ is defined to be the class of Λt in $C(\Lambda)$. If also $\alpha(x) = sxs^{-1}$, with $s \in U(\Sigma)$, then $t^{-1}sx = xt^{-1}s$ for all $x \in \Sigma$, hence $t^{-1}s \in U(K)$. Thus $\Lambda s = t^{-1}s(\Lambda t)$ is Λ^e-isomorphic with Λt, hence i is well defined on $\mathrm{Aut}(\Lambda)$. If α is an inner automorphism, then $\alpha(x) = txt^{-1}$ for some $t \in U(\Lambda)$, whence $\Lambda t = \Lambda$. Hence i is well defined on $O(\Lambda)$. If Λs, Λt are invertible divisors, then $\Lambda s \cdot \Lambda t = \Lambda st$ by 4.1, so i is a homomorphism. By 4.1 again, it is clear that $\ker(j) = \mathrm{im}(i)$. Finally, if $i(\alpha) = 1$, then Λt is Λ^e-isomorphic to Λ, where $\alpha(x) = txt^{-1}$ for all $x \in \Sigma$. Hence $\Lambda t = \Lambda r$, some $r \in K$. This entails $t = ur$, $st = r$, for some u, $s \in \Lambda$. Then $t = ur = ust$, so $us = 1$, and these are units of Λ. Then

$$\alpha(x) = txt^{-1} = (ur)x(ur)^{-1} = uxu^{-1}$$

since $r \in K$. Hence α is inner.

COROLLARY 4.4. *$O(\Lambda)$ is an abelian group.*

Let S be a maximal commutative subring of the tame order Λ, and $L = S \otimes_R K$. The group $O_S(\Lambda)$ is defined by the exactness of

$$0 \to U(S) \to \mathrm{Aut}_S(\Lambda) \to O_S(\Lambda) \to 0$$

where $\mathrm{Aut}_S(\Lambda)$ is the group of R-algebra automorphisms of Λ which leave S elementwise fixed, and the map $U(S) \to \mathrm{Aut}_S(\Lambda)$ associates with a unit u of S the inner automorphism $x \to uxu^{-1}$ of Λ. $O_S(\Lambda)$ is a subgroup of $O(\Lambda)$ since if $\alpha(x) = uxu^{-1}$ is an inner automorphism of Λ leaving S elementwise fixed, then u and u^{-1} both lie in S by definition of a maximal commutative subring. Hence we have a monomorphism $O_S(\Lambda) \to C(\Lambda)$ by 4.3. We will compute the cokernel of this map.

Let A, B be two-sided Λ-modules in Σ. Since $\Lambda \otimes S \subset \Lambda^e$, we can consider then as left $\Lambda \otimes S$-modules, where $\otimes$ means $\otimes_R$. Then

$$\mathrm{Hom}_{\Lambda \otimes S}(A, B) = \{x \in L; Ax \subset B\}.$$

Then A, B are $(\Lambda \otimes S)$-isomorphic $\Leftrightarrow B = Ax$ for $x \in L$. Define $C_S(\Lambda)$ to be the set of $\Lambda \otimes S$-isomorphism classes of invertible divisors. It is trivial that $C_S(\Lambda)$ is a group under $\otimes_\Lambda$.

COROLLARY 4.5.

$$0 \to O_S(\Lambda) \to C(\Lambda) \to C_S(\Lambda) \to 0$$

is an exact sequence of abelian groups.

Proof. We have only to show for an invertible ideal A of Λ, A is $(\Lambda \otimes S)$-isomorphic to $\Lambda \Leftrightarrow A = \Lambda t$ for some $t \in L$. By the above remarks and Theorem 4.3, this is trivial.

Another special computation is the case of a maximal order in a total matrix ring, i.e., by [3, Proposition 4.2], an order $\mathrm{End}_R(E)$, where E is a reflexive R-module. We begin with a general lemma.

LEMMA 4.6. *Let Γ, Λ be rings. Let A be any right Γ-module, C any left Λ-right Γ-bimodule, and B a finitely generated Λ-projective module. Then there is a natural isomorphism*

$$\alpha : B \otimes_\Lambda \mathrm{Hom}_\Gamma(A, C) \to \mathrm{Hom}_\Gamma(A, B \otimes_\Lambda C).$$

Proof. We define $[\alpha(b \otimes \varphi)](a) = b \otimes \varphi(a)$. If $x \in \Lambda$, then

$$[\alpha(bx \otimes \varphi)](a) = bx \otimes \varphi(a) = b \otimes x\varphi(a) = b \otimes (x\varphi)(a) = [\alpha(b \otimes x\varphi)](a)$$

so that α is well defined. For $B = \Lambda$, α is clearly an isomorphism, hence α is an isomorphism for any finitely generated right Λ-projective module B by a direct sum argument.

COROLLARY 4.7. *Let E be a reflexive R-module, $\Lambda = \mathrm{End}_R(E)$, and A a right Λ-projective module. Then*

$$\mathrm{Hom}_R(E, A \otimes_\Lambda E) \approx A \otimes_\Lambda \mathrm{Hom}_R(E, E) \approx A$$

as right Λ-modules. Consequently, for an invertible divisor A, $A \approx \Lambda$ as right Λ-modules $\Leftrightarrow A \otimes_\Lambda E \approx E$ as R-modules.

Proof. From 4.6, $\alpha : A \to \mathrm{Hom}_R(E, A \otimes_\Lambda E)$ is an isomorphism of abelian groups defined by $[\alpha(a)](e) = a \otimes e$. Then for $f \in \Lambda$,

$$[\alpha(af)](e) = af \otimes e = a \otimes fe = [\alpha(a) \cdot f](e),$$

so α is an isomorphism of right Λ-modules. Hence if $A \otimes_\Lambda E \approx E$ as R-modules, then

$$A \approx \mathrm{Hom}_R(E, A \otimes_\Lambda E) \approx \mathrm{Hom}_R(E, E) \approx \Lambda$$

as right Λ-modules. Conversely, if $A \approx \Lambda$ as right Λ-modules, then $A = t\Lambda$ for some $t \in U(\Sigma)$, and $A \otimes_\Lambda E = tE$ is R-isomorphic to E since t is by definition a monomorphism on E.

Let $C(E)$ be the set of R-isomorphism classes of $A \otimes_\Lambda E$, where A runs through the invertible divisors of $\Lambda = \mathrm{End}_R(E)$. A natural multiplication is given by

$$(A \otimes_\Lambda E)(B \otimes_\Lambda E) = AB \otimes_\Lambda E.$$

THEOREM 4.8. *$C(E)$ is an abelian group, and the sequence*

$$0 \to O(\mathrm{End}_R(E)) \to C(\mathrm{End}_R(E)) \to C(E) \to 0$$

is exact.

Proof. It follows immediately from 4.3 and 4.7.

II. Crossed products

1. Relative dimension. Let S be a commutative ring. Let G be a finite group represented as automorphisms of S. Let R be the fixed-point subring. Let $f \in Z^2(G, U(S))$ be a 2-cocycle, i.e., $f : G \times G \to U(S)$ such that

$$f(\tau, \rho)^\sigma f(\sigma, \tau\rho) = f(\sigma\tau, \rho)f(\sigma, \tau)$$

for all $\sigma, \tau, \rho, \in G$. (See [17] for definitions and properties.) The crossed product $\Delta(f, S, G)$, or Λ_f if there is no ambiguity, is defined to be the free left S-module generated by $\{u_\sigma ; \sigma \in G\}$, where

$$su_\sigma tu_\tau = st^\sigma f(\sigma, \tau)u_{\sigma\tau}$$

whenever $s, t \in S$ (t^σ means $\sigma(t)$). Associativity of this multiplication follows from the above formula for f. We will always assume that f is a normalized 2-cocycle, i.e., $f(\sigma, 1) = f(1, \sigma) = 1$ for all $\sigma \in G$, so that u_1 is the identity of the crossed product. Every element of $H^2(G, U(S))$ can be represented by a normalized cocycle.

If f and g are cohomologous 2-cocycles, it is well known that Λ_f and Λ_g are isomorphic, and the tower $R \subset S \subset \Lambda_f$ of rings is an invariant, up to isomorphism, of the class of f in $H^2(G, U(S))$.

