CYCLIC HOMOTOPIES[1]

BY
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1. Let X be a topological space with base-point $*$. We say that a homotopy $h_t : X \to X$ is *cyclic* if $h_0 = h_1 = 1$, the identity map of X, and the loop ω, given by $\omega(t) = h_t(*)$, is called the *trace* of h_t [5]. The elements of the fundamental group of X which may be represented by traces of cyclic homotopies form a subgroup $G(X)$ of $\pi_1(X)$ and, if X is a CW-complex, the property of a loop ω to be the trace of a cyclic homotopy depends only on the element in $\pi_1(X)$ represented by ω [5; Th. I.2 and Th. I.1]. Let $P(X)$ denote the subgroup of $\pi_1(X)$ consisting of all elements which operate trivially on every homotopy group $\pi_n(X)$, $n \geq 1$, and let $Z(G)$ stand for the centre of any group G. It is shown in [5; Th. I.4] that

$$G(X) \subset P(X) \subset Z(\pi_1(X)),$$

and it is asked whether a space X with $G(X) \neq P(X)$ exists [5; §4]; the question is motivated by the fact that

$$G(P^{2n+1}) = \pi_1(P^{2n+1}) \quad \text{and} \quad 0 = P(P^{2n})$$

if P^n denotes the real projective n-space [5; Th. II.5 and Cor. I.6]. Now, for any elements $\gamma \in \pi_1(X)$ and $\alpha \in \pi_n(X)$ with $n \geq 1$, one has $\gamma \cdot \alpha = [\alpha, \gamma] + \alpha$, where the dot denotes the operation of $\pi_1(X)$ on $\pi_n(X)$ and the bracket stands for the classical Whitehead product [7; p. 139]; also, it is well known (see e.g. [1; Th. 4.6]) that all Whitehead products vanish in a space whose loop space is homotopy commutative. Therefore, $P(X) = \pi_1(X)$ if X has such a loop space, and the affirmative answer to the above question is given by

THEOREM 1.1. *There exists a CW-complex X whose loop space is homotopy commutative and for which $\pi_1(X) = Z_2$ and $G(X) = 0$.*

Proof. Let B be an Eilenberg-MacLane CW-complex of type $(Z_2, 3)$ and let $v \in H^3(B, Z_2)$ be its fundamental class. Introduce the diagram

$$\text{(1)} \qquad \begin{array}{ccccc} F \times S^1 & \xrightarrow{i \times 1} & E \times S^1 & \xrightarrow{p \times 0} & B \times * \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \beta} \\ F & \xrightarrow{\quad i \quad} & E & \xrightarrow{\quad p \quad} & B \end{array}$$

where E has the homotopy type of an Eilenberg-MacLane CW-complex of type $(Z_2, 1)$ with fundamental class $u \in H^1(E, Z_2)$, and p is a fibre map

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with homotopy class uniquely determined by requiring that $p^*(v) = u^3$, the cup-cube in mod 2 cohomology; $F = p^{-1}(*)$ is the fibre of p, i is the inclusion map, S^1 is the circle, and the top row is the Cartesian product of the bottom row with the fibration $S^1 \to S^1 \to *$. Thus, $\pi_n(F) = 0$ for $n \geqq 3$, $\pi_2(F) = Z_2$, and $\pi_1(F) = Z_2$ with generator represented by some loop $\omega : S^1 \to F$. Suppose $G(F) = \pi_1(F)$. As a consequence, since F has the homotopy type of a CW-complex, there results [5; Remark I, p. 842] a map φ whose restriction to the axes of the Cartesian product is homotopic to the map $F \vee S^1 \to F$ defined by the identity map of F and ω. Let

$$\varepsilon = \mu \circ (1 \times i \circ \omega),$$

where $\mu : E \times E \to E$ is the multiplication in the H-space E. Then, $i \circ \varphi$ and $\varepsilon \circ (i \times 1)$ are homotopic when restricted to $F \vee S^1$ and, since the inclusion map $F \vee S^1 \to F \times S^1$ is 1-connected whereas $\pi_n(E) = 0$ for $n \geq 2$, the left hand square, itself, in (1) homotopy commutes (rel. base-point since E is an H-space). Let $j : E \times S^1 \to Q$ denote the inclusion map into the space obtained by erecting a cone over the subset $F \times S^1$ of $E \times S^1$, and let $r : Q \to B \times *$ extend $p \times 0$ by mapping the cone to the base-point; also, let $k : Q \to B$ be the map induced by φ, ε, and any based homotopy connecting $i \circ \varphi$ with $\varepsilon \circ (i \times 1)$ so that $k \circ j \simeq p \circ \varepsilon$. By the Serre theorem (see e.g. [4; 2.1]), r is 4-connected and, since $\pi_n(B) = 0$ for $n \geqq 4$, a standard obstruction argument yields a map β such that $\beta \circ r \simeq k$. There are only two homotopy classes of maps $B \to B$ so that, since $u^3 \neq 0$, $\beta \simeq 1$ and $p \times 0 \simeq p \circ \varepsilon$. Therefore, by the definition of ε,

$$u^3 \otimes 1 = (p \times 0)^*(v) = \varepsilon^* \circ p^*(v) = u^2 \otimes s + u^3 \otimes 1,$$

where s generates $H^1(S^1, Z_2)$. Since $u^2 \otimes s \neq 0$, we have reached a contradiction which reveals that $G(F) = 0$. Finally, since ΩE has the homotopy type of the 0-sphere, $\Sigma\Omega E \times \Sigma\Omega E$ has the homotopy type of a 2-dimensional torus and diagram (3) below homotopy commutes with $\phi = 0$, the constant map. The homotopy commutativity of the loop space of F follows then by the first part of 2.1 below, and the required CW-complex X is provided by the singular polytope of F.

