STRUCTURE OF MONOGENIC GROUPS

BY

M. Rajagopalan

Rubel [9] introduced the notion of a monogenic locally compact abelian group recently. This paper describes the structure of such groups. We introduce the notions of topologically divisible groups, canonical monogenic groups and amalgam of topological groups. We show that a totally disconnected monogenic locally compact abelian group is a direct product $L \times K$ where L is a topologically divisible group and K is either a compact monothetic group or a canonical monogenic group. If G is monogenic but not totally disconnected then it is either of the form $L \times K$ where L is topologically divisible and K is compact monothetic or is an amalgam $L + K$ of a compact monothetic group K and a group L such L/L_0 is topologically divisible where L_0 is the connected component of identity of L. The structure of topologically divisible groups and of canonical monogenic groups are described. In the totally disconnected case the structure is an exact generalization of the result of [7]. The notion of amalgam was discussed by B. H. Neumann for discrete groups.

Conventions and Notations. All groups occurring in this paper are assumed to be locally compact Hausdorff and abelian groups. All notions in abstract abelian groups are to be found in [3] and [4]. All notions in topological groups which are not defined here are to be found in [5] or [10]. $R^n (n \geq 0)$ denotes the usual real Euclidean group. If p is a prime then I_p^* denotes the group of p-adic integers, and J_p the group of p-adic numbers.

Definition 1. Let G be a group. A compact character χ of G is a continuous character of G which is also open.

Definition 2 (Rubel). A group G is called monogenic if there exists an element $x_0 \in G$ such that whenever H is a subgroup of G such that G/H is compact we have that $\varphi(x_0)$ generates G/H topologically where $\varphi : G \to G/H$ is the canonical map. Such an element x_0 is called a special element of G.

Definition 3. A group G is called topologically divisible if the only compact character of G is the identity character. (See also [8]).

Note 4. The group J_p is topologically divisible for all primes p and every discrete divisible group is also topologically divisible. A group G is topologically divisible if and only if whenever H is a closed subgroup of G such that G/H is compact we have that $H = G$. Loosely speaking a group G is topologically divisible if and only if it admits no non-trivial compact quotient groups. In this sense our definition of topologically divisible groups generalizes the

Received January 16, 1967.

notion of abstract divisible groups. Dixmier [2] and Ahern and Jewett [1] have described the class of injective locally compact abelian groups. From their results it follows that the only group which is topologically divisible and is also injective in the class of all locally compact abelian groups is the group containing only one element. But abstract divisible abelian groups are injective in the class of all discrete abelian groups and also have no non-trivial finite quotients. So we find that in the nondiscrete case no group has both these corresponding properties. We shall see later that every topologically divisible group can be obtained from the discrete divisible groups and the groups J_p by using duals, subgroups and products. Finally we note that every monogenic group contains a compact, open subgroup.

LEMMA 5. *If G is a monogenic group and H is a closed subgroup of G then G/H is also monogenic. If L is a topologically divisible closed subgroup of a group G and G/L is monogenic then G is monogenic.*

Proof. The first remark is easy to prove. The restriction to L of a compact character χ of G is a compact character of L. So we get the second statement.

DEFINITION 6. An element x_0 of a group G is said to be of compact order if the closed subgroup generated by x_0 is compact. If x_0 is not of compact order it is said to be of discrete order.

Remark. If an element x_0 of a group G is of discrete order then the closed subgroup that it generates is isomorphic to the group Z of integers.

LEMMA 7. *A group G is topologically divisible if and only if the dual G^* is torsion free as an abstract group and consists of elements of compact order only. If G is a topologically divisible group then G/H is also topologically divisible for every closed subgroup $H \subset G$. In particular if H is an open subgroup then G/H is divisible. If G is a topologically divisible group then it is totally disconnected.*

Proof. All statements are easy to prove.

LEMMA 8. *Let H be a compact, open subgroup of a group G. Let $G = H \times D$ where D is a discrete subgroup of G. Consequently if G is a topologically divisible group then it is of the form $S \times H$ where S is a discrete torsion free divisible group and H is topologically divisible and each element of H is of compact order.*

Proof. The statement is an easy consequence of theorem 25.21 page 410 of [12].

