THE SECOND DUAL OF THE SPACE OF CONTINUOUS FUNCTIONS AND THE RIEMANN INTEGRAL

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Introduction

For concreteness, let X be the closed interval $\{0 \leqq x \leqq 1\}$, μ the Lebesgue measure on X, $\mathfrak{L}^1 = \mathfrak{L}^1(\mu)$, and $\mathfrak{L}^\infty = \mathfrak{L}^\infty(\mu)$ (over the reals). As we know, $\mathfrak{L}^\infty$ is the norm-dual of $\mathfrak{L}^1$. $C = C(X)$ can be imbedded isometrically in $\mathfrak{L}^\infty$, so for the moment, let us consider it as a subspace of $\mathfrak{L}^\infty$. Finally, let $\mathfrak{R}$ be the image in $\mathfrak{L}^\infty$ of the Riemann integrable functions, that is, each element of $\mathfrak{R}$ is the equivalence class (modulo the essentially bounded measurable functions vanishing almost everywhere) determined by some Riemann integrable function. Thus $C \subset \mathfrak{R} \subset \mathfrak{L}^\infty$.

Denote by $B(\mathfrak{L}^1)$ the unit ball of $\mathfrak{L}^1$ and by $\Delta(\mathfrak{L}^1)$ the set

$$\{\phi \in \mathfrak{L}^1 \mid \phi \geqq 0, \|\phi\| = 1\}$$

(the "face" of $B(\mathfrak{L}^1)$ lying in the positive cone). By the Grothendieck theorem, we have:

The set of linear functionals on $\mathfrak{L}^1$ continuous on $B(\mathfrak{L}^1)$ under the weak topology $w(\mathfrak{L}^1, C)$ defined by C is C itself.

In the present paper we show the following:

The set of linear functionals on $\mathfrak{L}^1$ continuous on $\Delta(\mathfrak{L}^1)$ under $w(\mathfrak{L}^1, C)$ is $\mathfrak{R}$,

We compare these two theorems further. As was shown by Caratheodory [4], $\mathfrak{R}$ consists of those elements of $\mathfrak{L}^\infty$ each of which is simultaneously the infimum in $\mathfrak{L}^\infty$ of some subset of C and the supremum in $\mathfrak{L}^\infty$ of some subset of C. We describe this shortly by the statement: $\mathfrak{R}$ is the *Dedekind closure* of C in $\mathfrak{L}^\infty$. The above theorems can then be stated as follows: the linear functionals on $\mathfrak{L}^1$ which are $w(\mathfrak{L}^1, C)$-continuous on $B(\mathfrak{L}^1)$ constitute the norm-closure of C in $\mathfrak{L}^\infty$, those which are $w(\mathfrak{L}^1, C)$-continuous on $\Delta(\mathfrak{L}^1)$, the Dedekind-closure. (The norm-closure of C is C itself; we just state it this way for purposes of comparison.)

Caratheodory's characterization has been extensively used in recent years as an approach to the Riemann integral [2], [7], [13], [14]. In the present paper we make a study of the Dedekind closure of $C(X)$, X compact, in a general closed ideal of the bidual $M(X)$ of $C(X)$. Since for every Radon measure μ on X, $\mathfrak{L}^\infty(\mu)$ is a closed ideal in $M(X)$, our work includes the

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Riemann integral as a special case. Specifically, given a closed ideal I in $M = M(X)$, we denote the projection of $C = C(X)$ in I by C_I and call the Dedekind closure of C_I in I the *Riemann subspace* of I. The characterization stated above for $\mathcal{R}$ in $\mathcal{L}^\infty$ is obtained for this general Riemann subspace.

Part I is devoted to obtaining this characterization. It requires a study of uppersemicontinuity (or equivalently, of lowersemicontinuity) in the dual L of C, which is of interest in itself. Part II is devoted to carrying out—in M —Semadeni's generalized Riemann integration [15], [1].

Before turning to the work itself, we want to discuss a phenomenon which occurs frequently in going from a topic in topology to the corresponding topic in M. This is that the role played by *σ-closed* (set) ideals in the former turns out to be played in M by *norm-closed* (vector lattice) ideals. The concept of norm-closure in M has no correspond for sets. In working with sets, if a set ideal or ring is not large enough for one's purposes, one usually takes its σ-closure. In M, however, between an ideal and its σ-closure there lies its norm-closure, and it is this which often seems to be the correct enlargement.

We have already given one example of this in [11, §10], showing that a theorem in function theory which can ordinarily be stated only in terms of meagerness (i.e. first category) turns out in M to involve nothing more than rareness (i.e. nowhere-denseness). We give another example now.

As before, let us confine ourselves to $X = \{0 \leqq x \leqq 1\}$ and the Lebesgue measure. Also, for simplicity of illustration, we will work with the vector lattice of bounded real functions on X (endowed with the supremum norm) rather than with M. Consider the classic Lebesgue theorem: *a bounded function f is Riemann integrable if and only if its set of (points of) discontinuity has measure zero.* The correspond, in our vector lattice, to the set of discontinuity of f is the saltus function $\delta(f)$ of f. (Indeed $\delta(f)$ tells us more; not only is $\delta(f)(x) \neq 0$ if and only if x is a point of discontinuity, but the value $\delta(f)(x)$ is the magnitude of the discontinuity.) Writing the Lebesgue theorem in terms of $\delta(f)$—and thus eliminating all reference to sets—we have: *a bounded function f is Riemann integrable if and only if $\delta(f)$ has its Lebesgue integral equal to zero.*

But now we can immediately sharpen the theorem. $\delta(f)$ is always a non-negative uppersemicontinuous function, and for such functions, the Lebesgue integral vanishing is equivalent to the Riemann integral (existing and) vanishing. Thus the theorem becomes: *a bounded function f is Riemann integrable if and only if $\delta(f)$ has Riemann integral zero.* Besides being more satisfying esthetically, this final form illustrates our point. The bounded functions having Lebesgue integral zero form a σ-closed ideal, while those having Riemann integral zero form only a norm-closed ideal.

A more refined analysis leads to the same result. Let P_0 be the set ideal of sets of content zero, P_1 its σ-closure (i.e. the σ-ring generated by P_0), and P_2 the ideal of sets of measure zero. Thus $P_0 \subset P_1 \subset P_2$, the inclusions both proper. Now the Lebesgue theorem actually has P_1 in place of P_2 : f is

Riemann integrable if and only if its set of discontinuity is an element of P_1. Paralleling this with functions, let N_0 be the ideal generated by the characteristic functions of sets of content zero, N_1 its σ-closure, and N_2 the ideal generated by the characteristic functions of sets of measure zero (this last is simply the ideal of functions having Lebesgue integral zero). Again $N_0 \subset N_1 \subset N_2$, the inclusions both proper. But now we also have the norm-closure N of $N_0 : N_0 \subset N \subset N_1 \subset N_2$, all three inclusions proper. And it is N which is the ideal of functions having Riemann integral zero. Thus our final theorem in the preceding paragraph reads: f is Riemann integrable if and only if $\delta(f) \in N$.

Part I

1. Terminology. In general we follow the terminology and notation of the preceding papers [8], [9], [10], [11], [12]. If Y is a compact (Hausdorff) space, the Banach lattice of continuous real functions on Y, its dual, and its bidual are denoted by $C(Y)$, $L(Y)$, and $M(Y)$ respectively. As before, our concern is with a fixed compact space X, and $C(X)$, $L(X)$, $M(X)$ will be simply written C, L, M.

For $\mu \in L$ and $f \in M$ (hence in particular, for $f \in C$), $\langle f, \mu \rangle$ will denote the value of the bilinear functional giving the duality.

Adopting Bourbaki's term, we will call the weak* topology on L (that is, the weak topology $w(L, C)$ defined by C) the *vague* topology.

In general an element of M connot be considered a function on X. However there are two kinds of elements which are completely determined by their values on X, and therefore—with proper precautions—can often be treated as functions. These are the elements of U and the elements of M_0 [8]. Thus for example, if f and g are both in U, or both in M_0, then $f \leqq g$ if and only if $f(x) \leqq g(x)$ for all $x \in X$.

