THE FREE PRODUCT OF ALGEBRAS[1]

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Introduction

Let A and B be differential graded augmented algebras over a commutative ring K. Their free product $A * B$ is always defined; $A * B$ is a differential graded augmented K-algebra which together with canonical injections

$$A \xrightarrow{\iota_A} A * B \xleftarrow{\iota_B} B$$

forms a universal diagram in this category. In connection with certain topological questions, Berstein [1] first studied the free product of algebras and its homology; he showed for example that the homology of the loop space of $X_1 \vee X_2$ (where X_i are spaces with "nice" base point) is the free product $H(\Omega X_1) * H(\Omega X_2)$. We shall study the free product and its homology from a somewhat different viewpoint.

The first section is devoted to the definition and basic properties of the free product, including a consideration of Hopf algebras. Some of this material appears in Berstein [1], but is stated here for convenience since our notation is different and our definitions are somewhat more general (Berstein considers only positively graded connected K-algebras).

Palermo [10] and the author [5], [6] have studied the relationship between the vaious homologies $H(A)$, $H(B)$, and $H(A \otimes B)$. The chief purpose of this paper is to extend these investigations to $H(A * B)$. In particular since $A * B$ is defined in terms of the tensor product it seems natural to ask whether or not $H(A \otimes B)$ completely determines $H(A * B)$. Examples in Section 2 show that the answer is negative; furthermore neither does $H(A * B)$ determine $H(A \otimes B)$. For $K = Z$ and A, B torsion-free, it is known that the algebras $H(A)$ and $H(B)$ do not determine the algebra $H(A \otimes B)$; but $H(A \otimes B)$ is completely determined by the homology spectra of A and B (cf. Palermo [10], and [5]). The analogues of these facts are presented in Section 3: $H(A)$ and $H(B)$ are not sufficient to determine $H(A * B)$ (Example 3.4), but the algebra $H(A * B)$ is completely determined by the homology spectra of A and B (Theorem 3.3).

In the final sections, the work of Dold and Puppe [4] is used to develop a theory of derived functors for the nonadditive functor $A * B$. Not surprisingly these derived functors turn out to be closely related to the ordinary derived functors of the multiple tensor product (c.f [6]). Using these results we are able to state a "Künneth theorem" which relates the (additive) struc-

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ture of $H(A)$, $H(B)$ and $H(A * B)$ with the derived functors of $A * B$ (Theorem 5.2).

1. Definitions and basic properties

Let K be a (fixed) commutative ring with identity 1_K; $\otimes$ means $\otimes_K$ throughout. We shall use the terminology and definitions of chapter VI of MacLane [8], with one exception: we call an object *graded* if it is *Z-graded* in the sense of MacLane, "Differential graded augmented algebra" is abbreviated as DGA-algebra. Homomorphisms of DGA-algebras are called DGA-homomorphisms or DGA-maps. All algebras are assumed to be augmented, unless specifically stated otherwise. Direct sums are denoted by $+$ and/or $\sum$.

Let A be an algebra over K, with identity $I = I_A : K \to A$ and augmentation $\varepsilon = \varepsilon_A : A \to K$. Let $\bar{A} = \ker \varepsilon$; then $A \cong K + \bar{A}$. This is an isomorphism of DG-K-modules if A is a DGA-algebra.

If C and D are (differential graded) K-modules, for each $n \geq 1$, let $T_n(C, D)$ be the (differential graded) K-module given by

$$T_n(C, D) = C \otimes D \otimes C \otimes D \cdots \quad (n \text{ factors}).$$

DEFINITION 1.1. Let A and B be (augmented) algebras over K. The free product of A and B is the algebra $A * B$ given by

$$A * B = K + \textstyle\sum_{n \geq 1} T_n(\bar{A}, \bar{B}) + T_n(\bar{B}, \bar{A}).$$

The augmentation map is the projection onto the summand K; the identity map I is the injection of K into the sum $A * B$. The product is given as follows. Let $k, k' \in K$,

$$u = u_1 \otimes u_2 \otimes \cdots \otimes u_n \in T_n(\bar{A}, \bar{B}) \quad \text{or} \quad T_n(\bar{B}, \bar{A}),$$

and

$$v = v_1 \otimes v_2 \otimes \cdots \otimes v_m \in T_m(\bar{A}, \bar{B}) \quad \text{or} \quad T_m(\bar{B}, \bar{A});$$

then

$$k \cdot k' \text{ is given by multiplication in } K;$$

$$k \cdot u = (ku_1) \otimes u_2 \otimes \cdots \otimes u_n;$$

$$u \cdot k = u_1 \otimes u_2 \otimes \cdots \otimes (u_n k);$$

$$u \cdot v = u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_m,$$

$$\text{if } u_n \in \bar{B} \text{ and } v_1 \in \bar{A}, \text{ or } u_n \in \bar{A} \text{ and } v_1 \in \bar{B};$$

$$u \cdot v = u_1 \otimes u_2 \otimes \cdots \otimes u_{n-1} \otimes (u_n v_1) \otimes v_2 \otimes \cdots \otimes v_m,$$

$$\text{if } u_n \text{ and } v_1 \text{ are both in } \bar{A} \text{ or both in } \bar{B}.$$

If A and B are DGA-algebras, then $A * B$ as the direct sum of differential graded modules has an obvious grading and differential, and is a DGA-algebra.

