A MINIMAL REPRESENTATION FOR THE LIE ALGEBRA $\mathfrak{E}_7$

BY

ROBERT B. BROWN[1]

The smallest representation of a split Lie algebra of type E_7 over a field of characteristic zero has dimension fifty-six and is closely related to the split exceptional simple Jordan algebra. This representation has been studied by Freudenthal [4] and Seligman [11]. We show that it is possible to define a multiplication and a trace on the representation space so that the Lie algebra $\mathfrak{E}_7$ is realized by the derivations and left multiplications by elements of trace zero. The derivations alone form a Lie algebra of type E_6. The Killing forms of these Lie algebras are also presented. Later we slightly generalize the fifty-six dimensional algebras so as to obtain as algebras of derivations a class of Lie algebras of type E_6 including the "twisted" algebras. Although these include all real forms of $\mathfrak{E}_6$, no new forms of $\mathfrak{E}_7$ occur. Finally a method of twisting E_7's is given. This results in a class which contains all real forms of $\mathfrak{E}_7$.

Throughout, the characteristic of the ground field is not two or three.

1. Cayley and Jordan algebras

In this section we collect some facts about Cayley algebras and exceptional simple Jordan algebras. The following properties of Cayley algebras are proved in [6]. A Cayley algebra $\mathfrak{C}$ is an eight-dimensional central simple alternative algebra. It possesses an involution $x \to x^*$ such that the quadratic norm form $n(x) = xx^*$ permits composition: $n(xy) = n(x)n(y)$. We linearize $n(x)$ to obtain a non-degenerate symmetric bilinear form on $\mathfrak{C}$:

$$(x, y) = \tfrac{1}{2}[n(x + y) - n(x) - n(y)] = \tfrac{1}{2}(xy^* + yx^*).$$

A Cayley algebra has an identity 1 and every element is of the form $\alpha 1 + x_0$ with $(x_0, 1) = 0$. Then $(\alpha 1 + x_0)^* = \alpha 1 - x_0$. A basis can be chosen for $\mathfrak{C}$ so that the norm form becomes

$$n(x) = \xi_0^2 - \rho\xi_1^2 - \sigma\xi_2^2 + \rho\sigma\xi_3^2 - \tau(\xi_4^2 - \rho\xi_5^2 - \sigma\xi_6^2 + \rho\sigma\xi_7^2).$$

Here ρ, σ, τ are non-zero field elements. To exhibit them we will write $\mathfrak{C} = \mathfrak{C}(\rho, \sigma, \tau)$, even though they are not uniquely determined by $\mathfrak{C}$. A Cayley algebra with zero divisors is called split.

Received January 9, 1967.

[1] This is a revision of part of a dissertation written under Professor A. A. Albert at The University of Chicago. Thanks are due to Professor Albert for his encouragement and to the National Science Foundation for its support. Some of these results were announced in [3].

THEOREM. *Two Cayley algebras over the same ground field* (ch. $\neq 2$) *are isomorphic if and only if their norm forms are equivalent. Any two split Cayley algebras are isomorphic.*

We turn now to exceptional simple Jordan algebras [1], [2], [5], [8], [12], [14], [15]. An algebra $\mathfrak{J}$ is an exceptional simple Jordan algebra if there is an extension $\mathfrak{K}$ of the ground field such that $\mathfrak{J}_{\mathfrak{K}}$ is isomorphic to an algebra $\mathfrak{H}(\mathfrak{C}_3, \Gamma)$ of the following form. Let $\mathfrak{C}$ be a Cayley algebra over $\mathfrak{K}$, and Γ a diagonal matrix diag $\{\gamma_1, \gamma_2, \gamma_3\}$, where the γ_i are non-zero elements of $\mathfrak{K}$. The elements of $\mathfrak{H}(\mathfrak{C}_3, \Gamma)$ are 3×3 matrices of the form

$$(1) \qquad a = \begin{vmatrix} \alpha_1 & x & \gamma_1^{-1}\gamma_3 z^* \\ \gamma_2^{-1}\gamma_1 x^* & \alpha_2 & y \\ z & \gamma_3^{-1}\gamma_2 y^* & \alpha_3 \end{vmatrix}$$

where $\alpha_i \in \mathfrak{K}$; $x, y, z \in \mathfrak{C}$. The product in $\mathfrak{H}(\mathfrak{C}_3, \Gamma)$ is $ab = \frac{1}{2}(a\cdot b + b\cdot a)$, where $a\cdot b$ is ordinary matrix multiplication. Such a matrix algebra is called *reduced*, and every finite-dimensional exceptional simple Jordan algebra is either reduced or is a division algebra.

Although a reduced exceptional simple Jordan algebra may have different representations as an algebra of matrices, the Cayley algebras occurring are all isomorphic, and we refer to them as the coefficient algebra of $\mathfrak{H}(\mathfrak{C}_3, \Gamma)$. For (1) we will usually write

$$a = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + x_{12} + y_{23} + z_{31}.$$

The following define the multiplication. Let (ijk) be a cyclic permutation of (123). Then

$$e_i^2 = e_i, \; e_i e_j = 0, \qquad e_i x_{ij} = e_j x_{ij} = \tfrac{1}{2}x_{ij},$$

$$e_k x_{ij} = 0, \qquad x_{ij} y_{jk} = \tfrac{1}{2}\gamma_k^{-1}\gamma_i (xy)^*_{ki}, \qquad x_{ij} y_{ij} = \gamma_j^{-1}\gamma_i (x, y)(e_i + e_j).$$

The element $e = e_1 + e_2 + e_3$ is a unit element.