Remark 1.2. The simply connected covering space C of X is an Eilenberg-MacLane CW-complex of type $(Z_2, 2)$. Hence, there are only two homotopy classes of maps $C \to C$ and each homeomorphism of C onto itself is homotopic to the identity map. Therefore, the subgroup $\mathcal{H}(X)$ of those covering transformations which are homotopic to the identity map of C satisfies $G(X) \neq \mathcal{H}(X) = \pi_1(X)$. This answers a question raised in [5; §3] where examples with $G = \mathcal{H}$ are given.

Remark 1.3. A stronger property than that of having a homotopy commutative loop space is that of being an H-space; then, $G(X) = \pi_1(X)$ according to [5; Th. I.8].

2. Let Ω and Σ denote the loop and reduced suspension functors, respectively, and let $r : \Sigma\Omega X \to X$ be given by $r\langle t, \omega\rangle = \omega(t)$. Recall [9] that a CW-complex X has a homotopy commutative loop space if and only if there is a map

$$M : \Sigma\Omega X \times \Sigma\Omega X \to X \quad \text{with} \quad M \circ J \simeq \nabla \circ (r \vee r), \tag{2}$$

where $J : \Sigma\Omega X \vee \Sigma\Omega X \to \Sigma\Omega X \times \Sigma\Omega X$ is the inclusion of the axes in the Cartesian product and $\nabla : X \vee X \to X$ is the folding map given by $\nabla(x, *) = \nabla(*, x) = x$. Let X be a CW-complex with a single nontrivial Abelian homotopy group in some dimension $n \geqq 1$. Then X is an H-space with multiplication $\mu : X \times X \to X$ uniquely determined up to homotopy, and a standard obstruction argument reveals that $M = \mu \circ (r \times r)$ yields the unique homotopy class of maps fulfilling (2). Next, let

$$\mathfrak{F} : F \xrightarrow{i} E \xrightarrow{p} B$$

be a fibration of spaces having the homotopy type of CW-complexes, and consider the diagram

$$\begin{array}{ccc} \Sigma\Omega E \times \Sigma\Omega E & \xrightarrow{\Sigma\Omega p \times \Sigma\Omega p} & \Sigma\Omega B \times \Sigma\Omega B \\ \downarrow M_E & & \downarrow \phi \\ E & \xrightarrow{p} & B \end{array} \tag{3}$$

where M_E satisfies (2) so that E has a homotopy commutative loop space.

THEOREM 2.1. *If there is a map ϕ yielding homotopy commutativity in* (3), *then ΩF is homotopy commutative. Conversely, if ΩF is homotopy commutative, and if both E and B have a single non-trivial homotopy group in dimensions n and $m + 1$, respectively, with $m > n > 1$, then* (3) *homotopy commutes with $\phi = M_B$.*

We omit the proof since it is, essentially, dual to that given in [2; 3.3 and 3.4] and follows the general pattern described in [8]. The result is similar to the known fact [3] that a two-stage Postnikov system is an H-space if and only if its Eilenberg-MacLane invariant is primitive. In fact, let Y be a CW-complex with only two non-trivial homotopy groups in dimensions n and m with $m > n > 1$, and let $\mu, L, R : X \times X \to X$ denote the multiplication and the two projections in the CW-complex X of type $(\pi_n(Y), n)$ which results by killing off $\pi_m(Y)$. Then

COROLLARY 2.2. *ΩY is homotopy commutative if and only if*

$$(r \times r)^* \circ (\mu^* - L^* - R^*)(k) = 0,$$

where k is the Eilenberg-MacLane invariant of Y.

As is well known [10], r^* followed by a natural identification $H^{m+1}(\Sigma\Omega X) =$

$H^m(\Omega X)$ coincides with the cohomology suspension $H^{m+1}(X) \to H^m(\Omega X)$ for any coefficient group.

Remark 2.3. As before, let ΩE in (3) be homotopy commutative. Then it is shown in [6] that ΩF is homotopy commutative if p is homotopic to a composite

$$E \xrightarrow{f} Y_1 \times \cdots \times Y_n \xrightarrow{Q} Y_1 \# \cdots \# Y_n \xrightarrow{g} B,$$

where $n \geq 3$ and Q is the identification map which collapses to a point the subset T of $Y_1 \times \cdots \times Y_n$ consisting of all points that have at least one coordinate at the base-point. This is an immediate consequence of 2.1: $\Sigma\Omega E \times \Sigma\Omega E$ has (reduced) Lusternik-Schnirelmann category ≤ 2 so that, by [10], any map $\Sigma\Omega E \times \Sigma\Omega E \to Y_1 \times \cdots \times Y_n$ may be compressed into T, and (3) homotopy commutes with $\phi = 0$. In turn, the result in [6, Remark 2.16(c)] immediately yields the homotopy commutativity of ΩF in 1.1.

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