DEFINITION 9. Let p be a prime and K_p an index set. For each $i \in K_p$ let J_p^i be a group isomorphic to J_p. Let $\prod_{i \in K_p} J_p^i$ be the cartesian product of the abstract groups J_p^i. Let $I_p^{\#i}$ be an open, compact, subgroup for J_p^i for each $i \in K_p$. Then $I_p^{\#i}$ is isomorphic to $I_p^{\#}$ for each $i \in K_p$. Then $I_p^{\#i}$ is isomorphic to $I_p^{\#}$ for each $i \in K_p$. Let

$$A_p = \prod_{i \in K_p} I_p^{\#i}.$$

Let

$$B_p = \{x \mid x \in \textstyle\prod_{i \in K_p} J_p^i \quad \text{and} \quad nx \in A_p \quad \text{for some integer} \quad n\}.$$

Then B_p is divisible as an abstract group. (See also page 419 of [12].) We define a canonical p-group to be a subgroup $H_p \subset B_p$ such that $H_p \supset A_p$ and H_p is given a topology τ in which H_p is a topological group and A_p is compact and open and the relative topology on A_p as a subspace of (H_p, τ) coincides with the product topology of $\prod_{i \in K_p} I_p^{*i}$.

DEFINITION 10 (Vilenkin and Braconnier). Let K be an index set and let G_i be a group for each $i \in K$. Let $H_i \subset G_i$ be a compact, open subgroup for each $i \in K$. By the weak direct sum of the groups G_i modulo H_i we mean the group G defined as follows: As an abstract group $G \subset \prod_{i \in K} G_i$ and consists exactly of those elements whose i^{th} coordinate is in H_i except for a finite number of indices. The topology on G is so given that it is made a topological group and the group $\prod_{i \in K} H_i$ with its product topology becomes a compact, open subgroup of G. (Also see page 56–57 of [12].)

DEFINITION 11. Let p be a prime. In Definition 9 we introduced the notion of a canonical p-group H_p. H_p has a compact, open subgroup $\prod I_p^{*i}$ which we may call A_p. Let $\mathcal{P}$ be a collection of primes. Then we can talk about the weak direct sum of the groups H_p modulo A_p where $p \in \mathcal{P}$, as in Definition 10. We call this sum, a weak direct sum of canonical p-groups.

THEOREM 12. *Let G be a topologically divisible group where every element is of compact order. Then G is the dual of a group G^* which is the weak direct sum of canonical p-groups. The converse is also true. Consequently a group H is topologically divisible if and only if it is of the form $D \times G$ where D is a discrete torsion free divisible group and G is the dual of a group G^* as in the first statement.*

Proof. Now we notice that a canonical p-group is totally disconnected and torsion free as an abstract group and consists of elements of compact order only. From this it follows easily that if G^* is a weak direct sum of canonical p-groups then it consists of elements of compact order only and is torsion free as an abstract group and is totally disconnected. From this and Lemma 7 it follows that the group G which is dual of G^* is topologically divisible. Since G^* is totally disconnected we get that all elements of G are of compact order. This proves the converse of the first statement of the theorem.

Now let G be a topologically divisible group where all elements are of compact order. Let G^* be its dual. Let $H \subset G$ be a compact open subgroup of G which exists by Note 4 and Lemma 5. Then G/H is a discrete divisible group which is a torsion group because all elements of G are of compact order. So there is a collection $\mathcal{P}$ of primes and index sets K_p for each $p \in \mathcal{P}$ such that $G/H = \sum_{p \in \mathcal{P}} \sum_{i \in K_p} Z(p_\infty^i)$ where $Z(p_\infty^i)$ is the p-primary part of the group of rationals modulo 1 for each $i \in K_p$ and the sum $\sum \sum Z(p^i_\infty)$ is the weak direct sum of the abstract groups concerned. Since $H^\perp \subset G^*$ is the dual of G/H we get $H^\perp = \prod_{i \in \mathcal{P}} \prod_{i \in K_p} (I^\#_p)^i$ where $(I^\#_p)^i$ is isomorphic to $I^\#_p$ for all $i \in K_p$. Now G^* is torsion free by Lemma 7 and since G is totally disconnected every element of G^* is of compact order. Now put $L_p = \prod_{i \in K_p} (I^\#_p)^i$ for each $p \in \mathcal{P}$. Then L_p is a compact subgroup of G^* and is divisible for every prime $q \neq p$. Let

$$H_p = \{x \mid x \in G^*; \quad p^k x \in L_p \quad \text{for some} \quad k = 0, 1, 2, \cdots \}.$$

Then H_p is a closed subgroup of G^* for each $p \in \mathcal{P}$ and we get from the previous remarks that G^* is the weak direct sum of the groups H_p modulo L_p where $p \in \mathcal{P}$.