It is readily verified that $A * B$ is in fact a (DGA-) algebra with identity $1_K \in K$. Henceforth we shall deal for the most part with DGA-algebras.

We have

$$A * B = K + \bar{A} + \bar{B} + \textstyle\sum_{n\geq 2} T_n(\bar{A}, \bar{B}) + T_n(\bar{B}, \bar{A}).$$

Then the isomorphism $A \cong K + \bar{A}$ induces a map $\iota_A : A \to A * B$, which is readily seen to be a DGA-map; $\iota_B : B \to A * B$ is defined similarly.

THEOREM 1.2. *If A, B, C are DGA-algebras and $f : A \to C$, $g : B \to C$ are DGA-homomorphisms, then there is a unique DGA-homomorphism $\varphi : A * B \to C$ such that the diagram*

$$\begin{array}{ccccc} A & \xrightarrow{\iota_A} & A*B & \xleftarrow{\iota_B} & B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle\varphi} & \swarrow{\scriptstyle g} & \\ & & C & & \end{array}$$

is commutative; i.e.,

$$A \xrightarrow{\iota_A} A * B \xleftarrow{\iota_B} B$$

is a universal diagram with ends A, B in the category of DGA-algebras and DGA-maps.

Since f and g are DGA-maps (hence $\varepsilon_C f = \varepsilon_A$, $\varepsilon_C g = \varepsilon_B$)

$$\bar{f} = f \mid \bar{A} : \bar{A} \to \bar{C} \quad \text{and} \quad \bar{g} = g \mid \bar{B} : \bar{B} \to \bar{C}.$$

The theorem now follows immediately by defining $\varphi \mid K = I_C$ and $\varphi \mid T_n(\bar{A}, \bar{B})$ as the composition

$$T_n(\bar{A}, \bar{B}) \xrightarrow{T_n(\bar{f}, \bar{g})} T_n(\bar{C}, \bar{C}) \xrightarrow{\mu} C,$$

where μ is multiplication in C. We denote φ by $\langle f, g\rangle$.

$A * B$ can be considered as a covariant functor of two variables as follows: If $f : A \to A'$ and $g : B \to B'$, define

$$f * g : A * B \to A' * B' \quad \text{by} \quad f * g = \langle \iota_{A'} f, \iota_{B'} g\rangle.$$

Note however that it is not an additive functor.

The above definition of free product is somewhat more general than that given in [1] where consideration was restricted to positively graded connected DGA-algebras. In fact this seems to be as general as possible since if A and B are not augmented there may not be a universal diagram with ends A, B in the category of DG-algebras (of course the direct sum $A + B$ is universal in the category of DG-K-modules). For a trivial example of this let $K = Z$ and let $A = Z_2$ in dimension zero and 0 elsewhere; similarly let $B = Z_3$.

Since DGA-maps preserve the identity element, if there were a diagram

$$A \xrightarrow{\iota_A} D \xleftarrow{\iota_B} B,$$

the identity element of D would have (additive) order dividing 2 and 3. Hence $D = 0$ and the diagram would not be universal.

Next we consider the situation when A and B are Hopf algebras (as defined in VI.9 of MacLane [8]).

PROPOSITION 1.3. *If A and B are (differential graded) Hopf algebras, then $A * B$ is a (differential graded) Hopf algebra.*

Proof. Let $\Psi_A : A \to A \otimes A$ and $\Psi_B : B \to B \otimes B$ be the coproduct maps for A and B respectively. By the definition of a Hopf algebra Ψ_A and Ψ_B are DGA-maps. Therefore, by Theorem 1.2, the DGA-maps

$$(\iota_A \otimes \iota_A)\Psi_A : A \to (A * B) \otimes (A * B)$$

and

$$(\iota_B \otimes \iota_B)\Psi_B : B \to (A * B) \otimes (A * B)$$

induce a DGA-map

$$\Psi : A * B \to (A * B) \otimes (A * B).$$

PROPOSITION 1.4. *In the category of DG-Hopf algebras and DG-Hopf algebra maps, the diagram*

$$A \xrightarrow{\iota_A} A * B \xleftarrow{\iota_B} B$$

is universal with ends A, B.

Proof. First note that by the definition of Ψ, ι_A and ι_B are Hopf algebra maps. We need only show that the map $\varphi : A * B \to C$ defined in the proof of Theorem 1.2 is a map of DG-coalgebras when f and g are DG-Hopf algebra maps; this is a straightforward verification.