We define trace $(a) = \alpha_1 + \alpha_2 + \alpha_3$. The trace is not dependent on the particular matrix representation. The bilinear form $(a, b) =$ trace (ab) is symmetric and non-degenerate. It can be induced on a division algebra from a reduced extension. We have $(e, e) = 3$ and $(ab, c) = (a, bc)$. Furthermore, every element a satisfies a cubic equation

$$a^3 - (a, e)a^2 - \tfrac{1}{2}[(a, a) - (a, e)^2]a - N(a)e = 0,$$

where $N(a) = \frac{1}{3}(a^2, a) - \frac{1}{2}(a, a)(a, e) + \frac{1}{6}(a, e)^3$.

For a in (1)

$$N(a) = \alpha_1\alpha_2\alpha_3 + 2(xy, z^*) - \alpha_1\gamma_3^{-1}\gamma_2 n(y) - \alpha_2\gamma_1^{-1}\gamma_3 n(z) - \alpha_3\gamma_2^{-1}\gamma_1 n(x).$$

Linearization of $N(a)$ yields a symmetric trilinear form (a, b, c) such that

$(a, a, a) = N(a)$. The non-degeneracy of the trace form then defines on $\mathfrak{J}$ a commutative cross-product $a \times b$ by $(a \times b, c) = 3(a, b, c)$. It turns out that

$$(2) \quad \begin{aligned} a \times b &= ab - \tfrac{1}{2}(a, e)b - \tfrac{1}{2}(b, e)a - \tfrac{1}{2}[(a, b) - (a, e)(b, e)]e, \\ b \times (a \times a) &= \tfrac{1}{2}a^2b - a(ab) + \tfrac{1}{2}(a, b)a. \end{aligned}$$

The semi-norm preserving group $M(\mathfrak{J})$ is the group of all linear bijections S on $\mathfrak{J}$ such that for some fixed $\rho_S \neq 0$ in the field and for all a in $\mathfrak{J}$, $N(aS) = \rho_S N(a)$. The norm preserving group $L(\mathfrak{J})$ is the subgroup of those S with $\rho_S = 1$. The map

$$U_a : x \rightarrow 2a(ax) - a^2x$$

is in $M(\mathfrak{J})$; $U_a = U_a^*$; $N(xU_a) = N(a)^2N(x)$. (See [7].) Here * is the transpose with respect to the trace form on $\mathfrak{J}$. For reduced $\mathfrak{J}$, any non-zero α may play the role of ρ_S. For, if $a = \alpha^{-1}e_1 + e_2 + e_3$, let $S = \alpha U_a$. Then $\rho_S = \alpha$.

Let $R(a)$ denote the map $x \rightarrow xa$ in $\mathfrak{J}$. The set of maps $R(a) + D$ where trace $a = 0$ and D is a derivation of $\mathfrak{J}$ is closed under commutation and forms a Lie algebra $\mathfrak{L}(\mathfrak{J})$ of type E_6. $\mathfrak{L}(\mathfrak{J})$ is known to consist of all linear transformations A on $\mathfrak{J}$ such that for every a, b, c in $\mathfrak{J}$,

$$(3) \qquad (aA, b, c) + (a, bA, c) + (a, b, cA) = 0.$$

2. Definition of the algebra

Let $\mathfrak{J}$ be an exceptional simple Jordan algebra over a field $\mathfrak{F}$ of characteristic not two or three. We define the algebra $\mathfrak{N}(\mathfrak{J}; \delta, \omega)$ to be set of quadruples $x = (\alpha, \beta, a, b)$ for α, β in $\mathfrak{F}$ and a, b in $\mathfrak{J}$; $\delta, \omega \neq 0$ in $\mathfrak{F}$. We will write

$$(4) \qquad x = \alpha f_1 + \beta f_2 + a_{12} + b_{21}.$$

A bilinear multiplication is defined as follows. Say $i \neq j$, where $\{i, j\} = \{1, 2\}$.

$$(5) \quad \begin{aligned} f_i^2 &= f_i, \qquad f_i f_j = 0, \\ f_i a_{ij} &= \tfrac{1}{3}a_{ij}, \qquad f_j a_{ij} = \tfrac{2}{3}a_{ij}, \\ a_{ij} f_i &= 0, \qquad a_{ij} f_j = a_{ij}, \\ a_{ij} b_{ji} &= \delta(a, b) f_i, \\ a_{12} b_{12} &= 2\omega(a \times b)_{21}, \qquad a_{21} b_{21} = 2\delta\omega^{-1}(a \times b)_{12} \end{aligned}$$

Notice that $f = f_1 + f_2$ is an identity element for $\mathfrak{N}(\mathfrak{J}; \delta, \omega)$. A trace and a bilinear form can be defined on $\mathfrak{N}(\mathfrak{J}; \delta, \omega)$ as follows. For x in (4), $\operatorname{tr}(x) = \alpha + \beta$. If $y = \xi f_1 + \eta f_2 + c_{12} + d_{21}$, then

$$(x, y) = \operatorname{tr}(xy) = \alpha\xi + \beta\eta + \delta(a, d) + \delta(b, c).$$

This bilinear form is symmetric and non-degenerate, and $(x, e) = \operatorname{tr}(x)$. However, (x, y) is not an associative form; that is, $(xy, z) \neq (x, yz)$ in general.

For $(a_{12} f_1, b_{21}) = 0$, while $(a_{12}, f_1 b_{21}) = \frac{2}{3}\delta(a, b)$. This reflects the pathological non-associativity of $\mathfrak{N}(\mathfrak{J}; \delta, \omega)$.