Now it is easy to see that H_p is topologically isomorphic to a canonical p-group in a natural way. So we find that G^* is the weak direct sum of canonical p-groups. From this and Lemma 8 we get the theorem.

LEMMA 13. *Let G be a group and $H \subset G$ a closed subgroup which is a weak direct sum of canonical p-groups. Let G/H be finite cyclic and p-primary. Then G is either a weak direct sum of canonical p-groups or it is of the form $L \times T$ where $L \supset H$ and is a weak direct sum of canonical p-groups and T is finite cyclic. Actually T is the torsion subgroup of G.*

THEOREM 14. *Let G be a monogenic totally disconnected group with a special element x_0 of compact order. Then G is the direct product $L \times H$ of a compact monogenic group H and a topologically divisible group L.*

Proof. This follows by repeated use of Lemma 13, Theorem 12 and the fact that a discrete divisible subgroup is a direct summand of every abstract abelian group in which it is contained.

DEFINITION 15. Let $\mathcal{P}$ be a collection of primes. For each $p \in \mathcal{P}$ let G_p be either a finite p-primary cyclic group with discrete topology or the group $I^\#_p$ of p-adic integers. Let $G = \prod_{p \in \mathcal{P}} G_p$. Let $\bar{x} = (x_p)$ be an element of G whose p^{th} coordinate is x_p for all $p \in \mathcal{P}$. $\bar{x}$ is said to be a main diagonal of G if the closed subgroup generated by x_p is G_p for all $p \in \mathcal{P}$. Note that G is a typical o-dimensional compact monothetic group in the product topology and $\bar{x}$ its generator.

DEFINITION 16. A canonical monogenic group H is a locally compact group of the following type:

As an abstract group, H is a subgroup (not necessarily closed) of a group G as in Definition 15. Moreover H should be pure subgroup of the abstract group G containing a main diagonal and the torsion subgroup of G. Now we take some closed subgroup $F \subset G$ such that $F \subset H$. H is now given a topology τ such that (H, τ) is a topological group and F is an open subset of (H, τ) and the relative topology on F as a subspace of (H, τ) coincides with that when F is treated as a subspace of G.

The reason for using the term canonical monogenic group in the above definition is the following lemma:

LEMMA 17. *A canonical monogenic group G is a locally compact group and is a monogenic group with a main diagonal x_0 as a special element. The group G is also monogenic when treated as a discrete group. If $L \subset G$ is a compact open subgroup and $x_0 \notin L$ and is of infinite order then it is of discrete order.*

Proof. That G is locally compact is obvious from Definition 16. Now let $G \subset H$ where $H = \prod_{p \in \mathcal{P}} G_p$ as in Definition 15. Then the main result of [7] shows that G is monogenic as a discrete group with a main diagonal x_0 of H as a special element. Now every compact character χ of G maps G onto a finite subgroup of the circle. So it follows that G is monogenic as a locally compact group with x_0 as a special element. The third statement is obvious.

LEMMA 18. *Let G be a canonical monogenic group. Then nG is an open subgroup of G for all integers $n = 1, 2, 3, \cdots$. Moreover the set of compact characters separates points of G.*

Proof. Let $G \subset H$ be as in Definition 16. Then every continuous character χ of H gives a compact character of G by restriction. Now let $L \subset G$ be the compact, open subgroup of G as in the Definition 16. Then the dual L^* of L is isomorphic to a subgroup of the circle and L^* is a torsion group. So nL is an open subgroup of L and hence is an open subgroup of G for all integers $n = 1, 2, 3, \cdots$. So nG is an open subgroup of G for all $n = 1, 2, 3, \cdots$.