In line with this we adopt a convention. Given a subset A of X, we cannot, in general, talk about the "characteristic function", or rather " characteristic element" of A in M. For, if $f \in M$ satisfies $f(x) = 1$ for $x \in A$, $f(x) = 0$ for $x \notin A$, then also $f + g$ satisfies this for all $g \in M_1$. However, if A is measurable with respect to every Radon measure, then there is a unique $f \in U$ satisfying the above. Hence for such a set A, by the *characteristic element* of A we will mean this $f \in U$. In particular, for a closed set A, the characteristic element of A is the u.s.c. element satisfying the above.

Finally, for any set A in X, we may sometimes talk about the *characteristic element of A in M_0*.

2. The state space Δ and semicontinuity. We first record a simple theorem from function theory which we will need.

(2.1) *Let $\{K_1, \cdots, K_n\}$ be a covering of X by compact sets, and f an upper-semicontinuous function on X. Then for each $\varepsilon > 0$, there exists a continuous* *function h dominating f such that*

$$\sup_{x \in K_i} h(x) < \sup_{x \in K_i} f(x) + \varepsilon, \qquad i = 1, \cdots, n.$$

We will denote by Δ the set $\{\mu \in L \mid \mu \geqq 0, \|\mu\| = 1\}$ of "states" of L. Δ is vaguely compact, and its set of extreme points is precisely X. Thus by the Krein-Milman theorem, Δ is the vague closure of the convex envelope $\hat{X}$ of X. For our purposes, we want to specify, for each $\mu \in \Delta$, a "canonical" type of net in $\hat{X}$ which converges vaguely to μ.

Let Π be the set of all partitions of unity in C, that is, each $\pi \in \Pi$ is a finite set $\{f_1, \cdots, f_n\}$ of non-negative elements of C satisfying $\sum_1^n f_i = \mathbf{1}$. Define a partial order on Π as follows: $\pi < \pi'$ will mean that π' can be written

$$\{g_{ij} \mid i = 1, \cdots, n; j = 1, \cdots, m(i)\}$$

such that for each i, $\sum_{j=1}^{m(i)} g_{ij} = f_i$. It is easily verified that Π is a directed set under this order.

Now for each $\pi = \{f_1, \cdots, f_n\}$ let us fix a set $\{x_1, \cdots, x_n\}$ where x_i is an arbitrary element of the support of f_i, $i = 1, \cdots, n$. Then for each $f \in C$, set $f_\pi = \sum_1^n f(x_i) f_i$. While the f_π's depend on our choices of x_i's, (2.2) and (2.3) below are independent of these choices, in the sense that they hold for any choices.

Given $f \in C$, $\{f_\pi \mid \pi \in \Pi\}$ is a net in C with Π for index system, and $\lim_\Pi \|f_\pi - f\| = 0$ [3, Chap. III, §2, Lemme 2]. It follows that for every $\mu \in L$, $\lim_\Pi \langle f_\pi, \mu\rangle = \langle f, \mu\rangle$. We state this formally:

(2.2) *Given $f \in C$ and $\mu \in L$,*

$$\lim_\Pi \sum_\pi f(x_i)\langle f_i, \mu\rangle = \langle f, \mu\rangle.$$

On the other hand, given $\mu \in L$, let us set $\mu_\pi = \sum_1^n \langle f_i, \mu\rangle x_i$. Then, reading the summation in (2.2) as $\langle f, \mu_\pi\rangle$ rather than $\langle f_\pi, \mu\rangle$, we have that $\lim_\Pi \langle f, \mu_\pi\rangle = \langle f, \mu\rangle$ for every $f \in C$. Thus

(2.3) $\mu = \lim_\Pi \mu_\pi$ *in the vague topology.*

In particular, if $\mu \in \Delta$, then $\mu_\pi \in \hat{X}$ for all π, and we have our "canonical" type net of $\hat{X}$ converging vaguely to μ.

Now consider an u.s.c. element f of M. f is vaguely uppersemicontinuous on Δ, since it is the pointwise infimum there (indeed, on all of L_+) of some subset of C. Thus for each $\mu \in \Delta$, $\langle f, \mu\rangle \geqq \limsup \langle f, \nu\rangle$, where $\nu \in \Delta$, $\nu \to \mu$ vaguely. We actually have a stronger property:

(2.4) THEOREM. *Given an u.s.c. element f of M, then for each $\mu \in \Delta$,*

$$\langle f, \mu\rangle = \limsup_{\nu \in \hat{X}, \nu \to \mu \text{ vaguely}} \langle f, \nu\rangle.$$

Proof. We of course need only show $\leqq$. First, for each partition of unity $\pi = \{f_1, \cdots, f_n\}$, let us denote the support of f_i by K_i, $i = 1, \cdots, n$, and choose $\{x_1, \cdots, x_n\}$ such that

$$\text{(i)} \qquad f(x_i) = \sup_{x \in K_i} f(x).$$

(The x_i's will now be fixed for the remainder of the proof.) Now consider $\mu \in \Delta$. Since $\mu_\pi \in \hat{X}$ for all $\pi \in \Pi$, we need only show $\langle f, \mu \rangle \leqq \limsup_\Pi \langle f, \mu_\pi \rangle$. Denote this lim sup by λ and choose any $\varepsilon > 0$; we show $\langle f, \mu \rangle \leqq \lambda + 3\varepsilon$. Specifically, we will produce $h \in C$ such that $h \geqq f$ and $\langle h, \mu \rangle \leqq \lambda + 3\varepsilon$.

Choose $\pi = \{f_1, \cdots, f_n\}$ such that

$$\text{(ii)} \qquad \langle f, \mu_\pi \rangle \leqq \lambda + \varepsilon.$$

Now choose $h \in C$ to satisfy (2.1). Then $h \geqq f$ (since $h(x) \geqq f(x)$ for all $x \in X$), and (from (i))

$$\text{(iii)} \qquad \sup_{x \in K_i} h(x) \leqq f(x_i) + \varepsilon, \qquad i = 1, \cdots, n.$$

Finally, by (2.3), choose $\pi' > \pi$, $\pi' = \{g_{ij} \mid i = 1, \cdots, n; j = 1, \cdots, m(i)\}$, such that

$$\text{(iv)} \qquad | [h, \mu_{\pi'} \rangle - \langle h, \mu \rangle | \leqq \varepsilon.$$

Then

$$\begin{aligned} \langle h, \mu_{\pi'} \rangle &= \textstyle\sum_{i,j} \langle g_{ij}, \mu \rangle h(x_{ij}) \\ &= \textstyle\sum_i \sum_j \langle g_{ij}, \mu \rangle h(x_{ij}) \\ &\leqq \textstyle\sum_i \sum_j \langle g_{ij}, \mu \rangle (f(x_i) + \varepsilon) && \text{(from (iii))} \\ &= \textstyle\sum_i (f(x_i) + \varepsilon) \langle \sum_j g_{ij}, \mu \rangle \\ &= \textstyle\sum_i (f(x_i) + \varepsilon) \langle f_i, \mu \rangle \\ &= \textstyle\sum_i f(x_i) \langle f_i, \mu \rangle + \varepsilon \sum_i \langle f_i, \mu \rangle \\ &= \langle f, \mu_\pi \rangle + \varepsilon \\ &\leqq \lambda + 2\varepsilon && \text{(from ii))} \end{aligned}$$

Combining this with (iv), we obtain $\langle h, \mu \rangle \leqq \lambda + 3\varepsilon$, which completes the proof.

If we have a l.s.c. element f of M, we can apply the above theorem to $-f$. Thus

(2.5) COROLLARY. *Given a l.s.c. element f of M, then for each $\mu \in \Delta$,*

$$\langle f, \mu \rangle = \liminf_{\nu \in \hat{X}, \nu \to \mu \text{ vaguely}} \langle f, \nu \rangle$$

We are now in a position to prove

(2.6) THEOREM. *Given $f \in M$, the following are equivalent:*

1° *f is an u.s.c. element.*

2° *f is vaguely uppersemicontinuous on Δ.*

However, this is a special case of (4.4) below, and it is the latter which we will need.

3. Closed ideals in L and M. In this section we give a short review of those properties of ideals which we will need. By an ideal, we will always mean a vector lattice ideal, and the unmodified term "closed" will always mean with respect to order-convergence.