LEMMA. *$\mathfrak{N}(\mathfrak{J}; \delta, \omega)$ is not flexible or power-associative.*

Proof. Let e be the identity element of $\mathfrak{J}$. Then $(e_{12} e_{12})e_{12} = 6\omega\delta f_2$, while $e_{12}(e_{12} e_{12}) = 6\omega\delta f_1$.

LEMMA. *$\mathfrak{N}(\mathfrak{J}; \delta, \omega)$ is central simple.*

Proof. Let $\mathfrak{M}$ be a non-zero ideal in $\mathfrak{N}(\mathfrak{J}; \delta, \omega)$. If $\mathfrak{M}$ contains $a_{ij} \neq 0$ and b is an element of $\mathfrak{J}$ such that $(a, b) \neq 0$, then $a_{ij} b_{ji} + b_{ji} a_{ij} = \delta(a, b)f$ is in $\mathfrak{M}$, and $\mathfrak{M} = \mathfrak{N}(\mathfrak{J}; \delta, \omega)$. Next, if $x = \alpha f_i + \beta f_j + a_{ij} + b_{ji} \in \mathfrak{M}$ with $a_{ij} \neq 0$, then $[f_i(f_j x)]f_j = \frac{2}{9} a_{ij} \in \mathfrak{M}$. Finally, if $x = \alpha f_i + \beta f_j$ with $\beta \neq 0$ is in $\mathfrak{M}$, then $a_{ij} x = \beta a_{ij} \in \mathfrak{M}$. Thus $\mathfrak{M} = \mathfrak{N}(\mathfrak{J}; \delta, \omega)$.

3. Derivations and left multiplications

In this section we show how Lie algebras of types E_6 and E_7 can be realized as algebras of linear transformations on $\mathfrak{N}(\mathfrak{J}; \delta, \omega)$.

THEOREM. *The derivations D of $\mathfrak{N} = \mathfrak{N}(\mathfrak{J}; \delta, \omega)$ are given by*

(6) $\quad D: f_1 \to 0, \qquad f_2 \to 0, \qquad a_{12} \to (aD_{12})_{12}, \qquad b_{21} \to (-bD_{12}^*)_{21},$

where $D_{12} \in \mathfrak{L}(\mathfrak{J})$. Hence, the derivation algebra $\mathfrak{D}(\mathfrak{N})$ is a Lie algebra of type E_6.

Proof. To see that conditions (6) are necessary, we substitute various elements of $\mathfrak{N}$ into the equation $(xy)D = (xD)y + x(yD)$. First we let $x = y = f_1$. If $f_1 D = \xi f_1 + \eta f_2 + c_{12} + d_{21}$, we obtain

$$\xi f_1 + \eta f_2 + c_{12} + d_{21} = \xi f_1 + d_{21} + \xi f_1 + \tfrac{1}{3} c_{12} + \tfrac{2}{3} d_{21}.$$

Hence $\xi = \eta = 0$ and $c = d = 0$, so that $f_1 D = 0$. Similarly $f_2 D = 0$. Next for $i \neq j$ suppose that $a_{ij} D = \xi f_i + \eta f_j + c_{ij} + d_{ji}$. We set $x = f_i$ and $y = a_{ij}$ to obtain $\xi = \eta = 0$ and $d = 0$. Hence $a_{ij} D = (aD_{ij})_{ij}$ for linear transformations D_{ij} on $\mathfrak{J}$. Next we let $x = a_{12}$ and $y = b_{21}$ and obtain $0 = \delta(aD_{12}, b) + \delta(a, bD_{21})$, so that $D_{21} = -D_{12}^*$. Finally, with $x = a_{12}$ and $y = c_{12}$ we obtain

$$-2\omega(a \times c)D_{12}^* = 2\omega aD_{12} \times c + 2\omega a \times cD_{12}.$$

Hence for all $b \in \mathfrak{J}$,

$$((a \times c)D_{12}^*, b) + (aD_{12} \times c, b) + (a \times cD_{12}, b) = 0.$$

This implies that

$$(a, c, bD_{12}) + (aD_{12}, c, b) + (a, cD_{12}, b) = 0,$$

which amounts by (3) to saying that $D_{12} \in \mathfrak{L}(\mathfrak{J})$. Hence the conditions are necessary. To show that they are sufficient entails an obvious verification, which we omit.

Under the action of $\mathfrak{D}(\mathfrak{N})$ the space $\mathfrak{N}$ decomposes into the trivial representation on $\mathfrak{F}f_1 + \mathfrak{F}f_2$ and two inequivalent 27-dimensional representations on $\mathfrak{N}_{12} = \{a_{12} : a \in \mathfrak{J}\}$ and $\mathfrak{N}_{21} = \{b_{21} : b \in \mathfrak{J}\}$.

We turn now to Lie algebras of type E_7. Let $\mathfrak{L}(\mathfrak{N})$ be the linear space of transformations on $\mathfrak{N}$ spanned by $\mathfrak{D}(\mathfrak{N})$ and the left multiplications $L(x) : y \to xy$ where $\operatorname{tr}(x) = 0$.

THEOREM. *Dimension* $\mathfrak{L}(\mathfrak{N}) = 133$, $\mathfrak{L}(\mathfrak{N})$ *is closed under commutation, and the following products hold, where* a, b, c, d *have trace zero and* $D \in \mathfrak{D}(\mathfrak{N})$.

$$[L(x), D] = L(xD), \tag{7}$$

$$[L(f_1 - f_2), L(a_{12})] = -\tfrac{2}{3}L(a_{12}), \tag{8}$$

$$[L(f_1 - f_2), L(b_{21})] = \tfrac{2}{3}L(b_{21}), \tag{9}$$

$$[L(a_{12}), L(c_{12})] = [L(b_{21}), L(d_{21})] = 0, \tag{10}$$

$$[L(a_{12}), L(b_{21})] = \delta(-(a, b)L(f_1 - f_2) + E), \tag{11}$$

where $E_{12} = 2R(\tfrac{1}{3}(a, b)e - ab) + 2[R(a), R(b)]$.