If S is an integrally closed noetherian domain with quotient field L, then R is an integrally closed domain with quotient field $K = L^G$, and L is a Galois extension of K. If $f \in H^2(G, U(L))$, then $\Sigma_f = \Delta(f, L, G)$ is a K-central simple algebra split by L when G acts faithfully. If $f \in H^2(G, U(S))$ then Λ_f is an order over R in Σ_f, where we identify f with its image under

$$H^2(G, U(S)) \to H^2(G, U(L)).$$

It is this case which is the most interesting.

The map $t : S \to R$ given by $t(x) = \sum_\sigma x^\sigma$ is the trace in S/R with respect to G. In nice cases, t actually is the trace of multiplication in S. (see [4])

PROPOSITION 1.1. *Let $f \in H^2(G, U(S))$, and $\Lambda = \Lambda_f$. For every Λ-module A and S-module C,*

$$\mathrm{Ext}^*_S(A, C) \cong \mathrm{Ext}^*_\Lambda(A, \Lambda \otimes_S C).$$

Hence, $dh_S(A) \leq dh_\Lambda(A)$, and equality holds when $dh_\Lambda(A)$ is finite.

Proof (See Lemma 3.4 of [4]). Define

$$p : \mathrm{Hom}_S(A, C) \to \mathrm{Hom}_\Lambda(A, \Lambda \otimes_S C)$$

by $pg(a) = \sum_\sigma u_\sigma \otimes g(u_\sigma^{-1}a)$ whenever $a \in A$ and $g \in \mathrm{Hom}_S(A, C)$. Then for $s \in S$, $\tau \in G$,

$$\begin{aligned} pg(su_\tau a) &= \textstyle\sum_\sigma u_\sigma \otimes s^{\sigma^{-1}} \dfrac{f(\sigma^{-1}, \tau)f(\sigma^{-1}\tau, \tau^{-1}\sigma)}{f(\sigma^{-1}, \sigma)} g((u_{\tau^{-1}\sigma})^{-1}a) \\ &= s \cdot \textstyle\sum_\sigma u_{\tau\sigma} \otimes \dfrac{f(\sigma^{-1}\tau^{-1}, \tau)f(\sigma^{-1}, \sigma)}{f(\sigma^{-1}\tau^{-1}, \tau\sigma)} g(u_\sigma^{-1}a) \\ &= su_\tau \cdot \textstyle\sum_\sigma u_\sigma \otimes \dfrac{f(\sigma^{-1}\tau^{-1}, \tau)f(\sigma^{-1}, \sigma)}{f(\sigma^{-1}\tau^{-1}, \tau\sigma)f(\tau, \sigma)^{\sigma^{-1}\tau^{-1}}} g(u_\sigma^{-1}a) \\ &= su_\tau \cdot pg(a) \end{aligned}$$

since f is a 2-cocycle. Hence p is well defined. We have also

$$\varphi \in \mathrm{Hom}_S(\Lambda \otimes_S C, C)$$

given by

$$\varphi(\textstyle\sum u_\sigma \otimes c_\sigma) = c_1 \quad \text{and} \quad x = \textstyle\sum_\sigma u_\sigma \otimes \varphi(u_\sigma^{-1}x)$$

for all $x \in \Lambda \otimes_S C$. Hence given $g \in \mathrm{Hom}_\Lambda(A, \Lambda \otimes_S C)$, $\varphi g \in \mathrm{Hom}_S(A, C)$ and

$$[p(\varphi g)](a) = \textstyle\sum_\sigma u_\sigma \otimes \varphi g(u_\sigma^{-1}a) = \sum_\sigma u_\sigma \otimes \varphi(u_\sigma^{-1}g(a)) = g(a)$$

so p is onto. Since p is clearly injective, we have the desired isomorphism. Hence $\mathrm{Ext}_S(A, C) \cong \mathrm{Ext}_\Lambda(A, \Lambda \otimes_S C)$ so $dh_S(A) \leq dh_\Lambda(A)$ for all Λ-modules A. If $dh_\Lambda(A) = n < \infty$, then $\mathrm{Ext}^n_\Lambda(A, \Lambda) \neq 0$, and thus $\mathrm{Ext}^n_S(A, S) \neq 0$ and $dh_S(A) = n$.

PROPOSITION 1.2. *Let $f \in H^2(G, U(S))$ and $\Lambda = \Lambda_f$.*

(i) *$t(S) = R \Rightarrow$ every short exact sequence of Λ^e-modules which splits over $S \otimes \Lambda$ splits over Λ^e. The converse holds if G acts faithfully in S.*

(ii) *If $t(S) = R$, then every short exact sequence of Λ-modules which splits over S splits over Λ.*

Proof. (i) Suppose $t(S) = R$, and let $s \in S$ with $t(s) = 1$. Let

$$0 \to E' \to E \xrightarrow{g} E'' \to 0$$

be a short exact sequence of Λ^e-modules, and let $h \in \mathrm{Hom}_{S\otimes\Lambda}(E'', E)$ such that gh is the identity on E''. Define $p : E'' \to E$ by

$$p(e'') = \textstyle\sum_\sigma u_\sigma h(su_\sigma^{-1}e''), \; e'' \in E''.$$

The proof that $p \in \mathrm{Hom}_{\Lambda^e}(E'', E)$ is exactly analogous to the proof of Proposition 5.1, and will be omitted. Then

$$\begin{aligned} gp(e'') = \textstyle\sum_\sigma g(u_\sigma h(su_\sigma^{-1}e'')) = \sum_\sigma u_\sigma gh(su_\sigma^{-1}e''), &= \textstyle\sum_\sigma u_\sigma su_\sigma^{-1} \cdot e'' \\ &= (\textstyle\sum_\sigma s^\sigma)e'' = e'' \end{aligned}$$

and p splits the sequence over Λ^e.

Conversely, suppose that G acts faithfully in S, and that every short exact sequence of Λ^e-modules which splits over $S \otimes \Lambda$ splits over Λ^e. Consider the following exact sequence of Λ^e-modules:

$$0 \to J \to \Lambda \otimes_S \Lambda \xrightarrow{\varphi} \Lambda \to 0.$$

Then it is easy to check that for a Λ^e-module E,

$$\mathrm{Hom}_{\Lambda^e}(\Lambda, E) = \{e \in E;\ (\lambda \otimes 1 - 1 \otimes \lambda^\circ)e = 0 \quad \text{for all } \lambda \in \Lambda\},$$

$$\mathrm{Hom}_{S\otimes\Lambda}(\Lambda, E) = \{e \in E;\ (s \otimes 1 - 1 \otimes s^\circ)e = 0 \quad \text{for all } s \in S\}.$$

Since the above exact sequence splits over $S \otimes \Lambda$, it splits over Λ^e. Then there exists

$$\theta = \textstyle\sum_{\sigma,\tau} (a_{\sigma,\tau} u_\sigma \otimes u_\tau) \in \Lambda \otimes_S \Lambda$$

such that $(s \otimes 1 - 1 \otimes s^\circ)\theta = 0$, and $(u_\rho \otimes 1 - 1 \otimes u_\rho^\circ)\theta = 0$ for all $s \in S$, $\rho \in G$, and, also, $\varphi(\theta) = 1$. Now

$$(s \otimes 1)\theta = \textstyle\sum s a_{\sigma,\tau} u_\sigma \otimes u_\tau, \qquad (1 \otimes s^\circ)\theta = \textstyle\sum s^{\sigma\tau} a_{\sigma,\tau} u_\sigma \otimes u_\tau,$$

hence $a_{\sigma,\tau} \neq 0 \Leftrightarrow s = s^{\sigma\tau}$ for all $s \in S \Leftrightarrow \sigma\tau = 1$ since G acts faithfully in S. Rewriting accordingly, we have

$$\theta = \textstyle\sum_\sigma a_\sigma u_\sigma \otimes u_\sigma^{-1}$$

Now

$$\begin{aligned}
(u_\rho \otimes 1)\theta &= \textstyle\sum u_\rho a_\sigma u_\sigma \otimes u_\sigma^{-1} \\
&= \textstyle\sum (a_\sigma)^\rho f(\rho, \sigma) u_{\rho\sigma} \otimes u_\sigma^{-1} \\
&= \textstyle\sum (a_{\rho^{-1}\sigma})^\rho f(\rho, \rho^{-1}\sigma) u_\sigma \otimes (u_{\rho^{-1}\sigma})^{-1} \\
&= \textstyle\sum (a_{\rho^{-1}\sigma})^\rho \cdot \dfrac{f(\rho, \rho^{-1}\sigma) f(\rho^{-1}, \sigma)^\rho}{f(\rho, \rho^{-1})} u_\sigma \otimes u_{\sigma^{-1}} u_\rho \\
&= \textstyle\sum (a_{\rho^{-1}\sigma})^\rho u_\sigma \otimes u_{\sigma^{-1}} u_\rho
\end{aligned}$$

since f is a 2-cocycle. Also, $(1 \otimes u_\rho^\circ)\theta = \sum a_\sigma u_\sigma \otimes u_\sigma^{-1} u_\rho$. Comparing, we have $a_\sigma = (a_{\rho^{-1}\sigma})^\rho$ for all σ, $\rho \in G$. In particular, $a_\sigma = (a_1)^\sigma$ for all $\sigma \in G$. Finally, from $\varphi(\theta) = 1$, we get $t(a_1) = 1$, as desired.