Let E be a complete vector lattice. The importance of the closed ideals of E lies in the Riesz theorem: an ideal I of E gives rise to a decomposition of E, $E = I \oplus I_1$, where I_1 is also an ideal, if and only if I is closed, and in such case I_1 is precisely the (closed) ideal I' consisting of all elements of E disjoint from I.

It follows that if I is closed, then each $a \in E$ has a natural projection, or component, in I. We denote it by a_I. More generally, the projection in I of any subset A of E is denoted by A_I. If I is a principal closed ideal, that is, the closed ideal generated by a single $b \in E$, then we also write a_b for a_I and A_b for A_I; in particular $I = E_b$.

We turn to L and M. M is not only the norm-dual of L; it is also its dual with respect to order-convergence, that is, it consists of the linear functionals on L which are continuous under order-convergence. While L, on the other hand, is not the norm-dual of M, it *is* the latter's dual with respect to order-convergence. Thus L and M are reflexive with respect to order-convergence.

Unless otherwise specified, annihilators of sets in L or M will always be with respect to the dual system (M, L). Thus, for an ideal J of L, $J^\perp$ will denote its annihilator in M.

Given an ideal J in L, $J^\perp$ is a closed ideal in M; and given an ideal I in M, $I^\perp$ is a closed ideal in L. Moreover, if J is a *closed* ideal in L, the Riesz decomposition $L = J \oplus J'$ gives us the decomposition $M = (J')^\perp \oplus J^\perp$, and (again from the Riesz theorem) $(J')^\perp = (J^\perp)'$. We will call $(J')^\perp$ the ideal in M *dual* to J. Conversely, given a closed ideal I in M, we have $M = I \oplus I'$ and $L = (I')^\perp \oplus I^\perp$; we will call $(I')^\perp$ (which is also $(I^\perp)'$) the ideal in L *dual* to I. Clearly, given closed ideals J and I in L and M respectively, I is dual to J if and only if J is dual to I. For short, we will also simply call them *dual (closed) ideals*.

Given two dual ideals J in L and I in M, the relationship of L and M is inherited by J and I: I is both the norm-dual of J and its dual with respect to order-convergence, while J is the dual of I with respect to order-convergence.

Remark. If J is a principal closed ideal, $J = L_\mu$ for some $\mu \in L$, then J can be identified with $\mathcal{L}^1(\mu)$ (Radon-Nikodym theorem), and so its dual ideal $(J^\perp)'$ can be identified with $\mathcal{L}^\infty(\mu)$. Thus $\mathcal{L}^1(\mu)$ and $\mathcal{L}^\infty(\mu)$ are reflexive with respect to order convergence.

We turn to the duality between L and C. For an ideal J of L, $J^\perp \cap C$ is also an ideal, but even if J is closed, the decomposition $L = J \oplus J'$ does not in general give rise to a decomposition of C. To clarify the situation, consider the ideal $(J^{\perp} \cap C)^{\perp}$ in L. $(J^{\perp} \cap C)^{\perp}$ is of course the vague closure of J; let us denote it by $\mathbf{J}$. We will call the set $K = X \cap \mathbf{J}$ the *support* of J. If $J = L_{\mu}$ for some $\mu \in L$, then K is the support of μ as the latter is ordinarily defined.

We collect the basic relations between J, $\mathbf{J}$, and K in (3.1) below. The set $\Delta \cap J$ (for *any* ideal J in L) will be denoted by $\Delta(J)$.

(3.1) (a) $\Delta(\mathbf{J})$ *is vaguely compact, and K is its set of extreme points.*
(b) $C/(J^{\perp} \cap C) = C(K)$, *hence*

$$\mathbf{J} = L(K) \quad \textit{and} \quad w(L, C) \mid \mathbf{J} = w(L(K), C(K)).$$

(c) $\mathbf{J}_+$ *is the vague closure of* J_+, *hence* $\Delta(\mathbf{J})$ *is the vague closure of* $\Delta(J)$.

The only statement whose proof is not immediate is (c), and this follows from the

LEMMA. *If $\mu \in L_+$ is not in the vague closure of J_+, then there exists $f \in (J^{\perp} \cap C)_+$ such that $\langle f, \mu \rangle > 0$.*

Proof. By the Hahn-Banach theorem, there exists $g \in C$ such that

$$\sup_{\nu \in J_+} \langle g, \nu \rangle < \langle g, \mu \rangle.$$

Set $f = g^+$. Then

(i) $$0 \leqq \sup_{\omega \in J_+} \langle f, \omega \rangle = \sup_{\omega \in J_+} \sup_{0 \leqq \nu \leqq \omega} \langle g, \nu \rangle = \sup_{\nu \in J_+} \langle g, \nu \rangle < \langle g, \mu \rangle \leqq \langle f, \mu \rangle.$$

Since J_+ is a cone, it follows $\langle f, \omega \rangle = 0$ for all $\omega \in J_+$. But then $\langle f, \omega \rangle = 0$ for all $\omega \in J$, and we have $f \in J^{\perp}$.

Remark 1. We emphasize that $\Delta(J)$ and $\hat{K}$ (the convex envelope of K) are each in the vague closure of the other.

Remark 2. It is clear from (3.1) that in working with J we can restrict ourselves to the support of J instead of dealing with all of X. Otherwise stated, we can assume J is vaguely dense in L.

4. Semicontinuity on $\Delta(J)$.

Let I be a closed ideal in M, and consider the projection C_I of C in I. C_I is a norm-complete linear sublattice of I— in the Geba and Semadeni terminology [5], an M-subspace of I—just as C is one of M. Copying the definition of an u.s.c. element in M, we will call $f \in I$ an $(\text{u.s.c.})_I$ element if $f = \bigwedge B$ for some subset B of C_I. Similarly, f will be called a $(\text{l.s.c.})_I$ element if $f = \bigvee B$ for some subset B of C_I.

(4.1) LEMMA. *$f \in I$ is an $(u.s.c.)_I$ element if and only if $f = g_I$ for some u.s.c. element g of M. And similarly for a $(l.s.c.)_I$ element.*

Proof. Suppose $f = g_I$, g u.s.c. Then $g = \bigwedge A$, $A \subset C$, and since projection preserves suprema and infima, $f = \bigwedge A_I$. Thus f is $(\text{u.s.c.})_I$. Conversely, suppose $f = \bigwedge_\alpha h_\alpha$, $\{h_\alpha\} \subset C_I$ (for the proof in this direction, it is more convenient to use an indexed set $\{h_\alpha\}$ than a set denoted by a single letter B). For each α, $h_\alpha = (g_\alpha)_I$ for some $g_\alpha \in C$, so we have $f = \bigwedge_\alpha (g_\alpha)_I$. Now $\mathbf{1}_I$ is a strong order unit for I, hence $f \geqq \lambda \mathbf{1}_I$ for some λ. Then

$$\bigwedge_\alpha [g_\alpha \vee \lambda \mathbf{1}]_I = \bigwedge_\alpha [(g_\alpha)_I \vee \lambda \mathbf{1}_I] = [\bigwedge_\alpha (g_\alpha)_I] \vee \lambda \mathbf{1}_I = f \vee \lambda \mathbf{1}_I = f.$$

Thus, for simplicity, we can assume we already have $g_\alpha \geqq \lambda \mathbf{1}$ for all α, that is, the set $\{g_\alpha\}$ is bounded below. Setting $g = \bigwedge_\alpha g_\alpha$, we have g u.s.c. and $f = g_I$. This completes the proof.

Our object in this section is Theorem (4.4) below. We need another lemma.

(4.2) Lemma. *Let $J \subset L$ and $I \subset M$ be dual ideals. Then, given $f \in I$ and $h \in C$, the following are equivalent.*

1° $h_I \geqq f$.

2° *For every x in the support K of J,*

$$h(x) \geqq \limsup_{\mu \in \Delta(J), \mu \to x \text{ vaguely}} \langle f, \mu \rangle.$$

Proof. Set $g = f - h_I$. Since $\langle h_I, \mu \rangle = \langle h, \mu \rangle$ for all $\mu \in J$, the lemma reduces to showing that the following are equivalent:

1° $g \leqq 0$.