LEMMA 19. *Let G be a monogenic group with a special element of discrete order. Let $H \subset G$ be an open subgroup which is topologically divisible and such that G/H is a canonical monogenic group. Then G is of the form $H \times L$ where L is a discrete, canonical monogenic group.*

Proof. Let $K \subset H$ be a compact open subgroup of G which exists by Lemma 7 and the fact that H is open. Let $\varphi : G \to G/H$ be the canonical map and $\tilde{x}_0 \in G/H$ an element of finite order. Let $x_0 \in G$ be such that $\varphi(x_0) = \tilde{x}_0$. Let M be the subgroup of G generated by H and x_0. Then M is seen to be monogenic with x_0 as a special element. It is also clear that x_0 is of compact order. A close look at the proof of Theorem 14 gives that $M = H \times S$ where S is a cyclic group with generator y_0 and y_0 has the same order as $\tilde{x}_0$. Then it follows that $\varphi(y_0) = \tilde{x}_0$. Now H/K is divisible and hence $G/K = H/K \times G/H$. Let $\psi : G \to G/K$ be the canonical map and $F = \psi^{-1}(G/H)$. Then the previous argument shows that K is a pure, compact, open subgroup of F. So by Lemma 8 we have that $F = K \times L$ for some discrete subgroup L. Then it is obvious that $G = H \times L$.

THEOREM 20. *Let G be a monogenic group with a special element x_0 of discrete order. Then $G = M \times L$ where M is a closed subgroup of G which is* *topologically divisible and L is a closed subgroup of G which is a canonical monogenic group.*

Proof. First of all we observe that since the special element x_0 is of discrete order the group G should be totally disconnected. So the set of compact characters of G coincides with the torsion subgroup τG^* of the dual G^* of G. Let H be the annihilator of τG^* in G. Let $\varphi : G \to G/H$ be the canonical map. Let $\tilde{x}_0 = \varphi(x_0)$. Then $\tilde{x}_0$ separates τG^*. So τG^* is isomorphic to a subgroup of the circle. Moreover if $y \in G/H$ then it is possible to define a character χ_y of the discrete group τG^* by the rule $\chi_y(t) = t(y)$ for all $t \in \tau(G^*)$. Then we get a map $\psi : G/H \to (\tau G^*)_d^*$ where $(\tau G^*)_d^*$ is the dual of the group τG^* with discrete topology by putting $\psi(y) = \chi_y$ for all $y \in G/H$. Let us call $(\tau G^*)_d^* = F$. Then we claim that ψ is continuous and one-to-one from G/H into F. Actually the continuity follows from duality and the one-to-oneness follows from the fact that H is the annihilator of τG^* in G. From the structure of τG^* it follows that there is a collection $\mathcal{P}$ of primes such that $F = \prod_{p \in \mathcal{P}} F(p)$ where $F(p)$ is either a discrete p-primary, finite cyclic group or the group $I_p^\#$ of p-adic integers. Put $\psi(G/H) = S \subset F$ and give S the topology τ which makes $\psi : G/H \to S$ a homeomorphism. Then (S, τ) is a locally compact group. Now give the topology τ_1 on F which makes F a topological group with S as an open subgroup and such that $\tau_1 = \tau$ on S. Then τ_1 is a stronger topology on F than the product topology of $\prod_{p \in \mathcal{P}} F(p)$. So by Theorem 1 of [6] there is a closed subgroup A of F, (here F is taken with the product topology), such that $A \subset S$ and A is open in (S, τ) and the relative topology on A when A is considered as a subspace of (S, τ) or as a subspace of the product space $\prod_{p \in \mathcal{P}} F(p)$ is the same. Then (S, τ) is a canonical monogenic group. So G/H is topologically isomorphic to a canonical monogenic group.