(ii) The proof exactly duplicates (i) and will be omitted.

Remark. The situation of Proposition 1.2 is described in terms of relative cohomology in [13, ch. IX], i.e., every Λ^e-module is relatively $(\Lambda^e - S \otimes_R \Lambda)$-projective, etc.

THEOREM 1.3. *The following statements are equivalent:*

(i) $t(S) = R$.

(ii) *For every* $f \in H^2(G, U(S))$ *and every* Λ_f*-module* E, $dh_S(E) = dh_{\Lambda_f}(E)$.

(iii) *For every* Λ_1*-module* E, $dh_S(E) = dh_{\Lambda_1}(E)$.

Proof. (ii) $\Rightarrow$ (iii) is trivial. (iii) $\Rightarrow$ S is Λ_1-projective, so (iii) $\Rightarrow$ (i) is proved in [4, Proposition 3.5]. It remains to show (i) $\Rightarrow$ (ii).

Using Proposition 1.2(ii), it follows that for every left $\Lambda = \Lambda_f$-module E, E is a Λ-direct summand of $\Lambda \otimes_S E$, e.g., the sequence

$$\Lambda \otimes_S E \to E \to 0$$

is exact and split over S, so split over Λ. Thus for any left Λ-module F, $\mathrm{Ext}_\Lambda(F, E)$ is a direct summand of

$$\mathrm{Ext}_\Lambda(F, \Lambda \otimes_S E) \cong \mathrm{Ext}_S(F, E)$$

by 1.1. Hence, $dh_\Lambda(F) \leq dh_S(F)$. On the other hand, Λ is S-free, so $dh_\Lambda(F) \geq dh_S(F)$ always, and we have equality.

Remark. As an immediate corollary, we have Williamson's result [18].

COROLLARY 1.4. *Let R be local and $f \in H^2(G, U(S))$. If $t(S) = R$, then the radical $r(\Lambda_f)$ of Λ_f is equal to $r(S)\Lambda_f$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. Let $\mathfrak{M}_1, \cdots, \mathfrak{M}_g$ be the maximal ideals of S (the reader will recall that all algebras are finitely generated R-modules, so that S is an integral extension of R). It is well known that G acts transitively in the set $\mathfrak{M}_1, \cdots, \mathfrak{M}_g$, i.e., $\mathfrak{M}_i = \mathfrak{M}_1^\sigma$ for some $\sigma \in G$. Then we have $r(S) = \mathfrak{M}_1 \mathfrak{M}_2 \cdots \mathfrak{M}_g$, and $r(S)\Lambda = \Lambda r(S)$ is a two-sided ideal, $\Lambda = \Lambda_f$.

Now it is clear that we have $t(\bar{S}) = \bar{R}$, where we have written $\bar{S} = S/r(S)$ and $\bar{R} = R/\mathfrak{m}$. Applying Theorem 1.3 to the situation $\bar{R}, \bar{S}, G$, we find that $dh_{\bar{\Lambda}}(E) = dh_S(E) = 0$ for every $\bar{\Lambda} = \Delta(f, \bar{S}, G)$-module E. Hence every $\bar{\Lambda}$-module is $\bar{\Lambda}$-projective, and $\bar{\Lambda}$ is semi-simple using the characterization of [7]. The assertion of the corollary is clear

Actually, $\bar{\Lambda}$ is a separable algebra over $\bar{R}$ [5], i.e., $k \otimes_{\bar{R}} \bar{\Lambda}$ is semi-simple for every field k containing $\bar{R}$, by Theorem 1.3. The theorem is applied to the extension $k \subset k \otimes_{\bar{R}} \bar{S}$, in which G acts through its action on $\bar{S}$, using the following:

$$k \otimes_{\bar{R}} \bar{\Lambda} = k \otimes_{\bar{R}} \Delta(f, \bar{S}, G) = \Delta(1 \otimes f, k \otimes_{\bar{R}} \bar{S}, 1 \otimes G)$$

and

$$t(k \otimes_{\bar{R}} \bar{S}) = kt(\bar{S}) = k.$$

In an important special case, we have

COROLLARY 1.5. *Let R be an integrally closed noetherian domain with quotient field K, and let S be the integral closure of R in a finite Galois extension L of K, G the Galois group. The following statements are equivalent:*

(i) *S is a tamely ramified extension of R.*

(ii) *$\Delta(f, \overline{S_\mathfrak{m}}, G)$ is $\bar{R}_\mathfrak{m}$-separable for all $f \in H^2(G, U(S))$, and every maximal ideal $\mathfrak{m}$ of R.*

(iii) *$\Delta(1, \overline{S_\mathfrak{m}}, G)$ is $\overline{R_\mathfrak{m}}$-separable for every maximal ideal $\mathfrak{m}$ of R.*

Proof. It remains only to show (iii) $\Rightarrow$ (i). Since $\Delta = \Delta(1, \overline{S_\mathfrak{m}}, G)$ is

semi-simple, then $\overline{S_\mathfrak{m}}$ is Δ-projective, so by proposition 3.5 of (4], $t(\overline{S_\mathfrak{m}}) = \overline{R_\mathfrak{m}}$. Hence, $t(S_\mathfrak{m}) = R_\mathfrak{m}$. Since this is true for every maximal ideal $\mathfrak{m}$, we have $t(S) = R$.

Remark. The main use of the above theorems has been as a technique for generating examples of orders with given global dimensions and studying their properties. For example, one can prove:

(i) A maximal order over an integrally closed local domain R can have an arbitrary (finite) number of maximal two-sided ideals. The only bound seems to be the Krull dimension of R.

(ii) If R is the center of an order Λ of finite global dimension, then gl dim(R) may be infinite.

2. Modules over crossed products. We begin with a very formal lemma. The proof is a simple computation and will be omitted.

LEMMA 2.1. *Let $f, g \in H^2(G, U(S))$. Let A be a left Λ_f-module, B a left Λ_g-module. Then $A \otimes_S B$ is a left Λ_{fg}-module under*

$$(sw_\sigma)(a \otimes b) = su_\sigma a \otimes v_\sigma b$$

and $\mathrm{Hom}_S(A, B)$ *is a left $\Lambda_{f^{-1}g}$-module under*

$$(sw_\sigma \varphi)(a) = sv_\sigma \cdot \varphi(u_\sigma^{-1} a).$$

LEMMA 2.2. *Let $R \subset S$ be an integral extension of integrally closed noetherian domains. Then an S-module E which is R-reflexive is also S-reflexive.*

Proof. Let E be R-reflexive, and we use the characterization of I, Lemma 1.1. Since S/R is integral and E is torsion-free over R, then E is torsion-free over S. If p is a minimal prime of R, then S_p is a dedekind domain, and E_p is S_p-projective. Then $E_p = \bigcap_q (E_p)_q = \bigcap_q E_q$ where q runs through the minimal primes of S with $q \cap R = p$. Then

$$E = \bigcap_p E_p = \bigcap_p (\bigcap_{q/p} E_q) = \bigcap_q E_q$$

and E is S-reflexive

If $\mathbf{C}$ and $\mathbf{D}$ are abelian categories, an isomorphism of $\mathbf{C}$ and $\mathbf{D}$ will consist of covariant additive functors from $\mathbf{C}$ to $\mathbf{D}$ and from $\mathbf{D}$ to $\mathbf{C}$ which yield the respective identity functors on composition.