2° *For every x in the support K of J,*

$$\limsup_{\mu \in \Delta(J), \mu \to x \text{ vaguely}} \langle g, \mu \rangle \leqq 0.$$

We of course need only show that 2° implies 1°. Suppose $g^+ \neq 0$. Then there exists $\lambda > 0$ such that $(g^+ - \lambda \mathbf{1})^+ \neq 0$. Let I_2 be the closed ideal generated by $(g^+ - \lambda \mathbf{1})^+$ and J_2 its dual ideal in L, and set $e = \mathbf{1}_{I_2}$. By elementary vector lattice calculations, we have $0 < \lambda e \leqq g^+$. (c.f. the proof of (6.1) in [9].) This gives first of all that $I_2 \subset I$, hence $J_2 \subset J$, hence $\Delta(J_2) \subset \Delta(J)$, and thus the support K_2 of J_2 is contained in K. We will obtain a contradiction of 2° by showing that $\langle g, \mu \rangle \geqq \lambda$ for all $\mu \in \Delta(J_2)$. In effect, since $\lambda e \leqq g^+$, $e \wedge g^- = 0$; it follows g^- vanishes on J_2, hence for $\mu \in \Delta(J_2)$,

$$\langle g, \mu \rangle = \langle g^+, \mu \rangle \geqq \langle \lambda e, \mu \rangle = \lambda \langle e, \mu \rangle = \lambda \langle \mathbf{1}, \mu \rangle = \lambda.$$

We are now prepared to establish (4.4). We will make use of the following theorem from function theory.

(4.3) *Let f be a bounded real function on a subset A of a topological space Y, and define F on $\bar{A}$ by*

$$F(y) = \limsup_{z \in A, z \to y} f(z).$$

Then

(a) *F is uppersemicontinuous on $\bar{A}$;*

(b) *it is the smallest uppersemicontinuous function on $\bar{A}$ which dominates f on A;*

(c) *if f is uppersemicontinuous (on A), then F coincides with f on A.*

(4.4) THEOREM. *Let $J \subset L$ and $I \subset M$ be dual ideals. Then given $f \in I$, the following are equivalent:*

1° *f is an $(u.s.c.)_I$ element.*

2° *f is vaguely uppersemicontinuous on $\Delta(J)$.*

3° *f is $w(J, C_I)$-uppersemicontinuous on $\Delta(J)$.*

Proof. Since $\langle f, \mu\rangle = \langle f_I, \mu\rangle$ for every $f \in M$, $\mu \in J$, we have $w(L, C) | J = \omega(J, C_I)$. Thus 2° and 3° are equivalent. Suppose 1° holds. Then $f = g_I$ for some u.s.c. element g of M (4.1). g is vaguely uppersemicontinuous on Δ, hence in particular on $\Delta(J)$. Since $\langle f, \mu\rangle = \langle g, \mu\rangle$ for all $\mu \in J$, it follows f is also vaguely uppersemicontinuous on $\Delta(J)$. Thus 1° implies 2°.

Now suppose 2° holds. Following Remark 2 at end of §3, we will assume J is vaguely dense in L, and thus the support of J is X. By (4.3), we can extend $f | \Delta(J)$ to a vaguely uppersemicontinuous function F on Δ.

(i) *F is superlinear, that is, for $\mu, \nu \in \Delta$ and $\lambda, \kappa \geqq 0, \lambda + \kappa = 1$,*

$$F(\lambda\mu + \kappa\nu) \geqq \lambda F(\mu) + \kappa F(\nu).$$

To show this we choose (as is always possible) two nets $\{\mu_\alpha\}, \{\nu_\alpha\}$, with the same index system, on $\Delta(J)$ such that $\mu_\alpha \to \mu$, $\nu_\alpha \to \nu$ vaguely and $\langle f, \mu_\alpha\rangle \to F(\mu)$, $\langle f, \nu_\alpha\rangle \to F(\nu)$. Then the net $\{\lambda\mu_\alpha + \kappa\nu_\alpha\}$ is on $\Delta(J)$,

$$\lambda\mu_\alpha + \kappa\nu_\alpha \to \lambda\mu + \kappa\nu \quad \text{vaguely,}$$

and

$$\langle f, \lambda\mu_\alpha + \kappa\nu_\alpha\rangle = \lambda\langle f, \mu_\alpha\rangle + \kappa\langle f, \nu_\alpha\rangle \to \lambda F(\mu) + \kappa F(\nu).$$

Since F is vaguely uppersemicontinuous, this gives

$$F(\lambda\mu + \kappa\nu) \geqq \lambda F(\mu) + \kappa F(\nu),$$

which is (i).

Now $f \geqq k\mathbf{1}_I$ for some real number k. Let $g = \bigwedge \{h \in C \mid h \geqq \lambda\mathbf{1}, h_I \geqq f\}$. Since projections preserve infima, $g_I \geqq f$; we show the opposite inequality, which will give equality and establish 1°, thus completing the proof.

(ii) $$g(x) = F(x) \qquad \textit{for all} \quad x \in X.$$

To show this, consider $x \in X$ and $\lambda > F(x)$. Since F is uppersemicontinuous on X, there exists $h \in C$ dominating F on X such that $h(x) \leqq \lambda$. From (4.2), $h_I \geqq f$, hence by definition of g, $g(x) \leqq h(x) \leqq \lambda$. Since λ was arbitrary, this gives (ii).

(iii) $$\langle g, \mu\rangle \leqq F(\mu) \qquad \textit{for all} \quad \mu \in \Delta.$$

Assume first that $\mu \in \hat{X}$. Then $\mu = \sum_1^n \lambda_i x_i$, $\lambda_1, \cdots, \lambda_n \geqq 0$, $\sum_1^n \lambda_i = 1$; hence $\langle g, \mu\rangle = \sum_1^n \lambda_i g(x_i) = \sum_1^n \lambda_i F(x_i) \leqq F(\mu)$, this last inequality following from (i). Now consider any $\mu \in \Delta$. Applying Theorem (2.4),

$$\langle g, \mu\rangle = \limsup_{\nu \in \hat{X}, \nu \to \mu \text{ vaguely}} \langle g, \nu\rangle \leqq \limsup_{\nu \in \hat{X}, \nu \to \mu \text{ vaguely}} F(\nu) \leqq F(\mu),$$

and we have (iii).

Since $F(\mu) = \langle f, \mu\rangle$ for every $\mu \in \Delta(J)$ (4.3), (iii) gives us that $\langle g, \mu\rangle \leqq \langle f, \mu\rangle$ for all $\mu \in \Delta(J)$. It follows easily that $\langle g, \mu\rangle \leqq \langle f, \mu\rangle$ for all $\mu \in J_+$, hence $g_I \leqq f$. The proof of (4.4) is complete.

Replacing f by $-f$ in (4.4) gives us the corresponding theorem for lowersemicontinuity:

(4.5) *Let $J \subset L$ and $I \subset M$ be dual ideals. Then, given $f \in I$, the following are equivalent*:

1° *f is a $(l.s.c.]_I$ element.*

2° *f is vaguely lowersemicontinuous on $\Delta(J)$.*

3° *f is $w(J, C_I)$-lowersemicontinuous on $\Delta(J)$.*

Setting $J = L$ in (4.4) gives us (2.6). In the Appendix (8.1), we apply (2.6) to strengthen considerably an earlier theorem on homomorphisms of C.

5. Riemann subspaces. Let E be a vector lattice and F a linear sublattice of E. The set of elements of E each of which is simultaneously an infimum of some subset of F and a supremum of some subset of F will be called *the Dedekind closure of F in E*. We have immediately,

(5.1) *The Dedekind closure of F in E is again a linear sublattice (containing F).*

If the Dedekind closure of F in E is F itself, we will say F is *Dedekind closed in E*. The Dedekind closure of a set is always Dedekind closed.

Warning. A linear sublattice in a vector lattice E may be Dedekind closed yet not be complete (as a vector lattice), even when E is complete. C is Dedekind closed in M but in general is not complete.

Given a closed ideal I in M, consider the linear sublattices C_I and S_I. We will call the Dedekind closure of C_I in I the *Riemann subspace* of I and denote it by $\mathcal{R}(I)$; and we will call the Dedekind closure of S_I in I the *Lebesgue subspace* of I and denote it by $\mathcal{L}(I)$. These names stem from the case where I can be identified with $\mathcal{L}^\infty(\mu)$ for some $\mu \in L$. Under this identification, $\mathcal{R}(I)$ is the image in $\mathcal{L}^\infty(\mu)$ of the μ-Riemann-integrable functions on X and $\mathcal{L}(I)$ is all of $\mathcal{L}^\infty(\mu)$. In §6, we will define the "μ-Riemann-integrable" *elements* of M and see that their projection in $\mathcal{L}^\infty(\mu)$ is $\mathcal{R}(I)$. And a similar statement holds for $\mathcal{L}(I)$.