Proof. The truth of the dimension assertion is clear. To show that $\mathfrak{L}(\mathfrak{N})$ is closed, it is sufficient to verify the products. For (7),

$$y[L(x), D] = (xy)D - x(yD) = (xD)y.$$

Let $z = \xi f_1 + \eta f_2 + g_{12} + h_{21}$. For (8),

$$\begin{aligned} z[L(f_1 - f_2), L(a_{12})] &= a_{12}(\xi f_1 - \eta f_2 - \tfrac{1}{3}g_{12} + \tfrac{1}{3}h_{21}) \\ &\quad - (f_1 - f_2)(\eta a_{12} + 2\omega(a \times g)_{21} + \delta(a, h)f_1 \\ &= -\tfrac{2}{3}\delta(a, h)f_1 - \tfrac{2}{3}\eta a_{12} - \tfrac{4}{3}\omega(a \times g)_{21} \\ &= -\tfrac{2}{3}a_{12}z. \end{aligned}$$

Equation (9) is established in the same way. To prove (10) we first calculate $c_{12}(a_{12}z)$.

$$\begin{aligned} c_{12}(a_{12}z) &= c_{12}(\eta a_{12} + 2\omega(a \times g)_{21} + \delta(a, h)f_1) \\ &= 2\omega\eta(c \times a)_{21} + 6\omega\delta(c, a, g)f_1. \end{aligned}$$

This expression is symmetric in a and c. Hence $[L(a_{12}), L(c_{12})] = 0$. Similarly $[L(b_{21}), L(d_{21})] = 0$. Finally, to verify (11) we show that $z[L(a_{12}), L(b_{21})] + \delta(a, b)(f_1 - f_2)z = zE$.

$$\begin{aligned} b_{21}(a_{12}z) - a_{12}(b_{21}z) + \delta(a, b)(f_1 - f_2)z \\ = b_{21}(\eta a_{12} + 2\omega(a \times g)_{21} + \delta(a, h)f_1) \\ -a_{12}(\xi b_{21} + \delta(b, g)f_2 + 2\delta\omega^{-1}(b \times h)_{12}) \\ + \delta(a, b)(\xi f_1 - \eta f_2 - \tfrac{1}{3}g_{12} + \tfrac{1}{3}h_{21}) \end{aligned}$$

$$= 4\delta b \times (a \times g)_{12} + \delta(a, h)b_{21} - \delta(b, g)a_{12}$$
$$- 4\delta a \times (b \times h)_{21} + \tfrac{1}{3}\delta(a, b)h_{21} - \tfrac{1}{3}\delta(a, b)g_{12}$$
$$= \delta(gE_{12})_{12} - \delta(hE_{12}^{*})_{21},$$

where $gE_{12} = 4b \times (a \times g) - (b, g)a - \frac{1}{3}(a, b)g$. We must verify that

$$E_{12} = 2R(\tfrac{1}{3}(a, b)e - ab) + 2[R(a), R(b)].$$

We have

$$2gR(\tfrac{1}{3}(a, b)e - ab) + 2g[R(a), R(b)] - gE_{12}$$
$$= 2(\tfrac{1}{3}(a, b)g - g(ab) + (ga)b - (gb)a) \tag{12}$$
$$+ 2(-2b \times (a \times g) + \tfrac{1}{2}(b, g)a + \tfrac{1}{6}(a, b)g).$$

Linearizing (2) we obtain

$$2b \times (a \times g) = (ag)b - a(gb) - g(ab) + \tfrac{1}{2}(a, b)g + \tfrac{1}{2}(g, b)a.$$

Therefore the right side of (12) is zero.

THEOREM. $\mathfrak{L}(\mathfrak{N})$ *is central simple.*

Proof. Let $\mathfrak{A} \neq 0$ be an ideal in $\mathfrak{L}(\mathfrak{N})$. Suppose first that some $L(a_{ij}) \neq 0$ is in $\mathfrak{A}$. Since $\mathfrak{L}(\mathfrak{J})$ acts irreducibly on $\mathfrak{J}$,

$$[L(a_{ij}), \mathfrak{D}(\mathfrak{N})] = \mathfrak{N}_{ij} \subseteq \mathfrak{A}.$$

Then if e is the identity of $\mathfrak{J}$,

$$[L(e_{12}), L(e_{21})] = -3\delta L(f_1 - f_2) \in \mathfrak{A}.$$

Therefore,

$$[L(f_1 - f_2), \mathfrak{N}_{ji}] = \mathfrak{N}_{ji} \subseteq \mathfrak{A}.$$

Finally for any a of trace zero in $\mathfrak{J}$, $[L(e_{12}), L(a_{21})]$ is a non-zero element of $\mathfrak{D}(\mathfrak{N})$, which is simple. Hence, $\mathfrak{D}(\mathfrak{N}) \subseteq \mathfrak{A}$, and $\mathfrak{L}(\mathfrak{N}) = \mathfrak{A}$. We now show that any ideal $\mathfrak{A}$ contains some $L(a_{ij}) \neq 0$. Let

$$T = \alpha L(f_1 - f_2) + L(a_{12}) + L(b_{21}) + D \in \mathfrak{A}.$$

If $a \neq 0$, then

$$[L(f_1 - f_2), [L(f_1 - f_2), T] - \tfrac{2}{3}T] = (8/9)L(a_{12}) \in \mathfrak{A}.$$

If $b \neq 0$, then

$$[L(f_1 - f_2), [L(f_1 - f_2), T] + \tfrac{2}{3}T] = (8/9)L(b_{21}) \in \mathfrak{A}.$$

Finally, if $a = b = 0$, then for some c, $[L(c_{12}), T]$ will have a non-zero component in $\mathfrak{N}_{12}$. Hence $\mathfrak{L}(\mathfrak{N})$ is simple.