THEOREM 2.2. *Denote by $\mathbf{C}_f$ the category of left Λ_f-modules.*

(i) *If there exists a Λ_f-module P which is S-projective of rank one, then $\mathbf{C}_g$ and $\mathbf{C}_{fg}$ are isomorphic abelian categories for all $g \in H^2(G, U(S))$.*

(ii) *Let R be an integrally closed noetherian domain with quotient field K, and S the integral closure of R in a finite Galois extension L of K, with G the Galois group. Suppose that every S-reflexive module of rank one is S-projective, i.e., S is locally a UFD. Then the converse of* (i) *holds. In particular, $\mathbf{C}_f$ is isomorphic to $\mathbf{C}_1 \Leftrightarrow f \to 1$ in $H^2(G, U(L))$.*

Proof. (i) We observe first that if P is a Λ_f-module which is S-projective of rank one, then $\mathrm{Hom}_S(P, S)$ is a Λ_f^{-1} module which is S-projective of rank one, and then the canonical isomorphisms

$$S \xrightarrow{i} \mathrm{Hom}_S(P, P) \xleftarrow{\mu} P \otimes_S \mathrm{Hom}_S(P, S)$$

are also isomorphisms of Λ_1-modules, where

$$[\mu(a \otimes \varphi)](b) = \varphi(b)a, \qquad a, b \in P, \varphi \in \mathrm{Hom}_S(P, S);$$

$$[i(s)](a) = sa, \qquad s \in S, \quad a \in P.$$

Now if A is a Λ_g-module, then $P \otimes_S A$ is a Λ_{fg}-module. If B is a Λ_{fg}-module, then $\mathrm{Hom}_S(P, S) \otimes_S B$ is a Λ_g-module. Also

$$P \otimes_S \mathrm{Hom}_S(P, S) \otimes_S B \cong B$$

as Λ_{fg} modules,

$$\mathrm{Hom}_S(P, S) \otimes_S P \otimes_S A \cong A$$

as Λ_g-modules by the above observations. Moreover, the canonical map

$$\mathrm{Hom}_{\Lambda_g}(A, A') \to \mathrm{Hom}_{\Lambda_{fg}}(P \otimes_S A, P \otimes_S A')$$

is obviously an isomorphism.

(ii) It remains only to prove that, under these restrictive hypotheses, $\mathbf{C}_f$ and $\mathbf{C}_1$ are isomorphic $\Leftrightarrow f$ is a coboundary in $H^2(G, U(L))$. If $\mathbf{C}_f$ and $\mathbf{C}_1$ are isomorphic, let F be a covariant additive functor fom $\mathbf{C}_1$ to $\mathbf{C}_f$ which is an isomorphism. Since a functor must preserve composition of maps, it follows that F leads to a ring isomorphism of $\mathrm{End}_{\Lambda_1}(E)$ with $\mathrm{End}_{\Lambda_f}(F(E))$ for all Λ_1-modules E. In particular, let $P = F(S)$. Then $\mathrm{End}_{\Lambda f}(P) \cong R$, so P has rank one as an S-module. Hence, there exists a Σ_f-module of L-rank one, so Σ_f must be a total matrix algebra over K, i.e., f must be a coboundary in $H^2(G, U(L))$.

Conversely, suppose that f becomes a coboundary. Then Σ_f is a total matrix algebra over K, and Λ_f is an order over R in Σ_f. Let Γ be a maximal order containing Λ_f. Then by [3, Proposition 4.2], $\Gamma = \mathrm{End}_R(E)$, where E is a reflexive R-module. Then E is a reflexive S-module by Lemma 2.2. Moreover, by an elementary counting argument, $[E:S] = 1$. Since S is locally a UFD, E is a Λ_f-module which is S-projective of rank one, and the isomorphism follows from (i). This completes the proof.

We consider now the special case $f = 1$. Denote by $P(S, G)$ the set of Λ_1-isomorphism classes of Λ_1-modules which are S-projective of rank one. If P, Q are such modules, then $P \otimes_S Q$ is also such. Moreover, if

$$\alpha : P \to P' \quad \text{and} \quad \beta : Q \to Q'$$

are Λ_1-isomorphisms, then $\alpha \otimes \beta$ is again a Λ_1-isomorphism. Thus, we have a well-defined multiplication in $P(S, G)$. Associativity follows from the associativity of the tensor product, and commutativity is obvious from the

definition in Lemma 2.1. By the proof of 2.3, $S \approx P \otimes_S \mathrm{Hom}_S(P, S)$ as Λ_1-modules if $P \in P(S, G)$. We have proved

PROPOSITION 2.4. *The set $P(S, G)$ of Λ_1-isomorphism classes of Λ_1-modules which are S-projective of rank one forms an abelian group for which S is the identity.*

In the most interesting special case, we have the following canonical computation of $P(S, G)$: We assume as usual that R is an integrally closed noetherian domain and S is the integral closure of R in a finite Galois extension L of the quotient field K of R, $G = G(L/K)$. Then Λ_1 is an order over R in the total matrix ring $\Sigma_1 = \Delta(1, L, G)$. Since $P \otimes_R K$ is a simple Σ_1-module for $P \in P(S, G)$, it is clear that every element of $P(S, G)$ has a representative which is a Λ_1-submodule of L, i.e., a projective fractionary ideal of S which is closed under the action of G.

Let $D(S)$ denote the group of projective fractionary ideals of S, i.e., the subgroup of the group of divisors consisting of projective ideals. $D(S)$ is a G-module, where the action of G comes from its action on L.

PROPOSITION 2.5. *If R is a noetherian, integrally closed domain and S is the integral closure of R in a finite Galois extension L of its quotient field K, with $G = G(L/K)$, then there is an exact sequence*

$$0 \to U(K)/U(R) \to D(S)^G \to P(S, G) \to 0.$$

Proof. By the above remarks, $D(S)^G \to P(S, G)$ is onto, so we have only to compute the kernel. If $\mathfrak{a}$ is a G-invariant ideal which is Λ_1-isomorphic to S, then $\mathfrak{a} = Sa$ is a principal fractionary ideal generated by $a \in L$ such that $a^\sigma = a$ for all $\sigma \in G$, i.e., $a \in K$. The rest is immediate.

We next prove a theorem which is in part a reformulation of some results in [14]. It is true without restrictive hypotheses on S or G. Before stating the theorem, we make the following general observation: If E is an S-module and $\sigma \in G$, then we define a new S-module E^σ which is R-isomorphic to E, such that $s \cdot e = s^\sigma e$ for $s \in S$ and $e \in E^\sigma$. If $P(S)$ denotes the group of isomorphism classes of S-projective modules of rank one, then this defines the structure of G-module on $P(S)$.

THEOREM 2.6. *There exists an exact sequence*

$$0 \to H^1(G, U(S)) \to P(S, G) \to P(S)^G \to H^2(G, U(S)).$$

Proof. We define

$$\varphi_1 \colon H^1(G, U(S)) \to P(S, G)$$

as follows: Let $h : G \to U(S)$ be a 1-cocycle, i.e., $h(\sigma)h(\tau)^\sigma = h(\sigma\tau)$ for all $\sigma, \tau \in G$. $\varphi_1(h)$ will be the class of an S-free module Sx, where $u_\sigma sx = s^\sigma h(\sigma)x$ for all $s \in S$, $\sigma \in G$. This is easily seen to define a Λ_1-module. It is clear that φ_1 is well defined.

If $\varphi_1(h) = S$, then Sx is generated by some x' such that $u_\sigma x' = x'$ for all $\sigma \in G$. We have $x' = ax$, $x = a'x'$, for $a, a' \in S$, so $aa' = 1$, $a \in u(S)$. $u_\sigma x' = u_\sigma ax = a^\sigma h(\sigma)x = ax$, hence $h(\sigma) = a^{1-\sigma}$ and h is a 1-coboundary. Hence φ_1 is injective.

The map $P(S, G) \to P(S)^G$ simply forgets the Λ_1-module structure. Exactness at $P(S, G)$ is obvious.