Actually, of course, the above definition is for any closed ideal I in M. For such I, $\mathcal{R}(I)$ gives the projection in I of the elements of M which are μ-Riemann-integrable for every μ in the dual ideal J of I (some precautions have to be taken); and $\mathcal{L}(I)$ gives the projection of the elements of M which are μ-Lebesgue-integrable for every $\mu \in J$. For example, let I be M itself. $C_M = C$, which is Dedekind closed in M; so $\mathcal{R}(M) = C$. (This corresponds to the classic theorem that the functions on $\{0 \leqq x \leqq 1\}$ (say) which are Riemann-Stieltjes integrable with respect to every function of bounded variation are precisely the continuous functions.) And $S_M = S$, whose Dedekind closure in M is U; so $\mathcal{L}(M) = U$.

Our interest in the present paper lies in the Riemann subspaces rather than the Lebesgue and we will have nothing more to do with the latter.

We turn to our principal theorem in Part I. From its definition, $\mathcal{R}(I)$ consists of the elements of I which are simultaneously $(\text{u.s.c.})_I$ and $(\text{l.s.c.})_I$. Combining this with (4.4) and (4.5), we obtain

(5.2) THEOREM. *Let $J \subset L$ and $I \subset M$ be dual ideals. Then $\mathcal{R}(I)$ consists precisely of those linear functionals on J which are $w(J, C_I)$-continuous, or equivalently, vaguely continuous, on $\Delta(J)$.*

Proof. Since (4.4) and (4.5) are stated for elements of I only, we have to verify that a linear functional ϕ on J which is $w(J, C_I)$-continuous on $\Delta(J)$ lies in I. Now $w(J, C_I)$ is a coarser topology than $w(J, I)$, which in turn is coarser than the norm-topology. Thus ϕ is norm-continuous on $\Delta(J)$. It is not hard to show from this that ϕ is norm-continuous on J, hence lies in its norm-dual, which is I.

Setting $I = M$ in (5.2), and using the fact that C is Dedekind closed in M, we have the non-trivial

(5.3) COROLLARY. *A linear functional on L lies in C if and only if it is vaguely continuous on Δ.*

Returning to (5.2), if $J = L_\mu$ for some $\mu \in L$, then $J = \mathcal{L}^1(\mu)$, $I = \mathcal{L}^\infty(\mu)$, and we have the theorem of the introduction.

As in the introduction, replacing $\Delta(J)$ by the unit ball $B(J)$ of J, which is larger, gives us the smaller space C_I in place of $\mathcal{R}(I)$ (by the Grothendieck theorem).

J is not the full norm-dual of I. Suppose we set $\Omega = \Omega(I)$, this norm-dual, and consider J as imbedded in Ω. Then

$$\Delta(J) \subset \Delta(\Omega) = \{\rho \in \Omega \mid \rho \geqq 0, \|\rho\| = 1\}.$$

What happens if we replace $\Delta(J)$ in (5.2) by the larger set $\Delta(\Omega)$? It turns out again that we obtain C_I in place of $\mathcal{R}(I)$. To see this, let Y be the set of extreme points of $\Delta(\Omega)$ with the topology induced by $w(\Omega, I)$. Then Y is compact and by the Kakutani Representation theorem, $I = C(Y)$ and $\Omega = L(Y)$; the above statement then follows from

(5.4) *Given a compact space Y, a norm-closed linear sublattice E of $C(Y)$* containing **1**, *and a linear functional ϕ on $L(Y)$, the following are equivalent*:

1° $\phi \in E$.

2° *ϕ is $w(L(Y), E)$-continuous on $L(Y)$.*

3° *ϕ is $w(L(Y), E)$-continuous on $B(L(Y))$.*

4° *ϕ is $w(L(Y), E)$-continuous on $\Delta(L(Y))$.*

Proof. We need only prove 4° implies 1°. We note first that ϕ is $w(L(Y), C(Y))$-continuous on $\Delta(L(Y))$, since this latter topology is finer than $w(L(Y), E)$. It follows from (5.3) that $\phi \in C(Y)$. Now from 4°, ϕ is, in particular, $w(L(Y), E)$-continuous on Y. A straightforward application of the Stone-Weierstrass theorem [6, Theorem 16.4] gives us $\phi \in E$.

Part II

6. A second approach to Riemann subspaces. A closed ideal I in M is completely determined by its dual ideal J in L or by its disjoint ideal $I' = J^{\perp}$ in M. Hence of course $\mathcal{R}(I)$ is also. We examine the relation of $\mathcal{R}(I)$ to I'.

We recall [11] that for $f \in M$, $\delta(f) = u(f) - l(f)$. Since $u(f)$ and $l(f)$ correspond to the closure and interior of a set in topology, $\delta(f)$ corresponds to the frontier of a set.

Given a closed ideal I' in M, we will call

$$\delta^{-1}(I') = \{f \in M \mid \delta(f) \in I'\}$$

the *Riemann subspace modulo I' of M*, and denote it by $\mathcal{R}(M; I')$. We establish first that $\mathcal{R}(M; I')$ is an M-subspace of M:

(6.1) *Given any norm-closed ideal H in M, $\delta^{-1}(H)$ is a norm-closed linear sublattice of M containing C.*

Proof. If $\delta(f) \in H$, $\delta(g) \in H$, then from [11, §2],

$$0 \leqq \delta(f + g) \leqq \delta(f) + \delta(g) \in H,$$

$$0 \leqq \delta(f \vee g) \leqq \delta(f) \vee \delta(g) \in H,$$

and

$$\delta(\lambda f) = |\lambda| \, \delta(f) \in H \qquad \text{for all } \lambda.$$

Thus $\delta^{-1}(H)$ is a linear sublattice. For $f \in C$, $\delta(f) = 0 \in H$. Finally, suppose $\delta(f_n) \in H$, $n = 1, 2, \cdots$, and $\lim_n \|f_n - f\| = 0$. Then

$$\|\delta(f) - \delta(f_n)\| \leqq 2 \|f - f_n\|$$

[11, (2.21)], hence $\lim_n \|\delta(f) - \delta(f_n)\| = 0$. Since H is norm-closed, $\delta(f) \in H$, and we are through.

Remark. Speaking loosely, the common occurrence in vector lattices, as in classical algebra, is that under the usual mappings linear sublattices appear as images and ideals appear as inverse images. In contrast to this, (6.1) states that for the mapping $\delta : M \to M$, the inverse image of an *ideal* is a *linear sublattice.*

If the dual ideal J of I is an L_μ for some $\mu \in L$; then $\mathcal{R}(M; I')$ is the correspond in M of the functions in ordinary function theory which are Riemann integrable with respect to μ. Hence we will use the same terminology for $\mathcal{R}(M; I')$, that is, given $f \in \mathcal{R}(M; I')$, we will say f is *Riemann integrable with respect to μ.*

If I, H are closed ideals in M with $I \subset H$, then $H' \subset I'$, hence

$$\delta^{-1}(H') \subset \delta^{-1}(I').$$

Thus we have

(6.2) *If I, H are closed ideals in M with $I \subset H$, then $\mathcal{R}(M; H') \subset \mathcal{R}(M; I')$.*
Also

(6.3) *If $\{I_\alpha\}$ are closed ideals in M, and I is the closed ideal generated by their union, then $\mathcal{R}(M; I') = \bigcap_\alpha \mathcal{R}(M; I'_\alpha)$.*

For, $I' = \bigcap_\alpha I'_\alpha$, and for any $f \in M$, $\delta(f) \in I'$ if and only if $\delta(f) \in I'_\alpha$ for all α.

(6.4) COROLLARY. *Let I be a closed ideal in M and J its dual ideal. Then $\mathcal{R}(M; I')$ consists of those elements of M which are Riemann integrable with respect to every $\mu \in J$.*

For J is the closed ideal generated by $\bigcup_{\mu \in J} L_\mu$, hence I is the closed ideal generated by $\bigcup_{\mu \in J} (L_\mu)^{\perp\prime}$.