In the next section we show that the Killing form of $\mathfrak{L}(\mathfrak{N})$ is non-degenerate for characteristic not two or three. This will show that $\mathfrak{L}(\mathfrak{N})$ is a classical Lie algebra of type E_7 for algebraically closed fields of characteristic $p > 7$.

Actually, Seligman [11] has computed the root spaces for $\mathfrak{L}(\mathfrak{N})$ and has shown directly that they satisfy the axioms for classical Lie algebras [9], [10]. Hence $\mathfrak{L}(\mathfrak{N})$ is of classical type E_7 for algebraically closed fields of characteristic $\neq 2, 3$.

4. Killing form

The Killing form $K(\ ,\ )$ of a Lie algebra is defined by $K(A, B) = \operatorname{trace}(\operatorname{ad} A)(\operatorname{ad} B)$, where $T(\operatorname{ad} A) = [T, A]$. In case $\mathfrak{J}$ is reduced our representation enables us to calculate directly the Killing form of $\mathfrak{L}(\mathfrak{N})$. The details are tedious and uninteresting. Therefore only the results are given. The Killing form K for $\mathfrak{L}(\mathfrak{N})$ can be built up by first computing the Killing forms K_2 of $\mathfrak{D}(\mathfrak{J})$ and K_1 of $\mathfrak{L}(\mathfrak{J})$. Let $\mathfrak{J} = \mathfrak{H}(\mathfrak{C}_3, \Gamma)$ where $\mathfrak{C}$ has norm form $n(x)$. Let $n_0(x)$ be that norm form restricted to elements x such that $(x, 1) = 0$. Then K_2 is equivalent to

$$-2\,(\gamma_2^{-1}\gamma_1 n(x_1) + \gamma_3^{-1}\gamma_2 n(x_2) + \gamma_1^{-1}\gamma_3 n(x_3) + n_0(y_1) + n_0(y_2) + n_0(y_3) + n_0(y_4)).$$

For K_1 we have $K_1(R(a), G) = 0$ for $G \in \mathfrak{D}(\mathfrak{J})$. Further, the restriction of K_1 to $\{R(a) : \operatorname{tr}(a) = 0\}$ is equivalent to

$$6\xi_1^2 + 2\xi_2^2 + 6(\gamma_2^{-1}\gamma_1 n(x_1) + \gamma_3^{-1}\gamma_2 n(x_2) + \gamma_1^{-1}\gamma_3 n(x_3)).$$

K_1 is equivalent to

$$(13)\quad 6\xi_1^2 + 2\xi_2^2 + 6(\gamma_2^{-1}\gamma_1 n(x_1) + \gamma_3^{-1}\gamma_2 n(x_2) + \gamma_1^{-1}\gamma_3 n(x_3)) + 3K_2 .$$

Now let $Q(a) = (a, a) = \operatorname{trace}(a^2)$ be the quadratic form induced on $\mathfrak{J}$ by its trace. For K we have

$$\begin{aligned} K(L(f_1 - f_2), L(f_1 - f_2)) &= 24, \\ K(L(f_1 - f_2), \mathfrak{N}_{ij}) &= K(L(f_1 - f_2), \mathfrak{D}(\mathfrak{N})) = 0, \\ K(L(a_{ij}), L(b_{ij})) &= 0, \\ K(L(a_{12}), L(b_{21})) &= 36\delta(a, b), \\ K(D, E) &= (3/2)K_1(D_{12}, E_{12}), \end{aligned}$$

where D, E are in $\mathfrak{D}(\mathfrak{N})$. Thus K is equivalent to

$$(14)\qquad 6\xi^2 + 2\delta Q(a) - 2\delta Q(b) + 6K_1 .$$

For characteristic not two or three the discriminants of K_2, K_1, K are nonzero.

5. Classification of the algebras $\mathfrak{N}(\mathfrak{J};\ \delta,\ \omega)$

A characterization of the isomorphisms between $\mathfrak{N}_i = \mathfrak{N}(\mathfrak{J}_i;\ \delta_i, \omega_i)$, $i = 1, 2$ is given by the lemma which follows. Let f_1 and f_2 be the idempotents of $\mathfrak{N}_1$; f_1' and f_2' those of $\mathfrak{N}_2$.

LEMMA. *The isomorphisms of* $\mathfrak{N}_1$ *onto* $\mathfrak{N}_2$ *are the linear bijections* φ *of the following forms.*

(1) $f_1\varphi = f_1'$, $f_2\varphi = f_2'$, $a_{12}\varphi = (aS)_{12}$, $b_{21}\varphi = \delta_1\delta_2^{-1}(bS^{*-1})_{21}$, *where* $S:\mathfrak{J}_1 \to \mathfrak{J}_2$ *and* $N_2(aS) = \delta_1\omega_1(\delta_2\omega_2)^{-1}N_1(a)$.

(2) $f_1\varphi = f_2'$, $f_2\varphi = f_1'$, $a_{12}\varphi = \delta_1\delta_2^{-1}(aS^{*-1})_{21}$, $b_{21}\varphi = (bS)_{12}$, *where* $S:\mathfrak{J}_1 \to \mathfrak{J}_2$ *and* $N_2(aS) = \delta_1^2\omega_1^{-1}(\delta_2\omega_2)^{-1}N_1(a)$.