We define

$$\varphi_2 \colon P(S)^G \to H^2(G, U(S))$$

as follows: Let P be an S-projective module of rank one such that P and P^σ are S-isomorphic for every $\sigma \in G$, and let $g_\sigma \colon P \to P^\sigma$ be an S-isomorphism, i.e., $g_\sigma \in \mathrm{Aut}_R(P)$ with $g_\sigma s = s^\sigma g_\sigma$ whenever $s \in S$, $\sigma \in G$. Then $g_\sigma g_\tau g_{\sigma\tau}^{-1} = f(\sigma, \tau) \in \mathrm{Aut}_R(P)$ commutes with all of S, and hence $f(\sigma, \tau) \in U(S) = \mathrm{Aut}_S(P)$ for all $\sigma, \tau \in G$. It is clear that f must be a 2-cocycle. $\varphi_2(\mathrm{cl}(P))$ is defined to be $\mathrm{cl}(f)$. This clearly does not depend upon the choice of the isomorphism g_σ. Moreover, $\varphi_2(P)$ depends only on the isomorphism class of P, since if $\beta : Q \to P$ is an S-isomorphism, then $\beta^{-1} g_\sigma \beta$ is an S-isomorphism of Q with Q^σ and

$$(\beta^{-1} g_\sigma \beta)(\beta^{-1} g_\tau \beta)(\beta^{-1} g_{\sigma\tau} \beta)^{-1} = \beta^{-1} f(\sigma, \tau)\beta = f(\sigma, \tau).$$

Hence φ_2 is well defined. The exactness at $P(S)^G$ is now trivial once one observes that the g_σ give rise to a 2-coboundary if and only if they generate a Λ_1-module structure on P.

3. Invertible ideals and automorphisms. Let R be an integrally closed noetherian domain with quotient field K. Let L be a Galois extension of K with Galois group G, and assume that the integral closure S of R in L is tamely ramified over R. Then for all $f \in H^2(G, U(S))$, Theorem II.3 states that Λ_f is a tame order over R in Σ_f. These notations will persist throughout this section. The proof of the following lemma is an easy computation, which will be omitted. See 1.4.

LEMMA 3.1. *Suppose R is a discrete valuation ring, and M is the radical of S. Let $f \in H^2(G, U(S))$ and N be the radical of Λ_f. Then $N^{-1} = M^{-1}\Lambda_f$.*

PROPOSITION 3.2. *Let A be an invertible divisor of $\Lambda = \Lambda_f$ (see definition in Section* I.3). *Then $A \cap L$ is a Λ_1-module, hence defines an element of $P(S, G)$ and $A = (A \cap L)\Lambda$. Hence $C(\Lambda_f) \approx P(S, G)$ for every $f \in H^2(G, U(S))$.*

Proof. Write $A = \bigcap_p A_p$ over all minimal primes p of R. By I.2.3, II.1.4, and II.3.1, $A = \bigcap_p (r(S_p)^{m_p})\Lambda_p$, and this is equal to $(\bigcap_p r(S_p)^{m_p})\Lambda$, by the properties of localization, and because Λ is S-free. Clearly, $\bigcap_p r(S_p)^{m_p}$ is a Λ_1-module and is equal to $A \cap L$. Moreover, $A = (A \cap L)\cdot\Lambda$ is the direct sum of copies of $A \cap L$ as S-module, so A is S-projective $\Leftrightarrow A \cap L$ is S-projective, i.e., by tame ramification and II.1.3, A is Λ_f-projective $\Leftrightarrow A \cap L$ is Λ_1-projective. The last conclusion now follows from II.2.5, i.e., we have only to show that $A \approx B$ as Λ_f^e-modules $\Leftrightarrow A \cap L \approx B \cap L$ as Λ_1-modules. But $A \approx B$

as Λ_f^e-modules $\Leftrightarrow B = Aa$ for some $a \in K \Leftrightarrow (A \cap L)a = B \cap L$ since $A = (A \cap L)\Lambda$.

We will now compute $O_S(\Lambda)$, the group of "outer S-automorphisms", and $C_S(\Lambda)$, the group of $S \otimes_R \Lambda^\circ$-isomorphism classes of invertible divisors of $\Lambda = \Lambda_f$. (See I.4.5.)

Let $h : G \to U(S)$ be a one-cocycle. By Hilbert's Theorem 90 [17, X, Propositon 2], $h(\sigma) = s^{-1}s^\sigma$ for some $s \in U(L)$. If $\lambda = \sum_\sigma a_\sigma u_\sigma \in \Lambda$, then

$$s^{-1}\lambda s = \sum_\sigma s^{-1}s^\sigma a_\sigma u_\sigma \in \Lambda,$$

so the 1-cocycle h gives rise to an S-automorphism of Λ. A coboundary gives rise to an inner automorphism, clearly, so the map

$$H^1(G, U(S)) \to O_S(\Lambda)$$

is well defined. Actually, it is a monomorphism, for suppose that h gives rise to an inner automorphism. Then $s^{-1}\lambda s = t^{-1}\lambda t$ for all $\lambda \in \Lambda$, some $t \in U(S)$. Putting $\lambda = u_\sigma$, we get $s^{-1}s^\sigma = t^{-1}t^\sigma$ for all σ, as desired.

Let $P'(S)$ be the image of $P(S, G) \to P(S)$, using the map which assigns to each Λ_1-isomorphism class its S-isomorphism class. (See II.2.6.) Then $P'(S)$ is the group of S-isomorphism classes of Λ_1-projective modules which are S-modules of rank one, by II.1.4. There is a natural map $P'(S) \to C_S(\Lambda_f)$ which associates to the ambiguous ideal class $\mathfrak{a}$ the $S \otimes_R \Lambda_f^\circ$-isomorphism class of the invertible divisor $\mathfrak{a}\Lambda_f$.

THEOREM 3.3. *$H^1(G, U(S)) \to O_S(\Lambda_f)$ and $P'(S) \to C_S(\Lambda_f)$ are isomorphisms.*

Proof. We have the commutative diagram

$$\begin{array}{ccccccc} 0 \to & H^1(G, U(S)) & \to & P(S, G) & \to & P'(S) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & O_S(\Lambda_f) & \to & C(\Lambda_f) & \to & C_S(\Lambda_f) & \to 0 \end{array}$$

where the rows are exact by II.2.6 and I.4.5, and the middle vertical map is an isomorphism by II.3.2. The left vertical map is injective by the observations preceding the statement of the theorem. By the five lemma, it suffices to prove that $H^1(G, U(S)) \to O_S(\Lambda_f)$ is onto. By the Skolem-Noether theorem, and the fact that L is a maximal commutative subring of Σ_f, any S-automorphism β of Λ_f can be written as $\beta(\lambda) = x^{-1}\lambda x$ for some $x \in U(L)$, all $\lambda \in \Lambda_f$. Then $x^{-1}u_\sigma x = x^{-1}x^\sigma u_\sigma \in \Lambda_f$, so $x^{-1}x^\sigma \in S$ for all $\sigma \in G$. It is easily seen that this means $x^{-1}x^\sigma \in U(S)$, and we are done, since this is a 1-cocycle of G in $U(S)$.

III. Relations with Amitsur cohomology

1. $\Delta(S, G)$ and $\mathrm{End}_R(S)$. Throughout this section, R will be an integrally closed noetherian domain with quotient field K, S the integral closure of R in a finite Galois extension L of K, and $G = G(L/K)$.

The order $\Delta(1, S, G) = \Lambda$, is contained in the maximal order $\mathrm{End}_R(S)$; i.e., as usual we have $\Delta(1, S, G) \to \mathrm{End}_R(S)$ by $(\sum a_\sigma u_\sigma)(s) = \sum a_\sigma s^\sigma$, and this is injective since the quotient field extension is Galois. $\mathrm{End}_R(S)$ is a maximal order by [3, Proposition 4.2] once we know that S is a reflexive R-module. But by I.1.1, $S^{**} \supseteq S$ is a subring of L which is finitely generated as an R-module, hence $S = S^{**}$ since S is integrally closed. If the extension S/R is unramified [1], or S/R is a Galois extension in the terminology of [8], then $\Lambda_1 = \mathrm{End}_R(S)$. In this case, $O_S(\mathrm{End}_R(S)) = H^1(G, U(S))$ by 3.3. It is interesting to compute this group in a more general case. We will do so using the concept of Amitsur cohomology, the details of which are found in [16] or [9]. The following brief outline is reproduced from [10] for the convenience of the reader.