We turn to the relation between $\mathcal{R}(M; I')$ and $\mathcal{R}(I)$.

(6.5) $\mathcal{R}(M; I')_I \subset \mathcal{R}(I)$.

Proof. Consider $f \in \mathcal{R}(M; I')$. Since $l(f) \leqq f \leqq u(f)$, we have $l(f)_I \leqq f_I \leqq u(f)_I$. But $u(f) - l(f) \in I'$, hence $l(f)_I = u(f)_I =$ (therefore) f_I. Thus f_I is both a (l.s.c.)$_I$ and an (u.s.c.)$_I$, hence lies in $\mathcal{R}(I)$.

We have two important cases in which $\mathcal{R}(M; I')$ projects *onto* $\mathcal{R}(I)$.

(6.6) *If the dual ideal J of I is an L_μ for some $\mu \in L$, then $\mathcal{R}(M; I')_I = \mathcal{R}(I)$.*

Proof. From (6.5) we need only show $\mathcal{R}(I) \subset \mathcal{R}(M; I')_I$. Consider $f \in \mathcal{R}(I)$. f is both a supremum of some subset of C_I and an infimum of some subset of C_I. Using the norm $\|\cdot\|_\mu$ defined on I by μ [8, §11] ($\| g \|_\mu = \langle | g |, | \mu | \rangle$ for all $g \in I$), we can obtain two sequences $\{g_n\}$, $\{h_n\}$ in C_I such that $g_n \uparrow f$, $h_n \downarrow f$. For each n, choose $\bar{g}_n, \bar{h}_n \in C$ such that $g_n = (\bar{g}_n)_I$, $h_n = (\bar{h}_n)_I$.

(i) *$\{\bar{g}_n\}$, $\{\bar{h}_n\}$ can be chosen to satisfy*

$$\bar{g}_1 \leqq \cdots \leqq \bar{g}_n \leqq \cdots \leqq \bar{h}_n \leqq \cdots \leqq \bar{h}_1$$

The selection is carried out inductively. Choose any $\bar{g}_1 \in C$ such that $(\bar{g}_1)_I = g_1$. Choose any $h \in C$ such that $h_I = h_1$, then set $\bar{h}_1 = h \vee \bar{g}_1$, whence $(\bar{h}_1)_I = h_I \vee (\bar{g}_1)_I = h_1 \vee g_1 = h_1$. Assume $\bar{g}_1, \cdots, \bar{g}_n, \bar{h}_1, \cdots, \bar{h}_n$ have been chosen to satisfy (i). Choose any $g \in C$ such that $g_I = g_{n+1}$, then set

$$\bar{g}_{n+1} = (g \vee \bar{g}_n) \wedge \bar{h}_n,$$

whence

$$(\bar{g}_{n+1})_I = (g_I \vee (\bar{g}_n)_I) \wedge (\bar{h}_n)_I = (g_{n+1} \vee g_n) \wedge h_n = g_{n+1}.$$

This establishes (i).

From (i), $g = \bigvee_n \bar{g}_n$ and $h = \bigwedge_n \bar{h}_n$ both exist and satisfy $g \leqq h$. Clearly $g_I = h_I = f$; we show that $\delta(g) \in I'$, which will complete the proof. h is u.s.c., hence $h \geqq u(g)$, and g is l.s.c., hence $g = l(g)$. Thus $\delta(g) = u(g) - l(g) \leqq h - g$. But $h - g \in I'$ (since $(h - g)_I = h_I - g_I = 0$), and we are through.

(6.7) *If the dual J of I is vaguely dense in L, or equivalently, if* $C \cap I' = 0$, *then* $\mathcal{R}(M; I')_I = \mathcal{R}(I)$.

Proof. From $C \cap I' = 0$, it follows that the projection $C \to C_I$ is one-one, hence a vector-lattice isomorphism. Turning to the proof, we again need only show $\mathcal{R}(I) \subset \mathcal{R}(M; I')_I$. Given $f \in \mathcal{R}(I)$, consider the two sets

$$A = \{g \in C \mid g_I \leqq f\}, \qquad B = \{h \in C \mid h_I \geqq f\}.$$

From the isomorphism of C with C_I, every element of A is $\leqq$ every element of B, hence $g = \bigvee A$ and $h = \bigwedge B$ exist and satisfy $g \leqq h$. The remainder of the proof is the same as that of (6.6).

In general, we do *not* have $\mathcal{R}(M; I')_I = \mathcal{R}(I)$. We give an example. It is based on the fact that a compact space can have a non-normal subspace. Following [6, page 74] let ω be the first infinite ordinal, ω_1 the first uncountable ordinal, N the set of natural numbers,

$$N^* = N \cup \{\omega\}, \quad W = \{\alpha \mid \alpha < \omega_1\} \quad \text{and} \quad W^* = W \cup \{\omega_1\}.$$

Endow W^* and N^* with the order topology and take

$$X = W^* \times N^* \quad \text{and} \quad K = (W^* \times \{\omega\}) \cup (\{\omega_1\} \times N^*).$$

Since K is a compact subset of X, $M(K)$ can be identified with a closed ideal in M, hence the elements of $M(K)$ disjoint from the characteristic element of the single point (ω_1, ω) is also a closed ideal in M. We denote it by I. We show there exists $f \in I$ such that $f = g_I = h_I$ for some g l.s.c. and some h u.s.c. of M but that no such g and h can be found which also satisfy $g \leqq h$.

It is enough to show there exists a function f on the set $H = K \backslash \{(\omega_1, \omega)\}$ such that $f = g \mid H = h \mid H$ for some lowersemicontinuous function g on X and uppersemicontinuous function h on X but that no such g and h can be found which satisfy $g(x) \leqq h(x)$ for all $x \in X$. In short, the rest of the discussion is in ordinary function theory.

Let h be the characteristic function of $W^* \times \{\omega\}$, g the characteristic function of $X \backslash (\{\omega_1\} \times N^*)$, and f the characteristic function of $W \times \{\omega\}$. Then $f = g \mid H = h \mid H$. Suppose a g and h could be found with these properties and which also satisfied $g(x) \leqq h(x)$ for all $x \in X$. For every $n \in N$, $h(\omega_1, n) = f(\omega_1, n) = 0$, hence

$$\limsup_{\alpha \to \omega_1} h(\alpha, n) \leqq 0,$$

hence

$$\limsup_{\alpha \to \omega_1} g(\alpha, n) \leqq 0.$$

It follows there exists α_0 such that $g(\alpha, n) \leqq \frac{1}{2}$ for all n and all $\alpha \geqq \alpha_0$. This contradicts the fact that for every $\alpha < \omega_1$, $\liminf_{n \to \omega} g(\alpha, n) \geqq 1$.

In studying dual ideals J and I, there is no loss in confining ourselves to the support of J, instead of dealing with all of X (cf. (3.1) above), and we will do this. This is equivalent to assuming J is vaguely dense in L. We state it formally:

Henceforth, unless otherwise stated, J will always be a closed ideal which is vaguely dense in L.

As a consequence, we will always have $C \cap I' = 0$, C_I isomorphic with C, and $\mathcal{R}(M; I')_I = \mathcal{R}(I)$.

7. Zero content. I is a fixed closed ideal in M with $C \cap I' = 0$. We will denote $\mathcal{R}(M; I') \cap I'$ by $N(I)$, and its elements will be said to have *content zero with respect to I*. Since $\mathcal{R}(M; I')$ and I' are both norm-closed, $N(I)$ is also. Clearly $\mathcal{R}(M; I')/N(I)$ is isomorphic with $\mathcal{R}(I)$.

Before stating (7.1) we remark that one easy consequence of the condition $C \cap N = 0$ is that if an u.s.c. element is in I', then it is actually in $(I')_+$.