Now we claim that H is topologically divisible. If not, H will contain a closed subgroup M of G so that H/M is finite cyclic and p-primary and has more than one element. Let us put $G/M = F'$ and $H/M = T$. Then T is a finite subgroup of F'. Moreover every compact character of F' gives a compact character of G in a natural way. Thus we may identify τG^* with the set of all compact characters of F'. It is also easy to see that F'/T is algebraically isomorphic to G/H and hence is a reduced group. So F' should be a reduced group. But every compact character of F' is identically 1 on T. So by Lemma 18 we have that T is contained in a divisible subgroup D of F'. But then F' cannot be reduced. This is a contradiction. So we get that H is topologically divisible. Let $V \subset G/H$ be a compact open subgroup and $\varphi : G \to G/H$ the canonical map. Then by Theorem 14, we have that $\varphi^{-1}(V) = P \times W$ where P is topologically divisible and W is compact and monothetic. Then $\varphi(P)$ should be topologically divisible also and since the set of compact characters of G/H separates its points we have that $\varphi(P) = \{0\}$. So $P \subset H$. Now looking at $\varphi^{-1}(V)$ and using the same kind of argument as above using P we get that $H \subset P$. So $\varphi^{-1}(V) = H \times W$ where $\varphi(W) = V$. Then G/W has H as an open, topologically divisible subgroup and G/WH is a discrete, reduced, monogenic group. So by Lemma 19 we get that $G/W = H \times N$ where N is a discrete group. By taking the complete inverse of N in G we get that $G = H \times L$ where L is a canonical monogenic group. This gives the theorem.

Remark 21. Let G_1 and G_2 be two locally compact, Hausdorff groups which are not necessarily abelian. Let H_1 and H_2 be two closed subgroups of G_1 and G_2 respectively and let T be a topological isomorphism between H_1 and H_2. Now consider the free union $G_1 \cup G_2$ of G_1 and G_2. Let us define an equivalence relation S in $G_1 \cup G_2$ by putting $x \, S \, y$ where $x \in G_1$ and $y \in G_2$ if and only if either $x = y$ or $x \in H_1$ and $y \in H_2$ and $y = Tx$ or $x \in H_2$ and $y \in H_1$ and $x = T^{-1}(y)$. We call $F = (G_1 \cup G_2)/S$. If $a \in G_1$ and $b \in G_1$ then we define $\tilde{a}\tilde{b}$ to be equal to $(ab)^{\sim}$ where $\tilde{t}$ denotes the equivalence class in $G_1 \cup G_2$ to which an element t of $G_1 \cup G_2$ belongs. In a similar way we define $\tilde{a}\tilde{b}$ when $a, b \in G_2$. Then we get a binary operation defined among certain pairs of elements of F. It is clear that this operation in F is associative whenever it is defined. If F is given the quotient topology and if $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ are two nets in F such that $\tilde{a}_\alpha \to \tilde{a}_0$ and $\tilde{b}_\alpha \to \tilde{b}_0$ and $\tilde{a}_\alpha\tilde{b}_\alpha$ is defined for all α and $\tilde{a}_0\tilde{b}_0$ is defined then $\tilde{a}_\alpha\tilde{b}_\alpha \to \tilde{a}_0\tilde{b}_0$. We denote by $0^{\sim}$ the equivalence class of $G_1 \cup G_2$ to which the identity of G_1 belongs. Then for each $\tilde{a}$ in F there is an inverse element $(\tilde{a})^{-1}$ in F such that $(\tilde{a})(\tilde{a})^{-1} = 0^{\sim}$ and if $\tilde{a}_\alpha$ is a net in F which converges to an element $\tilde{a}_0$ of F then $(\tilde{a}_0)^{-1}$ converges to $(\tilde{a}_0)^{-1}$.

DEFINITION 22. Let G_1 and G_2 be two locally compact abelian groups and $H_1 \subset G_1$ and $H_2 \subset G_2$ closed subgroups. Let G_2 be compact. Let T be a topological isomorphism between H_1 and H_2. Let F be the quotient space described above. Now take a transversal K_1 of G_1/H_1 in G_1 and a transversal K_2 of G_2/H_2 in G_2 so that the identity is chosen in G_1 and G_2 as representatives for H_1 and H_2 respectively. We denote by $\tilde{K}_1$ and $\tilde{K}_2$ the equivalence classes to which the elements in K_1 and K_2 belong. Now we define the amalgam or topological amalgam G of G_1 and G_2 with respect to H_1, H_2 and T as follows:

G consists of all triplets $(\tilde{x}, \tilde{h}, \tilde{y})$ where $\tilde{x} \in \tilde{K}_1$, $\tilde{h} \in \tilde{H}_1$ and $\tilde{y} \in \tilde{K}_2$. We define the addition $(\tilde{x}_1, \tilde{h}_1, \tilde{y}_1) + (\tilde{x}_2, \tilde{h}_2, \tilde{y}_2)$ as the triple $(\tilde{x}_3, \tilde{h}_3, \tilde{y}_3)$ where $\tilde{x}_1 + \tilde{x}_2 = \tilde{x}_3 + \tilde{b}$; $\tilde{y}_1 + \tilde{y}_2 = \tilde{y}_3 + \tilde{g}$; and $\tilde{h}_3 = \tilde{h}_1 + \tilde{h}_2 + \tilde{b} + \tilde{g}$ and $\tilde{x}_3 \in \tilde{K}_1$; $\tilde{y}_3 \in \tilde{K}_2$ and $\tilde{b}, \tilde{g}, \tilde{h}_3 \in \tilde{H}_1$. Under this definition G becomes a group under addition. Now we define convergence in G as follows: Let $(\tilde{x}_\alpha, \tilde{h}_\alpha, \tilde{y}_\alpha)$ be a net in G directed by a set D. This net is defined to converge to an element $(\tilde{x}_0, \tilde{h}_0, \tilde{y}_0)$ in G if and only if given a subnet D_2 of D there is a subnet D_1 of D_2 such that $\tilde{x}_\alpha \to \tilde{x}_0$ and $\tilde{h}_\alpha \to \tilde{h}_0$ and $\tilde{y}_\alpha \to \tilde{y}_0$ in F along D_1.

With the definition of the topology and addition above it can be verified that G become a locally compact abelian group.

Remark 23. Let G be a group with two closed subgroups H_1 and H_2. Let H_2 be compact and $H = H_1 \cap H_2$ and $G = H_1 H_2$. Then the identity map id: $H \to H$ is a topological isomorphism and G is topologically isomorphic to the amalgam of H_1 and H_2 with respect to H, H and identity, in an obvious way. In such cases we shall only say that G is the amalgam of H_1 and H_2. In general, if G is a group with a compact subgroup H then G is topologically isomorphic to the amalgam of G and H. This provides us with the easiest examples of amalgams of two groups which are not direct products. If the groups H_1 and H_2 in Definition 22 consist of the identity alone then the amalgam of G_1 and G_2 reduces to their direct product. Now we come to the structure of monogenic groups which are not totally disconnected.

THEOREM 24. *Let $\mathcal{P}$ be a collection of primes. For each $p \in \mathcal{P}$ let D_p be an index set. Let $H = \prod_{p \in \mathcal{P}} \prod_{\alpha \in D_p} J_{p,\alpha}$ where $J_{p,\alpha}$ is the abstract group of p-adic numbers for each $\alpha \in D_p$ and $p \in \mathcal{P}$. Let $K_{p,\alpha}$ be a subgroup of $J_{p,\alpha}$ isomorphic to I_p^* for each $\alpha \in D_p$ and $p \in \mathcal{P}$. Give each $K_{p,\alpha}$ the usual topology of p-adic integers. Let $K = \prod_{p \in \mathcal{P}} \prod_{\alpha \in D_p} K_{p,\alpha}$ and give K the product topology. Now let τ be a topology for H making it a locally compact group in which K is open and is also such that the product topology on K coincides with the relative topology of it obtained from (H, τ). Let G^* be any open subgroup of H such that G^*/K has torsion free rank less than or equal to c (the cardinality of continuum). Then the dual G of G^* is such that G/G_0 is topologically divisible and G_0 is monothetic where G_0 is the connected component of identity of G. Consequently G is monogenic. Every monogenic locally compact abelian group G such that G/G_0 is topologically divisible is obtained in the above manner. If G is a monogenic locally compact abelian group with a non-trivial connected component G_0 of the identity then G is either the direct product $L \times K$ of a compact monothetic group K and a topologically divisible group L or is an amalgam $L + K$ of two groups L and K through their connected components L_0 and K_0 of identity and some topological isomorphism T between L_0 and K_0. In the latter case we also have that K is compact and L/L_0 is topologically divisible and L is monogenic.*

Proof. Now suppose G is a monogenic group and its connected component G_0 of identity is not trivial. Let $F = G/G_0$ and $\varphi : G \to F$ the canonical map. Then F is a totally disconnected monogenic group with a special element of compact order. So by Theorem 14 we have that $F = K_1 \times K_2$ where K_1 is topologically divisible and K_2 is compact monothetic. Let $\varphi^{-1}(K_1) = L$ and $\varphi^{-1}(K_2) = K$. Then K is a compact subgroup of G and L is such that L/G_0 is topologically divisible and $L \cap K = G_0$ and $LK = G$. Now it is clear that G is topologically isomorphic to the amalgam of L and K through G_0 where the isomorphism T from $G_0 \to G_0$ is identity. Since G_0 is compact monothetic and L/G_0 is topologically divisible it follows that L is also monogenic. If G_0 is topologically a direct summand of L then L can be written as $H \times G_0$ where H is topologically divisible and we get that G is the direct product $H \times K$ of H and K. So we get the last two statements of the theorem.