Let R be a commutative ring and T a commutative R-algebra, R not necessarily neotherian and T not necessarily finitely generated as an R-module. Let T^n denote the n-fold tensor product of T over R, in the category K of commutative R-algebras. For all $i = 0, 1, \cdots, n + 1$, we have R-algebra homomorphisms

$$\varepsilon_i\colon T^{n+1} \to T^{n+2}$$

defined by

$$\varepsilon_i(t_0 \otimes t_1 \otimes \cdots \otimes t_n) = t_0 \otimes \cdots \otimes t_{i-1} \otimes 1 \otimes t_i \otimes \cdots \otimes t_n .$$

The ε_i satisfy cosemisimplicial identities $\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i$ for $i \leq j$. Let F be a covariant functor from K to the category Ab of abelian groups. A cochain complex $C(T/R, F)$ is defined by

$$C^n(T/R, F) = F(T^{n+1}) \qquad \text{for} \quad n \geq 0.$$

with coboundary $\Delta^n : C^n(T/R, F) \to C^{n+1}(T/R, F)$ defined by

$$\Delta^n = \textstyle\sum_{i=0}^{n+1} (-1)^i F(\varepsilon_i).$$

The n-th cohomology group of this complex, written $H^n(T/R, F)$, is the n-th Amitsur cohomology group.

The Galois cohomology can be defined in a similar manner. Let G be represented as R-automorphisms of T, and let E_n denote the T-algebra of functions of n variables, defined on G, with values in T. As an R-algebra, E_n is isomorphic to the direct product of $n + 1$ copies of T. We define R-algebra homomorphisms $\theta_i\colon E_n \to E_{n+1}$ as follows: For $f \in E_n$,

$$\theta_0 f(\sigma_1, \sigma_2, \cdots, \sigma_{n+1}) = f(\sigma_2, \cdots, \sigma_{n+1})^{\sigma_1},$$

$$\theta_i f(\sigma_1, \cdots, \sigma_{n+1}) = f(\sigma_1, \cdots, \sigma_{i-1}, \sigma_i \sigma_{i+1}, \sigma_{i+2}, \cdots, \sigma_{n+1}) \qquad \text{for} \quad i = 1, 2, \cdots, n,$$

and

$$\theta_{n+1} f(\sigma_1, \cdots, \sigma_{n+1}) = f(\sigma_1, \cdots, \sigma_n).$$

A cochain complex $C(G, F)$ is defined by

$$C^n(G, F) = F(E_n) \qquad \text{for} \quad n \geq 0,$$

with coboundary $d^n : F(E_n) \to F(E_{n+1})$ given by

$$d^n = \sum_{i=0}^{n+1} (-1)^i F(\theta_i).$$

Suppose that F is a multiplicative functor from K to Ab (called "additive" in [8]), i.e., F commutes with direct products. This means that the canonical map

$$F(T_1 \oplus \cdots \oplus T_m) \to F(T_1) \times \cdots \times F(T_n),$$

generated by the projections $T_1 \oplus \cdots \oplus T_m \to T_j$, is an isomorphism of abelian groups for every finite set $T_1, \cdots, T_m$ of R-algebras. In this case, $F(E_n)$ is the abelian group of all functions of n variables, defined on G, with values in $F(T)$; it is then clear that the complex $C(G, F)$ is the standard non-homogeneous cochain complex of G with coefficients in $F(T)$, whose nth cohomology group is $H^n(G, F(T))$ [17].

We define a map $h_n : T^{n+1} \to E_n$ by

$$h_n(t_0 \otimes \cdots \otimes t_{n+1})(\sigma_1, \cdots, \sigma_n) = t_0(t_1)^{\sigma_1}(t_2)^{\sigma_1\sigma_2} \cdots (t_{n+1})^{\sigma_1\sigma_2\cdots\sigma_n}.$$

This is a map of R-algebras for all $n \geq 0$. Moreover, the following diagram commutes for all $n \geq 0$, all $i = 0, 1, \cdots, n + 1$:

$$\begin{array}{ccc} T^{n+1} & \xrightarrow{h_n} & E_n \\ \Big\downarrow{\scriptstyle \varepsilon_i} & & \Big\downarrow{\scriptstyle \theta_i} \\ T^{n+2} & \xrightarrow{h_{n+1}} & E_{n+1}. \end{array}$$

Therefore, the h_n define a chain map $h : C(T/R, F) \to C(G, F)$ for any multiplicative functor F. In [9], it is proved that h is an isomorphism of complexes when T/R is a Galois extension with group G. In general, we have a map

$$h_n^* : H^n(T/R, F) \to H^n(G, F(T)).$$

This map will be studied more closely in Section 2.

Let U denote the multiplicative functor which assigns to an R-algebra T its group of units $U(T)$, and UK the multiplicative functor which assigns to T the group $U(K \otimes_R T)$, where we assume that R is an integrally closed noetherian domain with quotient field K. The natural transformation of U to UK induced by the injection $R \to K$ defines maps on cohomology

$$H^n(T/R, U) \to H^n(T/R, UK) = H^n(T \otimes_R K/K, U).$$

We consider the special case $n = 1$.

Let S be the integral closure of R in a finite Galois extension L of K, $G = G(L/K)$. Then we have

$$H^1(S/R, UK) = H^1(L/K, U) \cong H^1(G, U(L)) = 0$$

by Hilbert's Theorem 90. Using the coboundary maps explicitly, we find that an Amitsur 1-cocycle is therefore an element $a \otimes a^{-1} \in U(S^2)$, where $a \in U(L)$, and that this is a coboundary if, and only if, $a \otimes a^{-1} = b \otimes b^{-1}$ for some $b \in U(S)$.

THEOREM 1.1. *There is a natural map*

$$\varphi : H^1(S/R, U) \to O_S(\mathrm{End}_R(S)),$$

which is a monomorphism. Moreover, φ is an isomorphism if S is R-projective as a module.

Proof. First we observe that $\mathrm{Hom}_R(S, S)$ is naturally an $S \otimes S$-module under

$$(s \otimes t)f(a) = sf(ta) \quad \text{for} \quad s, t, a \in S, \otimes = \otimes_R .$$

Moreover, $\mathrm{Ann}_{S\otimes S}(\mathrm{Hom}_R(S, S)) = (0)$, since $\mathrm{Hom}_R(S, S) \otimes_R K = \mathrm{Hom}_K (L, L)$ is an $(L \otimes_K L)$-free module generated by the trace $t_{L/K}$, by definition of a Galois extension.

It follows that if $a \in U(L)$ such that $a \otimes a^{-1} \in U(S^2)$, then

$$f \to (a \otimes a^{-1})f = afa^{-1}$$

is an S-automorphism of $\mathrm{End}_R(S)$. Moreover, if $afa^{-1} = bfb^{-1}$ for all $f \in \mathrm{Hom}_R(S, S)$, then $a \otimes a^{-1} = b \otimes b^{-1}$ in $U(S^2)$. In particular, a and b define the same element of $H^1(S/R, U)$. Finally, if we define $\varphi(a \otimes a^{-1})(f) = afa^{-1}$, then φ is a well-defined monomorphism of abelian groups by these remarks.

Next, suppose that S is R-projective and let $a \in U(L)$ such that $f \to afa^{-1}$ is an S-automorphism of $\mathrm{End}_R(S)$. It is clear that every S-automorphism must be of this type. Now

$$\mathrm{Hom}_R(S, S) \otimes S \to \mathrm{Hom}_{1\otimes S}(S \otimes S, S \otimes S)$$

is an isomorphism since S is R-projective. Hence every S-automorphism of $\mathrm{End}_R(S)$ can be extended to an $(S \otimes S)$-automorphism of $\mathrm{End}_S(S \otimes S)$. By [8, Lemma 3.9(c)], every $(S \otimes S)$-automorphism of $\mathrm{End}_S(S \otimes S)$ is inner. Hence, there exists $u \in U(S \otimes S)$ such that

$$u(f \otimes s)u^{-1}$$
$$= (a \otimes 1)(f \otimes s)(a \otimes 1)^{-1} = afa^{-1} \otimes s \quad \text{for all} \quad f \in \mathrm{End}_R(S), \quad s \in S.$$

By the same reasoning as above, $\mathrm{Hom}_L(L^2, L^2)$ is L^3-free on one generator. Thus,

$$a \otimes a^{-1} \otimes 1 = \varepsilon_1(u)\varepsilon_0(u^{-1}) \qquad \text{in} \quad L^3.$$

Hence, $a \otimes a^{-1} \otimes 1 \in U(S^3)$, by definition of u. Applying the natural degeneracy operator $S^3 \to S^2$ which maps $s_0 \otimes s_1 \otimes s_2$ onto $s_0 \otimes s_1 s_2$, it follows that $a \otimes a^{-1} \in U(S^2)$, as desired.