(7.1) *The following are two alternate definitions for $N(I)$:*
(a) $N(I) = \{f \in M \mid u(|f|) \in I'\}$.
(b) *$N(I)$ is the ideal generated by the u.s.c. elements in I'.*

Proof. Consider $f \in N(I)$; we show $u(|f|) \in I'$. $u(|f|) \leqq |f| + \delta(|f|)$, hence it is enough to show $|f|$ and $\delta(|f|)$ are in I'. That $|f| \in I'$ follows from $f \in I'$; that $\delta(|f|) \in I'$ follows from $\delta(f) \in I'$ and $\delta(|f|) \leqq \delta(f^+) + \delta(f^-) = \delta(f)$ [11, §2]. Next suppose $u(|f|) \in I'$. Since $u(|f|)$ is an u.s.c. element and $|f| \leqq u(|f|)$, f lies in the ideal defined in (b). Finally suppose f is in this last ideal. Then, from the remark preceding the present proposition, $u(|f|)$ is also. The inequality $\delta(f) \leqq 2u(|f|)$ then gives us that $f \in \mathcal{R}(M; I')$, hence

$$f \in \mathcal{R}(M; I') \cap I' = N(I),$$

and we are through

Since $\delta(f)$ is u.s.c., it follows from (7.1) that for any $f \in M$, $\delta(f) \in I'$ if and only if $\delta(f) \in N(I)$. Thus, setting $\mathcal{R}(M; N(I)) = \delta^{-1}(N(I))$, we have $\mathcal{R}(M; I') = \mathcal{R}(M; N(I))$. This gives in turn that the Riemann subspace $\mathcal{R}(I)$ of I is completely determined by $N(I)$. In particular, given two closed ideals I, H, if $N(I) = N(H)$, then $\mathcal{R}(I)$ is isomorphic with $\mathcal{R}(H)$.

We examine this in more detail. For convenience we denote $N(I)$ simply by N. Suppose the closure $\bar{N}$ of N is strictly smaller than I'. Then $N'(=\bar{N}')$ is strictly larger than I. Now it is easily shown that $N(N') = N$, whence $\mathcal{R}(N')$ is isomorphic with $\mathcal{R}(I)$; more exactly, the projection of N' onto I gives an isomorphic mapping of $\mathcal{R}(N')$ onto $\mathcal{R}(I)$. Moreover N' is the largest closed ideal with this property, that is, if I, H both give rise to N, then $I, H \subset N'$ and $R(N')$ projects isomorphically onto $R(I)$ and $R(H)$.

We can obtain more insight into N' by looking at its dual ideal $N^{\perp}$. The following theorem gives some characterizations of $N^{\perp}$, but because it has an independent interest, we state it in terms of a general ideal G of L.

(7.2) *Given an ideal G in L, the following ideals are identical:*
$G_1 = N^{\perp}$, *where N is the ideal generated by the u.s.c. elements in* $(G^{\perp})_+$;
$G_2 =$ *the* $w(L, S)$*-closure of* G;
$G_3 =$ *the* $w(L, B_o)$*-closure of* G;
$G_4 =$ *the* $w(L, U)$*-closure of* G.
For need of a name, we will call this ideal the $\omega(L, U)$*-closure of* G.

Proof. We have

$$G_2 = (S \cap G^{\perp})^{\perp}, \quad G_3 = (B_o \cap G^{\perp})^{\perp} \quad \text{and} \quad G_4 = (U \cap G^{\perp})^{\perp}.$$

Set $A = \{f \text{ u.s.c.} \mid f \in (G^{\perp})_+\}$. Then

$$A \subset N \subset S \cap G^{\perp} \subset B_o \cap G^{\perp} \subset U \cap G^{\perp},$$

hence $G_4 \subset G_3 \subset G_2 \subset N^{\perp} \subset A^{\perp}$. We show $A^{\perp} \subset G_4$, which will give equality and complete the proof. Consider $\mu \in A^{\perp}$; we show $\langle g, \mu \rangle = 0$ for all $g \in U \cap G^{\perp}$. It is enough to show this for $g \in (U \cap G^{\perp})_+$. Since $g \in U$, there exists a net $\{g_\alpha\}$ in M_+, consisting of u.s.c. elements, such that $g_\alpha \uparrow g$ [8, (7.3)]. Since μ is continuous on M, it follows $\langle g, \mu \rangle = \lim_\alpha \langle g_\alpha , \mu \rangle$. But $g_\alpha \in G^{\perp}$, hence $g_\alpha \in A$, hence $\langle g_\alpha , \mu \rangle = 0$ for all α, and we are through.

Summing up the above discussion, if we wish to study the various Riemann subspaces of M, the ideals of M dual to $w(L, U)$-closed ideals of L seem to be the natural domain for such study. With this in view the following theorems become of interest.

(7.3) *Every principal closed ideal of L is* $w(L, U)$*-closed.*

This was proved in [9, (5.5)].

(7.4) *Given an ideal G in L, if $x \in X$ is not in G, then it is not in the* $w(L, U)$*-closure of G.*

Otherwise stated, the intersection of X with the $w(L, U)$-closure of G is the same as its intersection of G. To prove this, suppose $x \notin G$ and let f be the characteristic element of x. Since f is u.s.c., it is an element of U not vanishing on x. Thus we need only show it vanishes on G. The one-dimensional linear subspace $R \cdot x$ generated by x is a closed ideal in L, hence $L = (R \cdot x) + (R \cdot x)'$, hence $G \subset (R \cdot x)'$. Since clearly $f \in (R \cdot x)'^{\perp}$, we have $f \in G^{\perp}$.

An ideal in L_0 always contains points of X; hence if an ideal in L intersects L_0, it contains points of X. Combining this with (7.4), we obtain the following corollary. Note first that the decomposition $L = L_0 \oplus L_1$ gives $G = (G \cap L_0) \oplus (G \cap L_1)$, hence the intersection of G with L_0 is also its projection G_0 in L_0.

(7.5) *Let G be an ideal in L and F its* $w(L, U)$*-closure. Then* $F_0 = G_0$.

This gives in turn

(7.6) L_1 *is* $w(L, U)$*-closed.*

In contradistinction, L_0 is $w(L, U)$-dense in L, since it is separating on U [8, (8.3)]. (Thus, interestingly, Δ is not only the vague closure of $\hat{X}$, it is also its $w(L, U)$-closure.)

We close this section with some examples.

Example 1. Let $I = M$. On the one hand, $\mathcal{R}(M) = C$; on the other $M' = 0$, giving $N(M) = 0$ and $\mathcal{R}(M; 0) = C$ ($\delta(f) = 0$ if and only if $f \in C$).

Example 2. Let I be the dual ideal of a principal closed ideal $J = L_\mu$ (J need not be vaguely dense for this example). This is the example from which our terminology comes. $N(I)$ corresponds in function theory to the functions whose Riemann integral (with respect to μ) is zero, $\mathcal{R}(M; I')$ to the functions which are Riemann integrable, and $\mathcal{R}(M; I')/N(I)$ and $\mathcal{R}(I)$ to the image in $\mathcal{L}^\infty(\mu)$ of the Riemann integrable functions.

We point out here how our approach gives a complete parallel between the Riemann integral and Jordan content. A set has content zero if its closure has measure zero; a non-negative element f of M has content zero in our terminology if $u(f)$ satisfies $\langle u(f), \mu \rangle = 0$. A set has content (is Jordan measurable) if its frontier has content zero; an element f of M lies in $\mathcal{R}(M; I')$ if $\delta(f)$ has content zero. Thus the "Riemann integrable" elements of M are those which "have content". (And we could make the same statement if we were confining ourselves to functions.)

Thus in our approach, the unsatisfying mixture in Lebesgue's theorem of Riemann integrability and Lebesgue measure has disappeared. The crucial point is the following. The frontier of a set is always closed, hence for it, having measure zero implies having content zero. Similarly $\delta(f)$ is always u.s.c., hence for it, lying in I' implies lying in $N(I)$. *However, even though* $\delta(f)$ *is u.s.c., the set* $\{x \mid \delta(f)(x) \neq 0\}$ *is not necessarily closed.*

Example 3. Let A be an arbitrary subset of X (again for the example, we need not assume A is dense in X). The closed ideal J generated by A as a subset of L is in L_0 and can be identified with $l^1(A)$, hence its dual ideal I is in M_0 and can be identified with $l^\infty(A)$. So for convenience of discussion, we confine ourselves to M_0 and consider it as the space of all bounded real functions on X. Then C_I consists of the restrictions to A of the continuous functions on X, and $\mathcal{R}(I)$ consists of the bounded continuous functions on A. $N(I)$ (really $N(I) \cap M_0$) is the ideal generated by the non-negative upper-semicontinuous functions vanishing on A. Finally, $\mathcal{R}(M_0 ; I')$ consists of the bounded functions whose points of discontinuity all lie in $X \backslash A$.