Proof. To prove that the stated conditions are necessary, we observe first that if φ is an isomorphism then $\mathfrak{D}(\mathfrak{N}_2) = \varphi^{-1}\mathfrak{D}(\mathfrak{N}_1)\varphi$. Hence $f_1\varphi$ must be an idempotent annihilated by $\mathfrak{D}(\mathfrak{N}_2)$; that is, $f_1\varphi = f_1'$ or f_2'. First suppose that $f_1\varphi = f_1'$. Then $f_2\varphi = f_2'$. If

$$a_{12}\varphi = \alpha f_1' + \beta f_2' + c_{12} + d_{21},$$

then

$$a_{12}\varphi = 3(f_1\varphi)(a_{12}\varphi) = 3\alpha f_1' + c_{12} + 2d_{21}.$$

Therefore $\alpha = \beta = 0$, $d = 0$, and $a_{12}\varphi = (aS)_{12}$. Similarly $b_{21}\varphi = (bT)_{21}$. Expanding $(a_{12}b_{21})\varphi = (a_{12}\varphi)(b_{21}\varphi)$, we obtain

$$\delta_1(a, b)f_1' = \delta_2(aS, bT)f_1',$$

so that $T = \delta_1\delta_2^{-1}S^{*-1}$. Finally from $(a_{12}b_{12})\varphi = (a_{12}\varphi)(b_{12}\varphi)$ we obtain

$$2\omega_1\delta_1\delta_2^{-1}(a \times b)S^{*-1} = 2\omega_2\, aS \times bS,$$

so that

$$\omega_1\delta_1(\omega_2\delta_2)^{-1}((a \times b)S^{*-1}, cS) = (aS \times bS, cS).$$

Setting $a = b = c$, we obtain $\omega_1\delta_1(\omega_2\delta_2)^{-1}N_1(a) = N_2(aS)$. The calculation in case $f_1\varphi = f_2'$ is similar. The verification that the conditions are sufficient is straightforward and is omitted.

COROLLARY. $\mathfrak{N}(\mathfrak{J};\ \delta,\ \omega)$ *is isomorphic to* $\mathfrak{N}(\mathfrak{J};\ 1,\ \alpha^3\delta\omega)$ *and to* $\mathfrak{N}(\mathfrak{J};\ 1,\ \alpha^3\delta^{-1}\omega^{-1})$ *for any* $\alpha \neq 0$.

Proof. In the lemma let $\delta_1 = \delta$, $\omega_1 = \omega$, $\delta_2 = 1$. In an isomorphism (1) we let $\omega_2 = \alpha^3\delta\omega$ and $S = \alpha^{-1}I$. In (2) we let $\omega_2 = \alpha^3\delta^{-1}\omega^{-1}$ and $S = \delta\alpha^{-1}I$.

More detailed information about isomorphisms from $\mathfrak{N}_1$ to $\mathfrak{N}_2$ is obtained from the semi-norm preserving maps S. Reduced exceptional simple Jordan algebras are characterized by the presence of non-zero elements a such that $N(a) = 0$ [12]. Hence if maps $S:\mathfrak{J}_1 \to \mathfrak{J}_2$ exist, then $\mathfrak{J}_1$ is reduced if and only if $\mathfrak{J}_2$ is reduced. Between reduced algebras the maps exist if and only if $\mathfrak{J}_1$ and $\mathfrak{J}_2$ have isomorphic coefficient algebras [5], [13]. Furthermore, we observe in section one that for a reduced algebra $\mathfrak{J}$ and any $\rho \neq 0$ a map $T:\mathfrak{J} \to \mathfrak{J}$ exists such that $N(aT) = \rho N(a)$. This establishes the following corollary.

COROLLARY. *Over a field of characteristic not two or three let* $\mathfrak{J}_1$ *be a reduced exceptional simple Jordan algebra and* $\mathfrak{J}_2$ *another exceptional simple Jordan* *algebra. Then* $\mathfrak{N}(\mathfrak{J}_1\,;\,\delta_1\,,\,\omega_1)$ *and* $\mathfrak{N}(\mathfrak{J}_2\,;\,\delta_2\,,\,\omega_2)$ *are isomorphic if and only if* $\mathfrak{J}_2$ *is reduced and* $\mathfrak{J}_1$ *and* $\mathfrak{J}_2$ *have isomorphic coefficient algebras.*

Hence, for $\mathfrak{J}$ reduced, $\mathfrak{N}(\mathfrak{J};\delta,\omega)$ is isomorphic to $\mathfrak{N} = \mathfrak{N}(\mathfrak{J};1,1)$. We have the following characterization of the automorphisms Aut $(\mathfrak{N})$ of $\mathfrak{N}$.

COROLLARY. *For* $\mathfrak{J}$ *reduced,* Aut $(\mathfrak{N})$ *consists of the maps*

$$S' : f_1 \to f_1\,, f_2 \to f_2\,, a_{12} \to (aS)_{12}\,, b_{21} \to (bS^{*-1})_{21}$$

and

$$S'' : f_1 \to f_2\,, f_2 \to f_1\,, a_{12} \to (aS^{*-1})_{21}\,, b_{21} \to (bS)_{12}\,,$$

where $S \in L(\mathfrak{J})$.

The subgroup $L(\mathfrak{J})' = \{S' : S \in L(\mathfrak{J})\}$ has index two in Aut $(\mathfrak{N})$ and is isomorphic to $L(\mathfrak{J})$. Jacobson [8] has shown that the center Z of $L(\mathfrak{J})$ consists of $\{\alpha I : \alpha^3 = 1\}$ and that for $\mathfrak{J}$ reduced $L(\mathfrak{J})/Z$ is simple.