COROLLARY 1.2. *If S is R-projective, then every S-automorphism of* $\mathrm{End}_R(S)$ *maps $\Delta(1, S, G)$ onto itself.*

Proof. This follows directly from the theorem, along with Theorem II.3.3.

2. A spectral sequence. The relation between the Galois cohomology and the Amitsur cohomology, studied briefly in the previous section, can be expressed in terms of a spectral sequence. The terminology of the last section will be used here without further definition.

Let S be an R-algebra and G a finite group represented as R-automorphisms of S. For any multiplicative functor F from commutative R-algebras to abelian groups, we define the functor FS from commutative R-algebras to G-modules by $FS(T) = F(S \otimes_R T)$. If F is multiplicative, then so is FS, since the tensor product of R-algebras distributes over finite direct products.

We define a double complex of cochains $C(S/R, G, *)$ as follows: For every pair p, q, of non-negative integers, let $C^{pq}(S/R, G, *)$ be the S-algebra of functions of q variables, defined on G, with values in S^{p+2}, i.e.,

$$C^{pq}(S/R, G, *) = E_q \otimes_R S^{p+1},$$

so $C(S/R, G, *)$ is the tensor product of the standard non-homogeneous cochain complexes $C(G, *)$ and $C(S/R, *)$ for Galois and Amitsur cohomology, respectively, as defined in the previous section. In the usual manner, the differentiations

$$\Delta^p : S^{p+1} \to S^{p+2}, \qquad d^q : E^q \to E_{q+1}$$

give rise to differentiations $D'_p = 1 \otimes \Delta^p$, $D''_q = (-1)^p d^q \otimes 1$ which are defined on $C^{pq}(S/R, G, *)$, with the total coboundary operator $D = D' + D''$, and these form a double complex. Let F be a multiplicative functor from R-algebras to abelian groups. Then the double complex $C(S/R, G, F)$ is given by

$$C^{pq}(S/R, G, F) = F[C^{pq}(S/R, G, *)].$$

This is the group of functions of q variables from G to $F(S^{p+2})$ or the G-module of functions of q variables from G to $FS(S^{p+1})$. The coboundary operators

$$D'_p : C^{pq}(S/R, G, F) \to C^{p+1,q}(S/R, G, F)$$

and

$$D''_q : C^{pq}(S/R, G, F) \to C^{p,q+1}(S/R, G, F)$$

are given by

$$D'_p = \textstyle\sum_{i=0}^{p+2} (-1)^i FS(\varepsilon_i) = \sum_{i=1}^{p+3} (-1)^{i+1} F(\varepsilon_i)$$
$$D''_q = \textstyle\sum_{j=0}^{q+1} (-1)^{p+j} FS(\theta_j)$$

where the ε_i and θ_j are defined in the previous section.

Differentiating first with respect to D'', we get, for every q, the Amitsur complex $C(S/R, F^q)$, where F^q is the multiplicative functor from R-algebras to abelian groups defined by

$$F^q(T) = H^q(G, F(S \otimes_R T))$$

for any R-algebra T. Differentiating next with respect to D', we get the bigraded cohomology $E_2^{pq} = H^p(S/R, F^q)$.

To compute the total cohomology of $C(S/R, G, F)$, we will show that the so-called first spectral sequence of $C(S/R, G, F)$ is degenerate, i.e., we differentiate first with respect to D', then with D''.

We begin with some observations about the complex $C(S \otimes_R S/S, *)$ for Amitsur cohomology in the extension $S \otimes_R S/S$. Let $\varphi : S^{p+2} \to S^{p+1}$ be the "degeneracy" given by

$$\varphi(s_0 \otimes s_1 \otimes \cdots \otimes s_{p+1}) = s_0 s_1 \otimes s_2 \otimes \cdots \otimes s_{p+1}.$$

The following identities are easily verified:

$$\varphi\varepsilon_1 = 1, \qquad \varphi\Delta^0 + \varepsilon_1\varphi = 1,$$

$$\varphi\varepsilon_i = \varepsilon_{i-1}\varphi \qquad \text{for } i > 1,$$

$$\varphi\Delta^p + \Delta^{p-1}\varphi = 1 \qquad \text{for } p \geq 1.$$

From these identities, we conclude immediately that φ is a contracting homotopy for the complex $C(S \otimes_R S/S, F)$, where F is any functor from S-algebras to abelian groups, and therefore:

$$H^n(S \otimes_R S/S, F) = 0 \qquad \text{for } n \geq 0,$$

$$\varepsilon_1 : F(S) \to_{\approx} H^0(S \otimes_R S/S, F).$$

We now return to the discussion of the double complex $C(S/R, G, F)$. Differentiating with respect to D', we have, for each $p > 0$, the complex

$$C(G, H^p(S/R, FS)) = C(G, H^p(S \otimes_R S/S, F)) = 0$$

and, for $p = 0$, the complex

$$C(G, H^0(S/R, FS)) = C(G, F(S)).$$

It follows that $H^n(\mathrm{Tot}\,(C(S/R, G, F))) = H^n(G, F(S))$ for all non-negative integers n. We summarize these results:

THEOREM 2.1. *Let G be a finite group represented as R-automorphisms of an R-algebra S. Let F be a multiplicative functor from R-algebras to abelian groups. Then there is a sectral sequence*

$$H^p(S/R, F^q) \underset{p}{\Rightarrow} H^n(G, F(S))$$

where $F^q(T) = H^q(G, F(S \otimes_R T))$ defines a multiplicative functor from R-algebras to abelian groups.

We next compute the first subgroup $F_1 H^n$ in the filtration of

$$H^n(\mathrm{Tot}\,(C(S/R, G, F))).$$

We observe that $F_1 C(S/R, G, F)$ is the subcomplex of $C(S/R, G, F)$ generated by all bihomogeneous elements of bidegree p, q, with $q \geq 1$. The quotient complex $C(S/R, G, F)/F_1 C(S/R, G, F)$ is canonically isomorphic with the complex $C(G, F(S \otimes_R S))$ for the Galois cohomology in $S \otimes_R S/S$. It follows that the natural projection

$$p : C(S/R, G, F) \to C(S/R, G, F)/F_1 C(S/R, G, F)$$

induces an exact sequence on cohomology

$$0 \to F_1 H^n \to H^n(G, F(S) \xrightarrow{p^*} H^n(G, F(S \otimes_R S))$$

Moreover, $p^* = \varepsilon_1^*$ is exactly the map generated by

$$\varepsilon_1 : S \to S \otimes_R S.$$

This can be most easily seen by studying the collapsing of the first spectral sequence of $C(S/R, G, F)$.

To obtain the main consequence of this spectral sequence, we state here a formal lemma, which is well known [9, Lemma 7.5].

LEMMA 2.2. *Let E_r be any cohomology spectral sequence, i.e., $E_2^{pq} = 0$ for $p < 0$ or $q < 0$. Let H be a graded, filtered abelian group such that*

$$E_2^{p,q} \underset{p}{\Rightarrow} H^n.$$

Then there is an exact sequence

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1}\, E_2^{2,0} \to F_1 H^2 \to E_2^{1,1} \to E_2^{3,0}$$

where $F_1 H^2$ is the first filtration subgroup of H^2.