Now suppose G^* is a locally compact abelian group with a compact, open subgroup K of the type described in the theorem. Let G^*/K be of torsion free rank less than or equal to c. Let M be the subgroup of all elements of compact order in G^*. Then M is an open subgroup of G^* and G^*/M is a torsion free subgroup of cardinality less than or equal to c. Now let G be the dual of G^*. Then G^*/M is the dual of the connected component G_0 of the identity of G. Since G^*/M is isomorphic to a subgroup of the reals mod 1, we have that G_0 is monothetic. Moreover M is the dual of G/G_0. Since M has no element of discrete order other than identity, we have that G/G_0 is topologically divisible. By reversing the above steps one can see that if G is a monogenic locally compact abelian group such that G/G_0 is topologically divisible then it can be obtained as in the first part of the theorem. This completes the proof of our theorem.

Example and Remark 25. The last theorem gives the structure of monogenic locally compact abelian groups which are not totally disconnected. We find that such a group is an amalgam of a compact group and another locally compact group. This amalgam does not always become a direct product and we give an example below to illustrate this point. Thus the previous theorem is the best possible result in this direction. For an example we take the group $I_p^{\#}$ of p-adic integers with usual addition and topology where p runs through all primes. We call the group $p\, I_p^{\#}$ to be H_p for each prime p. Then H_p is an open subgroup of $I_p^{\#}$ for each 'p'. Now put $G^* = \prod_{p\in\mathcal{P}} I_p^{\#}$ where $\mathcal{P}$ is the set of all primes. Now give a different topology τ on G^* in which it becomes a locally compact group and in which the group $H = \prod_{p\in\mathcal{P}} H_p$ is open and the relative topology on it from (G, τ) coincides with the product topology of $\prod_{p\in\mathcal{P}} H_p$. Let G be the dual of G^* with topology τ. Then the connected component G_0 of identity of G is monothetic and G/G_0 is topologically divisible and G_0 is not a topologically direct summand of G. Trivially then G is the amalgam of G_0 and G.

Acknowledgment. The author had support from the National Science Foundation through grant number GP 5370 when this research was done.

References

1. P. Ahern and J. Jewett, *Factorization of locally compact abelian groups*, Illinois J. Math., vol. 9 (1965), pp. 230–235.
2. J. Dixmier, *Quelques propriétés des groupes Abeliens localement compacts*, Bull. Sci. Math., vol. 81 (1957), pp. 38–48.
3. I. Kaplansky, *Infinite Abelian groups*, Univ. of Michigan Press, Ann Arbor, 1962.
4. B. H. Neumann, *Lectures on topics in the theory of finite groups*, Tata Institute of Fundamental Research, Bombay, 1960.
5. L. Pontrjagin, *Topological groups*, Princeton Univ. Press, Princeton, 1957.
6. M. Rajagopalan, *Topologies in locally compact groups*, Math. Ann., to appear,
7. M. Rajagopalan and J. J. Rotman, *A note on monogenic groups*, to appear.

8. L. C. Robertson, *Homogeneous dual pairs of locally compact abelian groups*, Thesis, Univ. of California, Los Angeles, 1965.
9. L. A. Rubel, *Uniform distribution in locally compact groups*, Comm. Math. Helv., vol. 39 (1965), pp. 253–258.
10. W. Rudin, *Fourier analysis on groups*, John Wiley, New York, 1962.
11. Y. Vilenkin, *Direct decomposition of topological groups I and II*, Amer. Math. Soc. Translations, sec. 1, vol. 8 (1962), pp. 78–186.
12. E. Hewitt and K. Ross, *Abstract harmonic analysis*, Springer-Verlag, Berlin, 1963.

University of Illinois
Urbana, Illinois