LEMMA. *For* $\mathfrak{J}$ *reduced the center* Z' *of* Aut $(\mathfrak{N})$ *is* $\{\alpha I : \alpha^3 = 1\}$.

Proof. It suffices to show that no T'' is in Z'. If T'' is a central element, then $T'' = I''T'$. For all S in $L(\mathfrak{J})$, $S'I'' = I''(S^{*-1})'$. Since $S'I''T' = I''T'S'$, we then have $S^{*-1}T = TS$. Suppose that

$$e_1 T = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_{12} + y_{23} + z_{31}\,.$$

Let $S = U_a$ where $N(a) = \pm 1$. Since $U_a = U_a^*$ it must be true that $T = U_a T U_a$. Select $a = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ where $\alpha_1\alpha_2\alpha_3 = \pm 1$. A direct computation shows that

$$e_1 U_a T U_a = \alpha_1^2(\alpha_1^2 \xi_1 e_1 + \alpha_2^2 \xi_2 e_2 + \alpha_3^2 \xi_3 e_3 + \alpha_1\alpha_2 x_{12} + \alpha_2\alpha_3 y_{23} + \alpha_3\alpha_1 z_{31}).$$

For all admissible choices of the α_i, $e_1 T = e_1 U_a T U_a$ only if $\xi_2 = \xi_3 = 0$, $x = y = z = 0$, and $\alpha_1^4 = 1$. Hence $e_i T = \eta_i e_i$ and the ground field is $GF(5)$. However if we let $a = \alpha e_2 + z_{31}$, then

$$e_1 U_a T U_a = \gamma_1^{-2}\gamma_3^2\, n(z)^2 \eta_3 e_1\,,$$

$$e_1 T = \eta_1 e_1\,.$$

Over a finite field, $n(z)$ is a universal form. Hence $e_1 T = 0$, a contradiction.

COROLLARY. *For* $\mathfrak{J}$ *reduced* $L(\mathfrak{J})'/Z'$ *is a simple normal subgroup of index two in* Aut $(\mathfrak{N})/Z'$.

6. $\mathfrak{N}$-forms

We define an algebra $\mathfrak{N}$ over a field $\mathfrak{F}$ to be an $\mathfrak{N}$-*form* if there is an extension field $\mathfrak{K}$ such that $\mathfrak{N}_{\mathfrak{K}} \cong \mathfrak{N}(\mathfrak{J};\delta,\omega)$ over $\mathfrak{K}$. We call $\mathfrak{N}(\mathfrak{J};\delta,\omega)$ a *reduced* $\mathfrak{N}$-form. To construct an example of a non-reduced $\mathfrak{N}$-form we let $\mathfrak{K} = \mathfrak{F}(\sqrt{\lambda})$ be a quadratic extension and let $\mathfrak{N} = \mathfrak{N}(\mathfrak{J};1,1)$ over $\mathfrak{F}$. We let $\mathfrak{N}_\lambda$ be the $\mathfrak{F}$-subspace of $\mathfrak{N}_{\mathfrak{K}}$ spanned by $f = f_1 + f_2$, $\sqrt{\lambda}(f_1 - f_2)$, $a_{12} + a_{21}$, $\sqrt{\lambda}(b_{12} - b_{21})$ for $a, b \in \mathfrak{J}$. Then $\mathfrak{N}_\lambda$ is a non-reduced $\mathfrak{N}$-form, and $(\mathfrak{N}_\lambda)_{\mathfrak{K}} \cong \mathfrak{N}_{\mathfrak{K}}$.

The derivation algebra of $\mathfrak{N}_\lambda$ is a twisted Lie algebra of type E_6. This can be shown as follows. Any derivation D of $\mathfrak{N}_\lambda$ extends to a derivation of $\mathfrak{N}_{\mathfrak{K}}$. Hence $D_{12} = R(c + \sqrt{\lambda}\, d) + E_1 + \sqrt{\lambda} E_2$, where trace (c) = trace $(d) = 0$, and $E_1, E_2 \in \mathfrak{D}(\mathfrak{J})$. We know $(a_{12} + a_{21})D$ is in $\mathfrak{N}_\lambda$. That element is

$$(ca)_{12} - (ca)_{21} + \sqrt{\lambda}((da)_{12} - (da)_{21}) + (aE_1)_{12} + (aE_1)_{21} + \sqrt{\lambda}((aE_2)_{12} + (aE_2)_{21}).$$

Hence $c = 0$ and $E_2 = 0$ and $D_{12} = \sqrt{\lambda} R(d) + E_1$. These derivations $\mathfrak{D}(\mathfrak{N}_\lambda)$ form a Lie algebra of type E_6 for which the 54-dimensional space spanned by the elements $a_{12} + a_{21}$ and $\sqrt{\lambda}(b_{12} - b_{21})$ is irreducible but not absolutely irreducible. The Killing form for $\mathfrak{D}(\mathfrak{N}_\lambda)$ can be read off from (13). It is equivalent to

$$(15) \quad 6\lambda\xi_1^2 + 2\lambda\xi_2^2 + 6\lambda(\gamma_2^{-1}\gamma_1 n(x_1) + \gamma_3^{-1}\gamma_2 n(x_2) + \gamma_1^{-1}\gamma_3 n(x_3)) + 3K_2.$$

By letting $\mathfrak{F}$ be the real field and letting $\mathfrak{J}$ and λ vary, we obtain the five real Lie algebras of type E_6.