THEOREM 2.3. *Let G be a finite group represented as R-automorphisms of a commutative R-algebra S. Let F be a multiplicative functor from R-algebras to abelian groups. Let F^q be the functor from R-algebras to abelian groups defined by*

$$F^q(T) = H^q(G, F(S \otimes_R T)).$$

Then there exists an exact sequence

$$\begin{aligned} 0 &\to H^1(S/R, F) \to H^1(G, F(S)) \to H^0(S/R, F^1) \\ &\to H^2(S/R, F) \to \ker\,[H^2(G, F(S)) \to H^2(G, F(S \otimes_R S))] \\ &\to H^1(S/R, F^1) \to H^3(S/R, F) \end{aligned}$$

Remark. If R is local and S is a finitely generated projective R-module then $H^2(S/R, U) \cong B(S/R)$ by [16], where $U(T)$ is the group of units for an

R-algebra T and $B(S/R)$ is the Brauer group [2]. Then the above spectral sequence computes the kernel and cokernel of the map

$$B(S/R) \to \ker [H^2(G, U(S)) \to H^2(G, U(S \otimes_R S))].$$

3. The Brauer group and crossed products. Let R be a local ring and S a commutative R-algebra which is a finitely generated R-module. Let Λ be an R-central separable algebra split by S [2], i.e., $\Lambda \otimes_R S \cong \mathrm{End}_S(E)$ for some faithfully S-projective module E. Using techniques similar to those of [16], we define a map

$$\theta : B(S/R) \to H^2(G, U(S))$$

as follows: for each $\sigma \in G$, E^σ is a $(\Lambda \otimes S)$-module defined by $(\lambda \otimes s)\cdot e = (\lambda \otimes s^\sigma)e$ for $e \in E^\sigma$, $s \in S$, $\lambda \in \Lambda$. It is clear that E^σ is faithfully S-projective, so E^σ is faithfully $(\Lambda \otimes S)$-projective by separability. Using the appendix of [3], we have $E \cong E^\sigma$ as $(\Lambda \otimes S)$-modules, since every projective module over the semi-local ring S is free. Let

$$g_\sigma : E \to E^\sigma, \qquad g_\sigma \in \mathrm{Aut}_\Lambda(E),$$

be the given isomorphism. Then

$$g_\sigma g_\tau g_{\sigma\tau}^{-1} = f(\sigma, \tau) \in \mathrm{Aut}_{\Lambda\otimes S}(E) \cong U(S)$$

gives a 2-cocycle, leading to an element of $H^2(G, U(S))$ represented by f. It is clear that this map is well defined. Actually, if we identify $B(S/R)$ with the Amitsur cohomology group $H^2(S/R, U)$ using [16], then θ is exactly the map in the exact sequence of Section 2.

We wish to identify the subgroup of $H^2(G, U(S))$ which is the image of $B(S/R)$ under θ:

PROPOSITION 3.1. *Given a 2-cocycle f, $\mathrm{cl}\,(f) \in I_m(\theta) \Leftrightarrow$ there is a Λ_f-module E which is faithfully S-projective such that $\mathrm{End}_{\Lambda_f}(E)$ is central R-separable.*

Proof. If $\mathrm{End}_{\Lambda_f}(E)$ is central R-separable, then

$$\mathrm{End}_{\Lambda_f}(E) \otimes_R S \cong \mathrm{End}_S(E).$$

Applying the construction of θ, we clearly retrieve the cocycle f. On the other hand, if $\Lambda \otimes_R S \cong \mathrm{End}_S(E)$ represents the splitting of an R-central separable algebra Λ with $\theta(\mathrm{cl}\,(\Lambda)) = \mathrm{cl}\,(f)$, then it is clear from the definition of θ that E is a Λ_f-module, and $\Lambda = \mathrm{End}_{\Lambda_f}(E)$.

In general, it is an open question to characterize those f which have this property. We proceed to examine a special case:

COROLLARY 3.2. *Let R, S be integrally closed noetherian domains and suppose S is an f.g. projective R-module. Then for $f \in H^2(G, U(S))$, $f \in \theta(B(S/R)) \Leftrightarrow \Lambda_f$ is contained in an R-separable order Γ which is S-projective.*

Proof. In case such Γ exists, then $\mathrm{End}_{\Lambda_f}(\Gamma) \cong \Gamma$, and Γ is split by S, since $\Gamma \otimes_R S \subseteq \mathrm{End}_S(\Gamma)$ by [2.4.1 and 7.1]. Moreover, $f = \theta(\Gamma)$.

On the other hand, let $f \in \theta(B(S/R))$ and E as in 3.1, $\Lambda = \Lambda_f$. Then we have a canonical R-algebra map

$$\Lambda \otimes_R \mathrm{End}_\Lambda(E) \to \mathrm{End}_R(E).$$

Since S is R-projective, then E is R-projective, and $\mathrm{End}_R(E)$ is R-central separable. By [2, Theorem 3.3], the commutant Γ of $\mathrm{End}_\Lambda(E)$ in $\mathrm{End}_R(E)$ is R-separable. Clearly, $\Lambda \subset \Gamma$. By [2, Theorem 2.1], R is a direct summand of $\mathrm{End}_\Lambda(E)$ as R-modules, hence Γ is a direct summand of $\mathrm{End}_R(E)$ as Γ-modules, thus as S-modules. Then Γ is S-projective since $\mathrm{Hom}_R(E, E)$ is S-projective by the appendix of [3].

Remark. The proof and construction of Γ, can be gotten from Amitsur cohomology using the techniques of [16].

THEOREM 3.2. *Let $R \subset S$ be a tamely ramified extension of a discrete rank one valuation ring R in a Galois extension $K \subset L$ of the quotient fields, where S is integrally closed and G is the Galois group. Let $f \in H^2(G, U(S))$. Then Λ_f is contained in a separable order over R if and only if Λ_f has exactly e maximal two-sided ideals, where $e = e(S/R)$ is the ramification index.*

Proof. By Theorem II.1.3, Λ_f is an hereditary order over R. Λ_f has a finite number of maximal two-sided ideals by standard methods.

As in [6, Chapter 5], we define the *different* of the hereditary order Λ to be the inverse of the invertible ideal

$$\mathfrak{D}_\Lambda^{-1} = \mathrm{Hom}_R(\Lambda, R) = \Lambda^* = \{x \in \Lambda \otimes_R K;\ \mathrm{trd}\,(x\Lambda) \subset R\}$$

where $\mathrm{trd} : \Lambda \otimes_R K \to K$ is the reduced trace [5]. If N is the Jacobson radical of Λ, we can write $\mathfrak{D}_\Lambda = N^{n-1}$ for some integer $n \geq 1$. If Γ is a maximal order containing Λ, and $\mathfrak{D}_\Gamma = r(\Gamma)^{m-1}$, then by Theorem 5.6 of [6], $mr = n$, where r is the number of maximal two-sided ideals of Λ. Furthermore, we apply [10, Satz 3, p. 84], and the fact that Λ is R-separable $\Leftrightarrow$ its completion is separable over the completion of R, to conclude that Γ is R-separable $\Leftrightarrow m = 1$. Hence, Γ is R-separable $\Leftrightarrow n = r$.

The reduced trace in a crossed product Σ_f defined in L/K can be computed from right multiplication by Σ_f on itself as an L-free module, using the L-isomorphism

$$L \otimes_K \Sigma_f^0 \to \mathrm{End}_L^l\,(\Sigma_f).$$

It follows that $\mathrm{trd}\,(au_\tau) = 0$ if $\tau \neq 1$, and $\mathrm{trd}\,(au_1) = \sum_\sigma a^\sigma = \mathrm{tr}_{S/R}(a)$. Therefore,

$$\mathrm{trd}\,(xSu_\tau^{-1}) = \mathrm{tr}_{S/R}(a_\tau S) \quad \text{for} \quad x = \textstyle\sum_\tau (a_\tau u_\tau) \in \Sigma_f\,.$$

It follows that $\mathrm{trd}\,(x\Lambda_f) \subset R \Leftrightarrow \mathrm{tr}_{S/R}(a_\tau S) \subset R$ for all τ. Hence, $\mathfrak{D}_{\Lambda_f} = \mathfrak{D}_{S/R} \cdot \Lambda_f$, where $\mathfrak{D}_{S/R}$ is the classical different in S/R from the trace map [1].

Finally, let $\mathfrak{m}$ be the radical of S, and write $\mathfrak{D}_{S/R} = \mathfrak{m}^{e-1}$ for some integer e. It is well known that $e = e(S/R)$ is the ramification index. Then $\mathfrak{D}_{\Lambda_f} = \mathfrak{m}^{e-1}\Lambda_f = (\mathfrak{m}\Lambda_f)^{e-1} = r(\Lambda_f)^{e-1}$ by II.1.4. The conclusion of the theorem now follows.

Remark. Using the results of [18], those f such that Λ_f has exactly e maximal two-sided ideals are exactly those f such that $f \in H^2(D_M, U(\overline{S_M}))$ is in the image of the inflation map

$$H^2(D_M/T_M, U(\overline{S_M})) \to H^2(D_M, U(\overline{S_M})),$$

for a maximal ideal M of S, D_M and T_M being respectively the decomposition and inertia subgroups of M in Λ. This means that $B(S/R)$ and $B(\bar{S}/\bar{R})$ have the same image in $H^2(D_M, U(\overline{S_M}))$.

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