2. $H(A * B)$ and $H(A \otimes B)$

In the next two sections we shall examine the homology of $A * B$ (the so called zero-stage homology; cf. MacLane [8]). If A and B are DGA-algebras, then so are $H(A * B)$, $H(A \otimes B)$, $H(A)$ and $H(B)$ (all with trivial differential). We shall study some of the relationships between them. First, one might ask for DGA-algebras E and F if $H(E * F)$ completely determines $H(E \otimes F)$, or vice versa. The answer to both questions is negative, as shown by the following examples.

Example 2.2. Let $K = Z$ and let A, B, C be differential graded algebras which are zero in all dimensions except 0, 1, 2 and are given there by

$$\begin{aligned}
A&: \; Z(a_0) \leftarrow Z(a_1) \leftarrow Z(a_2); \quad \partial a_1 = 0, \quad \partial a_2 = 4a_1;\\
B&: \; A(b_0) \leftarrow Z(b_1) \leftarrow Z(b_2); \quad \partial b_1 = 0, \quad \partial b_2 = 2b_1;\\
C&: \; Z(c_0) \leftarrow Z(c_1) \leftarrow Z(c_2); \quad \partial c_1 = 0, \quad \partial c_2 = 3c_1.
\end{aligned}$$

In each case the augmentation is the identity map $Z(x_0) = Z$ $(x = a, b, c)$; the multiplicative structure is given by

$$x_0 x_j = x_j x_0 = x_j \quad \text{for all } j;$$

$$x_i x_j = 0 \quad \text{for } i, j > 0; \ (x = a, b, c).$$

Now let $E = A * B$, $F = C$, $E' = A$, $F' = B * C$. By the associativity of the free product $E * F \cong E' * F'$, hence $H(E * F) \cong H(E' * F')$. However direct computation shows that $H_2(E \otimes F) \cong Z_2 + Z_2$, while $H_2(E' \otimes F') \cong Z_2$. Thus $H(E * F)$ does not determine $H(E \otimes F)$.

Example 2.3. Let $K = Z$ and A, B, C be as in the previous example. Let $E = A \otimes B$, $F = C$, $E' = A$, $F' = B \otimes C$. Then $E \otimes F \cong E' \otimes F'$ and hence $H(E \otimes F) \cong H(E' \otimes F')$. By using the fact that

$$\overline{A \otimes B} \cong \bar{A} \otimes \bar{B} + \bar{A} + \bar{B},$$

and some properties of the homology of tensor products of elementary complexes (cf. Lemma 3 of the Appendix of [5] and Lemma 3.2 of [6]) a straightforward calculation shows

$$H_2(E * F) \cong H_2(\bar{A} \otimes \bar{B}) \cong Z_2;$$

but

$$H_2(E' * F') \cong H_2(\bar{A} \otimes \bar{B}) + H_2(\bar{B} \otimes \bar{A}) \cong Z_2 + Z_2.$$

Thus $H(E * F) \not\cong H(E' * F')$ and hence $H(E \otimes F)$ does not determine $H(E * F)$.

3. The multiplicative structure of $H(A * B)$

The next question to be considered is whether or not $H(A)$ and $H(B)$ completely determine $H(A * B)$. The discussion will be restricted to the case $K = Z$, with A and B torsion-free. Analogous questions were considered in [5] with regard to $H(A)$, $H(B)$ and $H(A \otimes B)$ and, not surprisingly, many of these earlier results carry over to the present situation.

Recall that the *homology spectrum* of a torsion-free DGA-algebra A over Z consists of the rings $H(A, m) = H(A \otimes Z_m)$ (for all $m \geq 0$, where $Z_0 = Z$), together with the coefficient maps induced by the projections $Z_{mk} \to Z_m$ $(mk \geq 0)$ and injections $Z_m \to Z_{mk}$ $(mk > 0)$, and the Bockstein map $\mu_0^m : H(A, m) \to H(A) = H(A, 0)$ induced by the exact sequence

$$0 \to Z \xrightarrow{m} Z \to Z_m \to 0.$$

The homology spectrum is denoted by $\{H(A, m)\}$; for more details consult [5] and [6].

DEFINITION 3.1. Let A and B be torsion-free DGA-algebras over Z. The *free product of the homology spectra of A and B*, denoted

$\{H(A, m)\} * \{H(B, m)\}$, is the graded abelian group

$$Z + H(\bar{A}) + H(\bar{B}) + \sum_{n\geq 2} [\tilde{T}_n(\{H(\bar{A}, m)\}, \{H(\bar{B}, m)\}) + \tilde{T}_n(\{H(\bar{B}, m)\}, \{H(\bar{A}, m)\})]$$

where $\tilde{T}_n(\{H(\bar{A}, m)\}, \{H(\bar{B}, n)\})$ denotes the n-fold tensor product of homology spectra

$$\{H(\bar{A}, m)\} \otimes \{H(\bar{B}, m)\} \otimes \{H(\bar{A}, m)\} \otimes \{H(\bar{B}, m)\} \otimes \cdots,$$

as defined on page 261 of [6].