In order to work with the $w(L, U)$-closure of J, we cannot confine ourselves to L_0 and M_0, since this closure does not in general lie in L_0. However, from the discussion following (7.1), the Riemann subspace of the ideal dual to this $w(L, U)$-closure is still isomorphic with $\mathcal{R}(I)$ above, that is, with the space of bounded continuous functions on A.

Example 4. Let $J = L_1$, the ideal of diffuse Radon measures; hence

$I = M_1$. As we have pointed out, L_1 is $w(L, U)$-closed (7.6). Moreover, if X has no isolated points, L_1 is vaguely dense in L; we will assume this is the case (thus $C_1(=C_{M_1})$ is isomorphic with C).

$\mathcal{R}(M_1)$ is in general strictly larger than C_1. For example, if $X = \{0 \leqq x \leqq 1\}$ and f is the characteristic element of the closed set $\{x \in X \mid 0 \leqq x \leqq \frac{1}{2}\}$, then f_1 is an element of $\mathcal{R}(M_1)$ which is not in C_1. The exact description of $\mathcal{R}(M_1)$ is simple. It is the projection in M_1 of $\mathcal{R}(M; M_0)$, that is, of the elements of M which are Riemann integrable with respect to every diffuse measure (6.4).

We now have essentially obtained a theorem of C. Goffman (oral communication)—or rather its correspond in M. Goffman's theorem runs as follows:

(7.7) (Goffman) *A bounded real function on a closed interval is Riemann integrable with respect to every diffuse (i.e. purely non-atomic) regular measure if and only if its set of points of discontinuity is countable.*

We obtain this theorem (with "bounded real function" replaced by "element of M") from our above description of $\mathcal{R}(M; M_0)$ and the

(7.8) LEMMA. *Let $X = \{a \leqq x \leqq b\}$ and f be an u.s.c. element in M_0. Then f vanishes at all but a countable number of x's.*

Proof. $f \geqq 0$, from the remark preceding (7.1). If $f(x) \neq 0$ for an uncountable number of x's, then there exists $\lambda > 0$ such that $f(x) \geqq \lambda$ on an uncountable set A, which is moreover closed. But A contains a perfect set, hence supports some diffuse measure μ. Since then $\langle f, \mu \rangle > 0$, this contradicts the hypothesis that $f \in M_0$.

Appendix

8. Let E, F be two M-spaces with strong order units, both of which we denote by $\mathbf{1}$. By an *M-homomorphism*, or simply *homomorphism*, $h : E \to F$, we will mean a linear mapping which preserves the operations $\vee$, $\wedge$ and satisfies $h\mathbf{1} = \mathbf{1}$. Let X, Y be compact spaces. Given a homomorphism

$$h : C(X) \to C(Y),$$

let $h^t : L(Y) \to L(X)$ and $h^{tt} : M(X) \to M(Y)$ denote its transpose and bitranspose. h^{tt} is also a homomorphism and moreover is order-continuous [11].

By a theorem of M. H. Stone [16], for each continuous mapping $q : Y \to X$, the "transpose" $h : C(X) \to C(Y)$ defined by $(hf)(y) = f(qy)$ for all $f \in C(X)$ and $y \in Y$ is a homomorphism, and conversely every homomorphism h is the "transpose" of a continuous mapping q (specifically, $q = h^t \mid Y$). Moreover, h is one-one (hence an isomorphism into) if and only if q is onto.

We noted in [11, (6.7)] that, given a homomorphism

$$h : C(X) \to C(Y),$$

h^{tt} carries u.s.c. elements of $M(X)$ into u.s.c. elements of $M(Y)$. We now show that if h is one-one, the converse also holds, a property which we could state in [11, (12.1)] only under a strong additional condition.

(8.1) *Let $h: C(X) \to C(Y)$ be an isomorphism of $C(X)$ into $C(Y)$. Then for $f \in M(X)$, $h^{tt}f$ u.s.c. implies f u.s.c.*

Proof. By (2.6) we need only show that f is vaguely uppersemicontinuous on $\Delta(L(X))$. Suppose not; then there exists $\mu \in \Delta(L(X))$, $\lambda > \langle f, \mu\rangle$, and a net $\{\mu_\alpha\} \subset \Delta(L(X))$ such that $\mu_\alpha \to \mu$ vaguely and $\langle f, \mu_\alpha\rangle \geqq \lambda$ for all α. Since h is one-one, h^t carries Y *onto* X, as we have pointed out above; it follows easily that h^t carries $\Delta(L(Y))$ onto $\Delta(L(X))$. Thus, for each α we can choose $\nu_\alpha \in \Delta(L(Y))$ such that $h^t\nu_\alpha = \mu_\alpha$. Since $\Delta(L(Y))$ is vaguely compact, $\{\nu_\alpha\}$ has a convergent subnet; for simplicity of notation, we assume $\{\nu_\alpha\}$ itself is convergent: $\nu_\alpha \to \nu$ vaguely, for some $\nu \in \Delta(L(Y))$. Since h^t is vaguely continuous, $h^t\nu = \mu$. But then we have

$$\langle h^{tt}f, \nu_\alpha\rangle = \langle f, h^t\nu_\alpha\rangle = \langle f, \mu_\alpha\rangle \geqq \lambda \qquad \text{for all} \quad \alpha,$$

while

$$\langle h^{tt}f, \nu\rangle = \langle f, h^t\nu\rangle = \langle f, \mu\rangle < \lambda.$$

This contradicts the fact that $h^{tt}f$ is vaguely uppersemicontinuous on $\Delta(L(Y))$ (2.6).

Remark (8.1) of course also holds with "u.s.c." replaced by "l.s.c.".

Bibliography

1. A. Alexiewicz, *On Hausdorff classes*, Fund. Math., vol. 34 (1947), pp. 61–65.
2. H. Bauer, *Über die Beziehungen einer abstrakten Theorie des Riemann-Integrals zur Theorie Radonscher Masse*, Math. Zeit., vol. 65 (1956), pp. 448–482.
3. N. Bourbaki, *Integration*, Éléments de Mathématique, Livre VI, Chapters I–IV, Hermann, Paris, 1952.
4. C. Caratheodory, *Über die Fourierschen Koeffizienten der nach Riemann integrierbaren Funktionen*, Math. Zeit., vol. 1 (1918), pp. 309–320.
5. K. Geba and Z. Semadeni, *On the M-subspaces of the Banach spaces of continuous functions*, Zeszyty Naukowe Universytetu Im. A. Mickiewicza, vol. 2 (1960), pp. 53–68.
6. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, New York, 1960.
7. O. Haupt et Chr. Pauc, *Mesure et topologie adaptées. Espaces mesurés topologiques*, C.R. Acad. Sci. Paris, vol. 230 (1950), pp. 711–712.
8. S. Kaplan, *On the second dual of the space of continuous functions*, Trans. Amer. Math. Soc., vol. 86 (1957), pp. 70–80.
9. ———, *The second dual of the space of continuous functions II*, Trans. Amer. Math. Soc., vol. 93 (1959), pp. 329–350.
10. ———, *The second dual of the space of continuous functions III*, Trans. Amer. Math. Soc., vol. 101 (1961), pp. 34–51.
11. ———, *The second dual of the space of continuous functions IV*, Trans. Amer. Math. Soc., vol. 113 (1964), pp. 512–546.
12. ———, *Closure properties of C(X) in its second dual*, Proc. Amer. Math. Soc., vol. 17 (1966), pp. 401–406.

13. L. H. LOOMIS, *Linear functionals and content*, Amer. J. Math., vol. 76 (1954), pp. 168–182.
14. S. MARCUS, *La mesure de Jordan et l'integral de Riemann dans un espace mesuré topologique*, Acta Sci. Math., vol. 20 (1959), pp. 156–163.
15. Z. SEMADENI, *Functions with sets of points of discontinuity belonging to a fixed ideal*, Fund. Math., vol. 52 (1963), pp. 25–39.
16. M. H. STONE, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., vol. 41 (1937), pp. 375–481.

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