A NOTE ON INVARIANT SUBSPACES

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1. Introduction

In a recent paper [5], Wells and Kellogg used a dual extremal technique to characterize the closed simply invariant subspaces of the L^p spaces ($1 \leq p < \infty$) associated with a Dirichlet algebra. For $1 < p < \infty$, their methods were intrinsic in the sense that they had no need to use special Hilbert space facts associated with the space L^2. For $p = 1$, however, they needed the following lemma, whose proof depended on a Hilbert space argument.

LEMMA. *If f_n is a sequence in $H^1(dm)$ such that*

$$\| f_n \|_1 \leq 1 \quad \textit{and} \quad \int f_n \, dm \to 1$$

then

$$\int | 1 - f_n | \, dm \to 0.$$

The purpose of this note is to show how this lemma may be proved without recourse to Hilbert space techniques, thus yielding an "intrinsic" proof of the invariant subspace theorem for L^1.

2. Preliminaries

We assume the reader is familiar with [5]. Let X be a compact Hausdorff space and let A be a sup norm algebra on X, i.e., a uniformly closed complex linear subalgebra of $C(X)$ which separates points and contains the constants. Let m be a nonzero algebra homomorphism of A into the complex numbers. By the Hahn-Banach and Riesz representation theorems, there exists a Baire probability measure μ on X such that

$$m(f) = \int_X f \, d\mu$$

for all $f \in A$. We call such a μ a representing measure for m. Of course, for a fixed m there are, in general, many such μ. If A is a Dirichlet algebra (or a logmodular algebra) on X, however, each complex homomorphism has a unique representing measure. Actually, the existence of a single complex homomorphism which has a unique representing measure is enough to capture the full strength of the H^p theory (for an illuminating discussion of which

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see [2]). There are other, more general though perhaps less "natural" hypotheses under which one can obtain the H^p theory; see, for instance, [3] and [4]. In particular, the invariant subspace theorem remains valid in these more general situations.

We shall content ourselves with a discussion of the case in which some complex homomorphism of A has a unique representing measure. Accordingly, let m be such a homomorphism and let dm be its representing measure. We form the spaces $H^p(dm)$ $(1 \leq p < \infty)$ in the usual way: $H^p(dm)$ is the $L^p(dm)$ closure of A. We shall need the following two facts:

(A) Suppose $f \in H^1(dm)$ and $\int f\, dm \neq 0$. Then $f = Fg$, where F is an inner function and g is an outer function in $H^1(dm)$.

(B) If g is an outer function in $H^1(dm)$, then $g = k^2$, where k is an outer function in $H^2(dm)$.

Neither of these results depends on Hilbert space arguments. To prove (A) first note that by the Jensen inequality

$$\int \log |f|\, dm \geq \log \left| \int f\, dm \right| > -\infty;$$

then apply Theorem 13 of [4]. (Lemma 3 of [1] insures that some represent ing measure for m obeys Jensen's inequality for functions in A; by the unique ness assumption, that measure must be dm. The inequality for functions in $H^1(dm)$ then follows by a simple limiting process.) For (B), use Theorem 14 of [4].

3. The Proof

We can now prove the lemma. We may assume that $\int f_n\, dm$ is never 0. Each f_n then has, by (A), a factorization

$$f_n = F_n g_n \tag{1}$$

where F_n is an inner function and g_n is an outer function in $H^1(dm)$. Since the multiplicative property of m on the algebra A extends to each product in (1), we have

$$\int F_n\, dm \cdot \int g_n\, dm = \int F_n g_n\, dm = \int f_n\, dm \to 1 \tag{2}$$

Now

$$\left| \int F_n\, dm \right| \leq \int |F_n|\, dm = 1 \tag{3}$$

and

$$\left| \int g_n\, dm \right| \leq \int |g_n|\, dm = \int |f_n|\, dm \leq 1. \tag{4}$$

So

$$\left| \int F_n \, dm \right| \to 1, \qquad \left| \int g_n \, dm \right| \to 1 \tag{5}$$

by (2), (3), and (4). Multiplying by appropriate unimodular constants, we obtain

$$\int F_n \, dm \to 1, \qquad \int g_n \, dm \to 1. \tag{6}$$

Now

$$\begin{aligned} \int | 1 - F_n |^2 \, dm &= \int dm + \int | F_n |^2 \, dm - 2 \operatorname{Re} \int F_n \, dm \\ &= 2\left(1 - \operatorname{Re} \int F_n \, dm\right) \to 0 \end{aligned} \tag{7}$$

by (6). Thus

$$\int | 1 - F_n | \, dm \leq \left(\int | 1 - F_n |^2 \, dm \right)^{1/2} \to 0. \tag{8}$$

Since $g_n \in H^1(dm)$ is outer, we have, by (B), $g_n = k_n^2$, $k_n \in H^2(dm)$ an outer function. Then

$$\left(\int k_n \, dm \right)^2 = \int k_n^2 \, dm = \int g_n \, dm \to 1. \tag{9}$$

Here we have used (6) and the fact that m is multiplicative on $H^2(dm)$. From (9) we get, multiplying by -1 if necessary,

$$\int k_n \, dm \to 1. \tag{10}$$

It follows as in (7) that

$$\int | 1 - k_n |^2 \, dm \to 0. \tag{11}$$

Thus

$$\begin{aligned} &\int | 1 - g_n | \, dm \\ &\quad = \int | 1 - k_n^2 | \, dm \\ &\quad = \int |1 - k_n | \cdot | 1 + k_n | \, dm \\ &\quad \leq \left(\int |1 - k_n |^2 \, dm \right)^{1/2} \cdot \left(\int | 1 + k_n |^2 \, dm \right)^{1/2} \to 0 \end{aligned} \tag{12}$$

since $\| 1 + k_n \|_2 \leq 1 + \| k_n \|_2 = 1 + \| f_n \|_1^{1/2} \leq 2$. By (8) and (12) we have

$$\begin{aligned} \int | 1 - f_n | \, dm &= \int | 1 - F_n g_n | \, dm \\ &= \int | 1 - F_n + F_n - F_n g_n | \, dm \\ &\leq \int | 1 - F_n | \, dm + \int | 1 - g_n | \, | F_n | \, dm \\ &= \int | 1 - F_n | \, dm + \int | 1 - g_n | \, dm \to 0. \end{aligned}$$

This completes the proof.

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