THEOREM 3.2. *If A and B are torsion-free augmented DGA-algebras over Z then there is a natural isomorphism of graded groups:*

$$\{H(A, m)\} * \{H(B, m)\} \cong H(A * B).$$

Proof. Since H is an additive functor, the definition of $A * B$ implies that

$$H(A * B) \cong Z + H(\bar{A}) + H(\bar{B}) + \sum_{n\geq 2} H(T_n(\bar{A}, \bar{B})) + H(T_n(\bar{B}, \bar{A})).$$

But Theorem 3.1 of [6] states in slightly different notation that for each $n \geq 2$, there is a natural isomorphism

$$H(T_n(\bar{A}, \bar{B})) \cong \tilde{T}_n(\{H(\bar{A}, m)], \{H(\bar{B}, m)\}). \tag{1}$$

The theorem now follows immediately.

The next step is to define a product in $\{H(A, m)\} * \{H(B, m)\}$ so that it becomes not just a group but a graded ring in such a way that the isomorphism of Theorem 3.2 becomes a ring isomorphism. The construction of such a product is very similar *mutatis mutandis*, to the construction of the product in the tensor product of homology spectra as given in Section 3 of [5]; consequently the details are omitted here. We can summarize these facts as follows.

THEOREM 3.3. *If A and B are torsion-free DGA-algebras over Z, then the homology spectra of A and B completely determine $H(A * B)$; in particular, there is a natural isomorphism of graded rings:*

$$\{H(A, m)\} * \{H(B, m)\} \cong H(A * B).$$

Palermo [10] has given an example to show that for $K = Z$, the ring $H(A \otimes B)$ need not be completely determined by the rings $H(A)$ and $H(B)$. The same example serves to show that $H(A)$ and $H(B)$ alone do not determine $H(A * B)$.

Example 3.4. Let four identical complexes of abelian groups, A^1, A^2, A^3, A^4 be given as follows ($i = 1, 2, 3, 4$):

$$A^i_0 = Z(e_i); \qquad A^i_{-1} = Z(a_i) + Z(c_i); \qquad A^i_{-2} = Z(b_i);$$

$$A^i_j = 0 \quad \text{for } j \neq 0, -1, -2; \quad \partial e_i = 0; \quad \partial a_i = 2b_i; \quad \partial c_i = 0; \quad \partial b_i = 0.$$

In each A^i the augmentation map is the identity map $Z(e_i) \to Z$. The multiplicative structure is given as follows:

For $i = 1, 2$:

A^i	e_i	c_i	a_i	b_i
e_i	e_i	c_i	a_i	b_i
c_i	c_i	0	b_i	0
a_i	a_i	b_i	b_i	0
b_i	b_i	0	0	0

for $i = 3, 4$:

A^i	e_i	c_i	a_i	b_i
e_i	e_i	c_i	a_i	b_i
c_i	c_i	0	0	0
a_i	a_i	0	0	0
b_i	b_i	0	0	0

It is readily verified that each A^i is in fact a DGA-algebra and that there are algebra isomorphisms $H(A^1) \cong H(A^2) \cong H(A^3) \cong H(A^4)$. The situation is different when homology is taken mod 2. There are algebra isomorphisms $H(A^1, 2) \cong H(A^2, 2)$ and $H(A^3, 2) \cong H(A^4, 2)$, but there is no algebra isomorphism of $H(A^1, 2)$ and $H(A^3, 2)$, although all the additive structures are the same; (cf. Palermo [10] in slightly different terminology). We claim that there is no algebra isomorphism of $H(A^1 * A^2)$ and $H(A^3 * A^4)$. Thus $H(A)$ and $H(B)$ do not determine $H(A * B)$.

In low dimensions the additive structure of $H(A^i * A^j)$ is given as follows (where $(i, j) = (1, 2)$ or $(3, 4)$ and η denotes homology class).

$$H_0(A^i * A^j) = Z;$$

$$H_{-1}(A^i * A^j) = Z[\eta(c_i)] + Z[\eta(c_j)];$$

$$H_{-2}(A^i * A^j) = Z_2[\eta(b_i)] + Z_2[\eta(b_j)] + Z[\eta(c_i \otimes c_j)] + Z[\eta(c_j \otimes c_i)];$$

$$\begin{aligned} H_{-3}(A^i * A^j) = {} & Z_2[\eta(b_i \otimes c_j)] + Z_2[\eta(c_i \otimes b_j)] + Z_2[\eta(b_i \otimes a_j - a_i \otimes b_j)] \\ & + Z_2[\eta(b_j \otimes c_i)] + Z_2[\eta(c_j \otimes b_i)] + Z_2[\eta(b_j \otimes a_i - a_j \otimes b_i)] \\ & + Z[\eta(c_i \otimes c_j \otimes c_i)] + Z[\eta(c_j \otimes c_i \otimes c_j)]; \end{aligned}$$

$$H_{-4}(A^i * A^j) = Z_2[\eta(b_i \otimes b_j)] + Z_2[\eta(b_j \otimes b_i)] + \text{other terms}.$$