7. Lie algebras of type E_7

Any isomorphism between $\mathfrak{N}_1$ and $\mathfrak{N}_2$ induces an isomorphism between $\mathfrak{L}(\mathfrak{N}_1)$ and $\mathfrak{L}(\mathfrak{N}_2)$. Hence for reduced exceptional simple Jordan algebras, $\mathfrak{L}(\mathfrak{N}_1)$ and $\mathfrak{L}(N_2)$ are isomorphic if the coefficient algebras of the Jordan algebras are isomorphic. As the following proposition shows, we do not obtain any additional Lie algebras of type E_7 as $\mathfrak{L}(\mathfrak{N}_\lambda)$.

PROPOSITION. $\mathfrak{L}(\mathfrak{N}_\lambda) \cong \mathfrak{L}(\mathfrak{N})$.

Proof. $\mathfrak{L}(\mathfrak{N}_\lambda)$ is the $\mathfrak{F}$-subalgebra of $\mathfrak{L}(\mathfrak{N})_{\mathfrak{K}}$ spanned by

$$\sqrt{\lambda}L(f_1 - f_2), \qquad L(a_{12} + a_{21}), \qquad \sqrt{\lambda}L(b_{12} - b_{21}), \qquad \sqrt{\lambda}R(c), \qquad D.$$

A direct computation shows that the following map is an isomorphism of $\mathfrak{L}(\mathfrak{N}_\lambda)$ onto $\mathfrak{L}(\mathfrak{N})$. Let e be the identity of $\mathfrak{J}$.

$$\begin{aligned}
\sqrt{\lambda}L(f_1 - f_2) &\to \tfrac{1}{3}L(\lambda e_{12} + e_{21}),\\
L(e_{12} + e_{21}) &\to 3L(f_1 - f_2),\\
L(a_{12} + a_{21}) &\to 2R(a) \quad \text{if } (a, e) = 0,\\
\sqrt{\lambda}L(b_{12} - b_{21}) &\to -L(\lambda b_{12} - b_{21})\\
\sqrt{\lambda}R(a) &\to \tfrac{1}{2}L(\lambda a_{12} + a_{21}) \quad \text{if } (a, e) = 0,\\
D &\to D.
\end{aligned}$$

There is a way of putting an effective twist into $\mathfrak{L}(\mathfrak{N})$. Let $\mathfrak{F}(\sqrt{\lambda}) = \mathfrak{K}$ be a quadratic extension of the ground field and defined $\mathfrak{L}(\mathfrak{N})_\lambda$ to be the $\mathfrak{F}$-subspace of $\mathfrak{L}(\mathfrak{N})_{\mathfrak{K}}$ spanned by $\sqrt{\lambda}L(f_1 - f_2)$, $\sqrt{\lambda}L(a_{12} + a_{21})$, $L(b_{12} - b_{21})$, $\sqrt{\lambda}R(c)$, D, where $a, b, c \in \mathfrak{J}$ (over $\mathfrak{F}$), trace $(c) = 0$, and $D \in \mathfrak{D}(\mathfrak{J})$. Then

$\mathfrak{L}(\mathfrak{N})_\lambda$ is an $\mathfrak{F}$-subalgebra with Killing form

$$6\lambda\xi^2 + 2\delta\lambda Q(a) - 2\delta Q(b) + 6K_1'$$

where K_1' is the form (15). If $\mathfrak{F}$ is the real field, we obtain all four real forms of $\mathfrak{E}_7$ by letting $\mathfrak{J}$ and λ vary.

REFERENCES

1. A. A. ALBERT, *A construction of exceptional Jordan division algebras*, Ann. of Math., vol. 67 (1958), pp. 1–28.
2. A. A. ALBERT AND N. JACOBSON, *On reduced exceptional simple Jordan algebras*, Ann. of Math., vol. 66 (1957), pp. 400–417.
3. R. B. BROWN, *A new type of nonassociative algebras*, Proc. Nat. Acad. Sci. U. S. A., vol. 50 (1963), pp. 947–949.
4. H. FREUDENTHAL, *Beziehungen der $\mathfrak{E}_7$ und $\mathfrak{E}_8$ zur Oktavenebene. I*, Nederl. Akad. Wetensch. Proc. Ser. A, vol. 57 (1954), pp. 218–230.
5. N. JACOBSON, *Exceptional Lie algebras*, Yale University, dittoed notes, 1957.
6. ———, *Composition algebras and their automorphisms*, Rend. Circ. Mat. Palermo (II), vol. 7 (1958), pp. 1–26.
7. ———, *Some groups of transformations defined by Jordan algebras. I*, J. Reine Angew. Math., vol. 201 (1959), pp. 178–195.
8. ———, *Some groups of transformations defined by Jordan algebras. III*, J. Reine Angew. Math., 207 (1961), 61–85.
9. W. H. MILLS, *Classical type Lie algebras of characteristic* 5 *and* 7, J. Math. Mech., vol. 6 (1957), pp. 559–566.
10. W. H. MILLS AND G. B. SELIGMAN, *Lie algebras of classical type*, J. Math. Mech., vol. 6 (1957), pp. 519–548.
11. G. B. SELIGMAN, *On the split exceptional Lie algebra E_7*, Yale University, dittoed notes, 1962.
12. T. A. SPRINGER, *On a class of Jordan algebras*, Nederl. Akad. Wetensch. Proc. Ser. A, vol. 62 (1959), pp. 254–264.
13. ———, *The projective octave plane. I*, Nederl. Akad. Wetensch. Proc. Ser. A, vol. 63 (1960), pp. 74–88.
14. ———, *Characterization of a class of cubic forms*, Nederl. Akad. Wetensch. Proc. Ser. A, 65 (1962), 259–265.
15. ———, *Oktaven, Jordan-Algebren und Ausnahmegruppen*, University notes, Gottingen, 1963.

UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA