A KNOTTED CELL PAIR WITH KNOT GROUP Z

BY

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In [2], L. C. Glaser and I proved that a locally flat cell pair (C, C') of type (n, k) was unknotted if $n - k$ was either 1 or was greater than 2, $n \geq 4$. We also proved that if $n - k = 2$ and both $C - C'$ and Bd C − Bd C' have the homotopy type of the 1-sphere, then (C, C') is unknotted, $n \geq 6$. Theorem 3 of this paper gives examples of cell pairs (C, C') of type $(n, n - 2)$, $n \geq 6$, such that $\pi_1(C - C') = Z$, but $\pi_1(\text{Bd } C - \text{Bd } C') \neq Z$.

It should be noted that the results obtained here can also be obtained in a piecewise linear or differentiable setting rather than the locally flat setting. In proving Theorem 4 in the differentiable case, one must apply the so-called "smoothing the corners" process; otherwise the proofs are not significantly different from what is done here.

Let C be an n-cell and C' be a k-cell, then (C, C') is called a cell pair of type (n, k) if C' is a spanning cell of C; that is, the boundary of C' is contained in the boundary of C and the interior of C' is contained in the interior of C. The boundary of a cell D is denoted by Bd D. A cell pair, or a sphere pair, is called unknotted if it is homeomorphic to the appropriate standard cell pair, or standard sphere pair. Finally, let (E^n, E^k) and (E^n_+, E^k_+) denote the standard Euclidean space pair of type (n, k) and the standard closed Euclidean half-space pair of type (n, k) respectively. A manifold pair (W, W') is called locally flat if each point w of W' has a neighborhood homeomorphic to (E^n, E^k) or (E^n_+, E^k_+) according to whether w is in the interior of W' or on the boundary of W'.

THEOREM 1 (Hudson and Sumners, see Cor. 2 of [3]). *For $n \geq 4$ there exists a locally flat sphere pair (S, S') of type $(n, n - 2)$ such that*

(1) *(S, S') is unknotted,*

(2) *$S = E \cup F$ n-cells with* Bd E = Bd $F = E \cap F$,

(3) *$S' = E' \cup F'$ where E' and F' are locally flat spanning $(n - 2)$-cells of E and F respectively, and*

(4) *$\pi_1(E - E') \neq Z$ and $\pi_1(F - F') \neq Z$.*

As stated above, this theorem was proved by J. F. P. Hudson and D. W. L. Sumners in [3]. They gave a method of constructing such cell pairs. We suggest here an alternative construction. This method, using twist spinning,

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is due to E. C. Zeeman, [4], but the relationship between twist spinning and such cell pairs does not seem to have been pointed out.

Let (C, C') be a smooth knotted $(n-1, n-3)$ cell pair with $(\mathrm{Bd}\, C, \mathrm{Bd}\, C')$ unknotted. The pair (S, S') is generated by spinning (C, C') with one twist. Then by [3, Corollary 2, page 487], (S, S') is unknotted. The cell pair (E, E') is generated by spinning (C, C') half way around with a half twist. The cell pair (F, F') is generated by spinning (C, C') the other half of the way around with the other half twist. It is easy to check that the cell pairs (E, E') and (F, F') have the desired properties. Clearly their union gives the unknotted sphere pair, (S, S'), and each of (E, E') and (F, F') has the same homotopy type as (C, C').

Before proceeding to the main result of this paper we need a theorem about the fundamental group of a manifold obtained by spinning. The following notation will be used in the next theorem. Let M be a topological space and let A be a subset of M. By the space obtained by spinning M about A, denoted $\mathrm{Sp}\,(M, A)$, we mean the following

$$\mathrm{Sp}\,(M, A) = M \times S^1 \cup_{A \times S^1} A \times D^2,$$

where $A \times S^1$ is considered as a subset of each piece and the two copies of $A \times S^1$ are identified under the identity map. The next theorem gives a condition under which $\pi_1(\mathrm{Sp}\,(M, A))$ is isomorphic to $\pi_1(M)$. A more geometric proof can be given using triangulated spaces or CW complexes. It is not as general, of course, but it is much easier to follow, so we give a brief indication of how it goes. Let (M, A) be a connected triangulated pair, a CW complex structure is sufficient. Let T be a maximal tree in the triangulation. Then $\mathrm{Sp}\,(M, A)$ has a natural CW complex structure, obtained by spinning the cells of M. Furthermore the tree T is maximal in the cell structure on $\mathrm{Sp}\,(M, A)$. Hence $\pi_1(\mathrm{Sp}\,(M, A))$ has a presentation with one generator for each edge that is not in T and one relation for each 2-cell. But the edges of $\mathrm{Sp}\,(M, A)$ are precisely the edges of M plus the edges obtained by spinning verticies of M, and any edge obtained by spinning a vertex is easily shown to bound a 2-cell. It is not difficult to show that the fundamental groups of M and $\mathrm{Sp}\,(M, A)$ are isomorphic.

THEOREM 2. *Let M be a pathwise connected topological space. Let A be a pathwise connected closed subset of M. Suppose that there exists an open set 0 such that $A \subseteq 0$ and 0 deformation retracts to A. Then $\pi_1(\mathrm{Sp}\,(M, A))$ is isomorphic to $\pi_1(M)$.*

Proof. Let

$$M_1 = (M \times S^1) \cup (A \times (D^2 - \{0\})) \quad \text{and} \quad M_2 = (A \times D^2) \cup (0 \times S^1).$$

Then $M_0 = M_1 \cap M_2 = (A \times (D^2 - \{0\})) \cup (0 \times S^1)$. Let i_1 and i_2 denote the inclusion induced homomorphisms of $\pi_1(M_0)$ into $\pi_1(M_1)$ and $\pi_1(M_2)$ respectively. Then van Kampen's theorem [1, Theorem 3.1] implies that

$\pi_1(\mathrm{Sp}\,(M, A))$ is isomorphic to

$$\frac{\pi_1(M_1) * \pi_1(M_2)}{[\pi_1(M_0)]},$$

where $[\pi_1(M_0)]$ denotes the smallest normal subgroup of $\pi_1(M_1) * \pi_1(M_2)$ that contains all the elements of the form $i_1(\gamma) * i_2(\gamma^{-1})$, where $\gamma \in \pi_1(M_0)$.

Since M_1 deformation retracts to $M \times S^1$, M_2 deformation retracts to $A \times D^2$ and M_0 deformation retracts to $A \times S^1$ it is easy to see that $\pi_1(\mathrm{Sp}\,(M, A))$ is isomorphic to

$$\frac{\pi_1(M \times S^1) * \pi_1(A \times D^2)}{N}$$

where N is defined as below.

Let i_* and j_* denote the inclusion induced homomorphisms $\pi_1(A \times S^1)$ into $\pi_1(M \times S^1)$ and $\pi_1(A \times D^2)$, respectively. Let p_* denote the inclusion induced homomorphisms of $\pi_1(A)$ into $\pi_1(M)$ and let q_* denote the inclusion induced homomorphism, namely the identity isomorphism, of $\pi_1(A)$ onto $\pi_1(A)$. Then, using the product theorem for fundamental groups, we see that $i_* = p_* \otimes \mathrm{id}$ and $j_* = q_* \otimes 0$, where id denotes the identity isomorphism of Z onto itself, 0 denotes the zero homomorphism of Z onto the zero group and $\otimes$ denotes the direct product. Let $G = (\pi_1(M) \otimes Z) * (\pi_1(A) \otimes 0)$, let $H = \pi_1(M)$ and let N denote the smallest normal subgroup of G that contains all elements of G of the form $(p_*(\alpha), k)$, $(q_*(\alpha^{-1}), 0)$, where $\alpha \in \pi_1(A)$ and $k \in Z$. That is, N is generated by the relations $i_*(\alpha, k) = j_*(\alpha, k)$, for every $(\alpha, k) \in \pi_1(A) \otimes Z = \pi_1(A \times S^1)$.

We wish to show that G/N is isomorphic to H. Let η be the quotient homomorphism of G onto G/N. Let ϕ be the homomorphism of G onto H defined as follows. Let

$$w = (\beta_1, k_1), (\alpha_1, 0), (\beta_2, k_2), (\alpha_2, 0), \cdots, (\beta_r, k_r), (\alpha_r, 0)$$

be a typical element of G, where $\beta_i \in \pi_1(M)$, $\alpha_i \in \pi_1(A)$ and $k_i \in Z$. Then

$$\phi(w) = \beta_1 \cdot (p_* \circ q_*^{-1}(\alpha_1)) \cdot \beta_2 \cdot (p_* \circ q_*^{-1}(\alpha_2)) \cdot \cdots \cdot \beta_r \cdot (p_* \circ q_*^{-1}(\alpha_r)).$$

Let ψ be the inclusion induced homomorphism of H into G; that is, if $\beta \in H$ then $\psi(\beta) = (\beta, 0)$.

First we note that $N \subseteq$ kernel of ϕ. To prove this it suffices to show that $\phi(w) = 1$ for each $w \in G$ of the form $w = (p_*(\alpha), k), (q_*(\alpha^{-1}), 0)$. But this is true because $\phi(w) = p_*(\alpha) \cdot (p_* \circ q_*^{-1} \circ q_*(\alpha^{-1})) = 1$. Thus $\phi \circ \eta^{-1}$ is a well defined homomorphism of G/N into H.

Next we note that $\phi \circ \psi =$ identity. Let $\beta \in H$. Then $\phi \circ \psi(B) = \phi((\beta, 0)) = \beta$. Hence $(\phi \circ \eta^{-1}) \circ (\eta \circ \psi) =$ identity.

Finally, if $w \in G$, then $\eta \circ \psi \circ \phi(w) = \eta(w)$. It suffices to show this for words of the form $w = (\beta, k)$, where $\beta \in \pi_1(M)$ and $k \in Z$, and $w' = (\alpha, 0)$, where $\alpha \in \pi_1(A)$.

$$\eta \circ \psi \circ \phi(w) = \eta((\beta, 0)) = \eta((\beta, 0)) \cdot \eta((1, k)) = \eta((\beta, k)) = \eta(w),$$

because $(1, k) \equiv (p_*(1), k) \cdot (q_*(1), 0) \in N$.

Similarly

$$\begin{aligned} \eta \circ \psi \circ \phi(w') &= \eta((p_* \circ q_*^{-1}(\alpha), 0)) \\ &= \eta((p_* \circ q_*^{-1}(\alpha), 0)) \cdot \eta((p_*(q_*^{-1}(\alpha^{-1})), 0), (q_*(q_*^{-1}(\alpha)), 0)) \\ &= \eta((q_*(q_*^{-1}(\alpha), 0)) = \eta((\alpha, 0)) \end{aligned}$$

because $(p_*(q_*^{-1}(\alpha^{-1})), 0), (q_*(q_*^{-1}(\alpha)), 0)) \in N$. Thus $\eta \circ \psi \circ \phi = \eta$ and hence $(\eta \circ \psi) \circ (\phi \circ \eta^{-1}) =$ identity.

Therefore, $\eta \circ \psi$ is an isomorphism of H onto G/N.

THEOREM 3. *For $n \geqq 6$ there exists a locally flat cell pair (C, C') of type $(n, n - 2)$ such that*

(1) $\pi_1(C - C') = Z$,

(2) $\pi_1(\mathrm{Bd}\, C - \mathrm{Bd}\, C') \neq Z$.

Proof. Let (S, S') be a smooth sphere pair of type $(n - 2, n - 4)$ that satisfies the conclusion of Theorem 1. Let (D, D') denote the cell pair obtained by taking the cone over the pair (S, S'). We obtain the pair (C, C') by spinning (D, D') around the $(n - 2)$-cell, F, that lies on the boundary of D. That is, $C = \mathrm{Sp}\,(D, F)$ and $C' = \mathrm{Sp}\,(D', F')$. Thus $C - C' = \mathrm{Sp}\,(D - D', F - F')$ and $\mathrm{Bd}\, C - \mathrm{Bd}\, C' = \mathrm{Sp}\,(E - E', \mathrm{Bd}\, E - \mathrm{Bd}\, E')$. Therefore, by Theorem 3,

$$\pi_1(C - C') = \pi_1(D - D') = Z \quad \text{and} \quad \pi_1(\mathrm{Bd}\, C - \mathrm{Bd}\, C') = \pi_1(E - E) \neq Z.$$

Question. What are the higher homotopy groups of $C - C'$?

REFERENCES

1. R. H. CROWELL AND R. H. FOX, *Introduction to knot theory*, Ginn, Boston, 1965·
2. L. C. GLASER AND T. M. PRICE, *Unknotting locally flat cell pairs*, Illinois J. Math., vol. 10 (1966), pp. 425–430.
3. J. F. P. HUDSON AND D. W. L. SUMNERS, *Knotted ball pairs in unknotted sphere pairs*, J. London Math. Soc., vol. 41 (1966), pp. 717–722.
4. E. C. ZEEMAN, *Twisting spun knots*, Trans. Amer. Math. Soc., vol. 115 (1956), pp. 471–495.

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