Suppose there were an isomorphism of graded algebras

$$f : H(A^1 * A^2) \to H(A^3 * A^4).$$

Then it is easy to see that $f \mid Z$ is the identity. Since f preserves degrees

$$f[\eta(c_1)] = x\eta(c_3) + y \cdot \eta(c_4)$$

for some $x, y \in Z$. Since $c_1^2 = 0$, $f[\eta(c_1)]^2 = 0$; but

$$f[\eta(c_1)]^2 = xy\eta(c_3 \otimes c_4) + xy\eta(c_4 \otimes c_3).$$

This will be zero if and only if $x = 0$ or $y = 0$. It follows that $f[\eta(c_1)] = \pm\eta(c_3)$ or $\pm\eta(c_4)$; by changing indices if necessary we can assume $f[\eta(c_1)] = \pm\eta(c_3)$. Then the same argument shows that $f[\eta(c_2)] = \pm\eta(c_4)$.

A similar argument shows that $f[\eta(b_1)] = \eta(b_3)$ or $\eta(b_4)$. The facts that $f[\eta(c_1)] = \pm\eta(c_3)$ and $c_1 \cdot b_1 = 0$ imply that $f[\eta(b_1)] = \eta(b_3)$, since $\eta(c_3) \cdot \eta(b_4) = \eta(c_3 \otimes b_4) \neq 0$ in $H(A^3 * A^4)$. Likewise $f[\eta(b_2)] = \eta(b_4)$.

Let $u = (b_1 \otimes a_2 - a_1 \otimes b_2)$. In $H(A^1 * A^2)$,

$$\eta(c_1) \cdot \eta(u) = \eta(b_1 \otimes b_2) \neq 0.$$

Hence

$$f[\eta(c_1)] \cdot f[\eta(u)] = \eta(c_3) \cdot f[\eta(u)] = \eta(b_3 \otimes b_4).$$

Since each A^i is torsion-free, the homology product maps (which define the product in $H(A^3 * A^4)$)

$$\alpha : H_{-1}(\bar{A}^3) \otimes H_{-3}(\bar{A}^4 \otimes \bar{A}^3) \rightarrow H_{-4}(\bar{A}^3 \otimes \bar{A}^4 \otimes \bar{A}^3),$$

and

$$\alpha : H_{-1}(\bar{A}^3) \otimes H_{-3}(\bar{A}^4 \otimes \bar{A}^3 \otimes \bar{A}^4) \rightarrow H_{-4}(\bar{A}^3 \otimes \bar{A}^4 \otimes \bar{A}^3 \otimes \bar{A}^4)$$

are both monic (cf. MacLane [9]; this means that in this case the product of nonzero elements is nonzero. Since $f[\eta(u)]$ must have additive order 2, the only possibility, therefore, for $f[\eta(u)]$ is a linear combination of

$$\eta(b_3 \otimes c_4), \quad \eta(c_3 \otimes b_4), \quad \eta(b_3 \otimes a_4 - a_3 \otimes b_4).$$

But the product of $\eta(c_3)$ (on the left) with each of these terms is zero in $H(A^3 * A^4)$. Hence $f[\eta(c_1)] \cdot f[\eta(u)] \neq \eta(b_3 \otimes b_4)$, a contradiction. Therefore there can be no algebra isomorphism between $H(A^1 * A^2)$ and $H(A^3 * A^4)$.

4. Derived functors

In an abelian category (with sufficient proper projectives) the derived functors of an additive functor T are always defined (cf. [8]). Furthermore, Dold and Puppe [4] have defined the derived functors of an arbitrary functor T on an abelian category in such a way that they agree with the usual derived functors if T is additive. Unfortunately, however, the cateogry of augmented algebras (or augmented K-modules) is not abelian and the functor $A * B$ is not additive. On the other hand, the functor $T(\bar{A}, \bar{B}) = A * B$, considered as a functor of the K-modules $\bar{A}$ and $\bar{B}$, is a (nonadditive) functor of two variables on the abelian category of K-modules. Also, it is clear that a DGA-algebra A completely determines the K-module $\bar{A}$. Since some analogue of derived functors may prove useful for $A * B$, it seems reasonable to *define* the derived functors of $A * B$ to be the derived functors of $T(\bar{A}, \bar{B})$ and apply the definitions and results of [4].

The reader should consult [4] or [8] for the definition of a semisimplicial object; $\mathbf{k}X$ will denote the chain complex determined by the semisimplicial (s.s.) K-module X. If X and Y are s.s. objects on an abelian category A with face and degeneracy operators d_i^X, s_i^X, d_i^Y, s_i^Y respectively and F is a covariant functor of two variables from the category A to itself, then $F(X, Y)$ is the s.s. object given by $F_n(X, Y) = F(X_n, Y_n)$, with face and degeneracy operators $d_i = F(d_i^X, d_i^Y)$ and $s_i = F(s_i^X, s_i^Y)$; similarly for functors of more than two variables. If F is the tensor product of K-modules then the Eilenberg Zilber Theorem states that there is a natural chain equivalence of complexes:

$$\mathbf{k}F(X, Y) \leftrightarrow \mathbf{k}X \otimes \mathbf{k}Y. \tag{1}$$

DEFINITION 4.1. Let A be a K-module and $n \geq 0$ an integer. A *projecitve semi-simplicial resolution of* (A, n) is an s.s. module X such that: X_i is projective for all i; $X_i = 0$ for $i < n$; $H_i(\mathbf{k}X) = 0$ for $i > n$; $H_n(\mathbf{k}X) \cong A$.

DEFINITION 4.2. Let A be a graded K-module; for each $m \in Z$, let X^m be a projective s.s. resolution of (A_m, m). The direct sum $X = \sum^m X_m$ is called a *projective s.s. resolution of* A.

This is an extension of definition 4.8 of [4]. The work in [4] is all done in the context of an abelian category; hence arbitrary direct sums may not exist. The technique of taking X^m to be a projective s.s. resolution of (A_m, m) rather than of $(A_m, 0)$ insures that for positively graded objects A, the projective s.s. resolution X of A is well defined, since for each $q \geq 0$ X_q is a finite sum $\sum_{m \leq q} X_q^m$. However, since infinite direct sums do exist in the category of K-modules, the definition can be extended in this case to arbitrarily graded K-modules.

The following facts are proved in [4]. For every K-module A and every $n \geq 0$ there is a projective s.s. resolution of (A, n). Hence every graded K-module has a projective s.s. resolution. If X and Y are projective s.s resolutions of A and B respectively, then every K-module map $f : A \to B$ can be lifted to an s.s. map $\bar{f} : X \to Y$; $\bar{f}$ is unique up to chain homotopy. If F is a (not necessarily additive) covariant functor of two variables on the category of K-modules, then $H(\mathbf{k}F(X, Y))$ is determined up to natural isomorphism and depends only on F, A and B, and not on X or Y. The same facts hold if A and B are graded K-modules.

DEFINITION 4.3. Let $F(A, B)$ be a covariant functor from the category of graded K-modules to itself. The *q-th left derived functor of* F is

$$L_q F(A, B) = H_q(\mathbf{k}F(X, Y)),$$

where X and Y are projective s.s. resolutions of A and B respectively.

The preceding remarks show that $L_q F(A, B)$ is well defined. Note that since negative gradings are allowed for A and B, the derived functors $L_q F(A, B)$ are defined for negative as well as positive q.

Before applying Definition 4.3 to the functor $T(\bar{A}, \bar{B}) = A * B$, we first consider the n-fold tensor product of K-modules, $A^1 \otimes A^2 \otimes \cdots \otimes A^n$. It is a covariant, additive, right exact functor of n variables on the category of K-modules, whose i-th left derived functor is generally denoted by $\mathrm{Mult}_i^{K,n}(A^1, \cdots, A^n)$. For $n = 2$, $\mathrm{Mult}_i^{K,2}$ is just Tor_i^K and for every n,

$$\mathrm{Mult}_0^{K,n}(A^1, \cdots, A^n) = A^1 \otimes A^2 \otimes \cdots \otimes A^n.$$

If K is a hereditary ring, then $\mathrm{Mult}_i^{K,n}(A^1, \cdots, A^n) = 0$ for $i \geq n$.

We shall use the following notation. Let A and B be graded K-modules;

p, n, i integers ($i \geq 0$). Then

$$\mathrm{Mult}_i^{K,n}[A, B]_p = \sum \mathrm{Mult}_i^{K,n}(A_{p_1}, B_{p_2}, A_{p_3}, B_{p_4}, \cdots) \quad (n \text{ factors}),$$

where the sum is taken over all $(p_1, \cdots, p_n)$ such that $\sum_{r=1}^n p_r = p$. We also adopt the conventions that

$$\mathrm{Mult}_i^{K,1}[A, B]_p = A_i \quad \text{for} \quad p = 0; \qquad 0 \quad \text{for} \quad p \neq 0;$$

$$\mathrm{Mult}_i^{K,0}[A, B]_p = K \quad \text{for} \quad i = p = 0; \qquad 0 \quad \text{otherwise}.$$

We can consider $\sum_{i \geq 0} \mathrm{Mult}_i^{K,n}[A, B] = \sum_{i \geq 0} \sum_p \mathrm{Mult}_i^{K,n}[A, B]_p$ as a (bi)graded K-module, with an element of $\mathrm{Mult}_i^{K,n}[A, B]_p$ having bidegree (i, p) and total degree $i + p$. Now let $T_n(A, B)$ be the n-fold tensor product of alternate copies of A and B as above.

THEOREM 4.4. *If A and B are graded K-modules, then there is a natural isomorphism of graded K-modules*:

$$\sum_q L_q T_n(A, B) \cong \sum_{i \geq 0} \mathrm{Mult}_i^{K,n}[A, B];$$

in particular,

$$L_q T_n(A, B) = \sum_{j \geq 0} \mathrm{Mult}_j^{K,n}[A, B]_{q-j}.$$

Proof. We shall use the following notation:

$$\sum_{(p_j)=k} M_{p_1} \otimes M_{p_2} \otimes \cdots \otimes M_{p_n}$$

is the sum over all $(p_1, \cdots, p_n)$ such that $\sum_{j=1}^n p_j = k$. Let X and Y be s.s. projective resolutions of A and B respectively; then, by the Eilenberg-Zilber Theorem and the appropriate definitions,

$$\begin{aligned}
L_q T_n(A, B) &= H_q(\mathbf{k}T_n(X, Y)) \\
&\cong H_q(T_n(\mathbf{k}X, \mathbf{k}Y)) \\
(2) \qquad &\cong H_*(\textstyle\sum_{(q_j)=q} \mathbf{k}X_{q_1} \otimes \mathbf{k}Y_{q_2} \otimes \mathbf{k}X_{q_3} \otimes \mathbf{k}Y_{q_4} \otimes \cdots) \\
&= H_*[\textstyle\sum_{(q_j)=q} (\sum_{m_1} \mathbf{k}X_{q_1}^{m_1} \otimes \sum_{m_2} \mathbf{k}Y_{q_2}^{m_2} \otimes \sum_{m_3} \mathbf{k}X_{q_3}^{m_3} \\
&\qquad \otimes \cdots)] \\
&= \textstyle\sum_{m_1} \cdots \sum_{m_n} H_*(\sum_{(q_j)=q} \mathbf{k}X_{q_1}^{m_1} \otimes \mathbf{k}Y_{q_2}^{m_2} \otimes \cdots) \\
&= \textstyle\sum_{i \leq q} \sum_{(m_j)=i} H_*(\sum_{(q_j)=q} \mathbf{k}X_{q_1}^{m_1} \otimes \mathbf{k}Y_{q_2}^{m_2} \otimes \cdots);
\end{aligned}$$

the first sum is actually over all $i \in Z$, but for each $(m_1, \cdots, m_n)$ such that $\sum_{r=1}^n m_r = i > q$ and each $(q_1, \cdots, q_n)$ such that $\sum_r q_r = q$, some $m_j > q_j$ and thus $X_{q_j}^{m_j}$ (or $Y_{q_j}^{m_j}$) is 0 since each X^m is a s.s. projective resolution of (A_m, m). Now by using this last fact, (2) becomes

$$\sum_{i \leq q} \sum_{(m_j)=i} \mathrm{Mult}_{q-i}^{K,n}(A_{m_1}, B_{m_2}, A_{m_3}, B_{m_4}, \cdots) = \sum_{i \leq q} \mathrm{Mult}_{q-i}^{K,n}[A, B]_i.$$

Now a change of indices ($j = q - i$) gives

$$\sum_{j\geq 0} \mathrm{Mult}_j^{K,n}\, [A,\, B]_{q-j}$$

as desired.

We are now in a position to compute the left derived functors of $A * B = T(\bar{A}, \bar{B})$, which we denote by $L_q(A * B)$. From the appropriate definitions and Theorem 4.4 we immediately obtain:

THEOREM 4.5. *If A and B are DGA-algebras, then the left derived functors of $A * B$ are given by*

$$L_0(A * B) = K + \sum_{n\geq 1} \sum_{i\geq 0} (\mathrm{Mult}_i^{K,n}\, [\bar{A},\, \bar{B}]_{-i} + \mathrm{Mult}_i^{K,n}\, [\bar{B},\, \bar{A}]_{-i}),$$

and for $q \neq 0$,

$$L_q(A * B) = \sum_{n\geq 1} \sum_{i\geq 0} (\mathrm{Mult}_i^{K,n}\, [\bar{A},\, \bar{B}]_{q-i} + \mathrm{Mult}_i^{K,n}\, [\bar{B},\, \bar{A}]_{q-i}).$$

5. Künneth theorems

If A and B are DGA-algebras over K, then so are $H(A)$, $H(B)$, and $H(A * B)$. Furthermore $\overline{H(A)} = H(\bar{A})$ and $\overline{H(B)} = H(\bar{B})$. Define a map

$$\alpha_*: H(A) * H(B) \rightarrow H(A * B)$$

as follows. $\alpha_* \mid K$, $\alpha \mid \overline{H(A)}$, $\alpha \mid \overline{H(B)}$ are the respective identity maps on K, $\overline{H(A)} = H(\bar{A})$, $\overline{H(B)} = H(\bar{B})$. $\alpha \mid T_n[\overline{H(A)}, \overline{H(B)}]$ is the usual homology product map

$$\alpha : T_n[H(\bar{A}), H(\bar{B})] \rightarrow H[T_n(\bar{A}, \bar{B})];$$

similarly for $\alpha_* \mid T_n[\overline{H(B)}, \overline{H(A)}]$. It can verified be that α_* is a DGA-map. In general, of course, it is not an isomorphism; however, we do have:

THEOREM 5.1. *If A and B are DGA-algebras over K such that the modules of cycles and the homology modules $Z_n(A)$, $H_n(A)$, $Z_n(B)$, $H_n(B)$ are projective K-modules for every n, then*

$$\alpha : H(A) * H(B) \rightarrow H(A * B)$$

is a DGA-isomorphism.

Proof. We need only show that α_* is an isomorphism of graded K-modules. Under the hypothesis that just one of A or B have projective cycles and homology in every dimension, the ordinary Künneth theorem (see, for example, Theorem V.10.1 of MacLane [6]) states that the homology product map $\alpha : H(\bar{A}) \otimes H(\bar{B}) \rightarrow H(\bar{A} \otimes \bar{B})$ is an isomorphism of graded K-modules. An inductive procedure then shows that the map

$$\alpha : T_n[H(\bar{A}), H(\bar{B})] \rightarrow H[T_n(\bar{A}, \bar{B})]$$

ia a graded K-module isomorphism for all $n \geq 2$. Therefore from the definition of α_* we see that α_* is an isomorphism.

If K is a hereditary ring we can also state a Künneth theorem of sorts for the free product, corresponding to similar theorems for the tensor product, as given in [3], [6], and elsewhere.

THEOREM 5.2. *If A and B are flat DGA-algebras over a hereditary ring K, then there is a (nonnatural) isomorphism of graded K-modules*:

$$H(A * B) \cong \sum_q L_q(H(A) * H(B)).$$

Proof. The multiple Künneth theorem of [6] states that for $K = Z$ and A, B torsion-free (i.e. Z-flat) there is for each q and each $n \geq 2$ a (nonnatural) isomorphism of graded groups:

$$(1) \qquad H_q(T_n(\bar{A}, \bar{B}) \cong \sum_{i=0}^{n-1} \mathrm{Mult}_i^{K,n}[H(\bar{A}), H(\bar{B})]_{q-i}.$$

Since for hereditary rings, $\mathrm{Mult}_i^{K,n}(-)$ is 0 for $i \geq n$, (for $q \neq 0$) Theorem 4.5 gives

$$\begin{aligned} H_q(A * B) &= \sum_{n \geq 1} H_q(T_n(\bar{A}, \bar{B})) + H_q(T_n(\bar{B}, \bar{A})) \\ &= \sum_{n \geq 1} \sum_{i=0}^{n-1} \mathrm{Mult}_i^{K,n}[H(\bar{A})H(\bar{B})]_{q-i} \\ &\quad + \mathrm{Mult}_i^{K,n}[H(\bar{B}), H(\bar{A})]_{q-i} \\ &= L_q(H(A) * H(B)). \end{aligned}$$

The proof for $q = 0$ is similar. Essentially the same multiple Künneth theorem as above is given for an arbitrary hereditary ring K in Dold [3]. Although the theorem is stated there only for the case $n = 2$, the proof given applies equally well, *mutatis mutandis*, to the case $n > 2$. Hence the theorem follows as above.

COROLLARY 5.3. *If A and B are flat DGA-algebras over a hereditary ring K, then there is an exact sequence of graded K-modules*:

$$\begin{aligned} 0 \to H(A) * H(B) &\xrightarrow{\alpha *} H(A * B) \\ &\to \sum_q \sum_{n \geq 1} \sum_{i-1}^{n-1} (\mathrm{Mult}_i^{K,n}[H(\bar{A}), H(\bar{B})]_{q-i} \\ &\qquad + \mathrm{Mult}_i^{K,n}[H(\bar{B}), H(\bar{A})]_{q-i}) \to 0. \end{aligned}$$

where α_* is the DGA-map defined above.

Proof. It is shown in [6] that the isomorphism (1) in the proof of the theorem is given for each q by the identity map on the summands $H_q(\bar{A})$, $H_q(\bar{B})$ (and Z, if $q = 0$) and for each n by the homology product α on the summand

$$\mathrm{Mult}_0^{K,n}[H(A), H(B)]_q = (T_n(H(A), H(B))_q.$$

Hence the isomorphism $H(A * B) \cong \sum_q L_q(H(A) * H(B))$ is given on

$$\begin{aligned} H(A) * H(B) &= K + H(\bar{A}) + H(\bar{B}) + \textstyle\sum_{n \geq 2} T_n(H(\bar{A}), H(\bar{B})) \\ &\quad + T_n(H(\bar{B}), H(\bar{A})) \\ &= K + \textstyle\sum_q \sum_{n \geq 1} \mathrm{Mult}_0^{K,n} [H(\bar{A}), H(\bar{B})]_q \\ &\quad + \mathrm{Mult}_0^{K,n} [H(\bar{B}), H(\bar{A})]_q \\ &\subseteq \textstyle\sum_q L_q(H(A) * H(B)) \end{aligned}$$

by the map α_* and the corollary follows immediately.

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