RATIONAL FUNCTIONS H^∞ AND H^p ON INFINITELY CONNECTED DOMAINS

BY

STEPHEN D. FISHER[1]

Introduction

Let D be a domain (an open connected set) in the complex plane and for $1 \leq p < \infty$, let $H^p(D)$ be those analytic functions f on D for which $|f|^p$ has a harmonic majorant on D. Fix t_0 in D and put $\| f \|_p = [u(t_0)]^{1/p}$ where u is the least harmonic majorant of $|f|^p$ on D. Then $\| \ \|_p$ is a norm on $H^p(D)$ which depends upon the point t_0 although the resulting topology on $H^p(D)$ does not. Let $H^\infty(D)$ be the algebra of bounded analytic functions on D with the uniform norm.

These H^p spaces, which generalize the classical Hardy H^p spaces in the unit disc U for $1 \leq p < \infty$, were introduced by Parreau in 1951 [5] and independently by Rudin in 1955 [6]. In his paper Rudin showed that if D is bounded by a finite number of disjoint circles then the rational functions with poles off $\bar{D}$ are dense in $H^p(D)$, $1 \leq p < \infty$, and hence $H^\infty(D)$ is dense in $H^p(D)$ for $1 \leq p < \infty$. Further, if D_1 is conformally equivalent to D, then $H^p(D_1)$ and $H^p(D)$ are isometrically isomorphic for $1 \leq p \leq \infty$; hence $H^\infty(D)$ is dense in $H^p(D)$ on all bounded domains with only finitely many complementary components. The aim of this paper is to show that $H^\infty(D)$ is dense in $H^p(D)$ on two types of infinitely-connected domains. These two types of domain are very different and the techniques of the proof differ vastly from one to the other. One type is treated in Section 1 and the other in Section 2. The author would like to thank Profs. F. Forelli and M. Voichick for several helpful conversations regarding the contents of this paper.

1.

If $H^\infty(D)$ contains non-trivial functions, then the unit disc U is the universal covering surface of D and hence there is an analytic function w from U onto D which is locally one-to-one and may be used to lift paths uniquely from D to U. If $f \in H^p(D)$, $1 \leq p \leq \infty$, then the analytic function $g(z) = f(w(z))$ is in $H^p(U)$ and if $w(0) = t_0$ (which we may assume without loss of generality), then $\| g \|_p = \| f \|_p$, where

$$\| g \|_p^p = \sup_{0 < r < 1} \left[\frac{1}{2\pi} \int_0^{2\pi} | g(re^{i\theta}) |^p \, d\theta \right] \qquad \text{for } 1 \leq p < \infty$$

and $\| g \|_\infty = \sup_{z \in U} | g(z) |$ is the usual H^p norm in the disc.

Let G be the group of linear fractional transformations T of U onto U such that $w(T(z)) = w(z)$ for all z in U. If $f \in H^p(D)$ and if $g = f \circ w$, then g is

nvariant under the group G. For if $T \epsilon G$ and $z \epsilon U$, then $g(T(z)) = f(w(T(z))) = f(w(z)) = g(z)$. Conversely, if g is in $H^p(U)$ and g is invariant under G, then there is an $f \epsilon H^p(D)$ with $g = f \circ w$. Each function h in $H^p(U)$ has non-tangential limits h^* almost everywhere with respect to arc-length on the unit circle Γ and the mapping $h \rightarrow h^*$ is an isometry of $H^p(U)$ onto a closed subspace, H^p, of $L^p(\Gamma, \sigma)$ where σ is Lebesgue measure on Γ. The elements of G are analytic homeomorphisms of Γ onto Γ; let H^p/G be those elements of H^p that are invariant under G; that is, $f \epsilon H^p/G$ if and only if $f \circ T = f$ a.e. σ for all $T \epsilon G$ and $f \epsilon H^p$. Hence, $H^p(D)$ is isometrically isomorphic to the (closed) subspace H^p/G of $L^p(\Gamma, \sigma)$.

A character on the group G is a homomorphism of G into the group of unimodular complex numbers. If χ is a character on G, an analytic function f on U is said to be automorphic with character χ if $f \circ T = \chi(T)f$ for each $T \epsilon G$. There is an intimate relation between automorphic functions on U and certain multiple-valued analytic functions on D. This relation is described below.

The group G is isomorphic to the fundamental group of D. Let $\{\alpha_j\}$ be a set of generators for the fundamental group of D with base point t_0. Each α_j can be lifted uniquely by w to a path $\tilde{\alpha}_j$ in U which begins at 0 and ends at some point z_j. The isomorphism of $\pi_1(D, t_0)$ onto G sends α_j to that linear fractional transformation of U onto U which sends 0 to z_j. This implies that if f is analytic on U and automorphic with character χ, then $f \circ w^{-1}$ is a multiple valued analytic function on D whose modulus is single-valued. If F is a branch of $f \circ w^{-1}$ in a neighborhood of t_0, then continuation of F along α_j leads to the value $\chi(T_j)F$ at t_0. The converse of this is also true. If F is a multiple valued analytic function on D whose modulus is single-valued and if continuation of a branch, F, of f in a neighborhood of t_0 along α_j leads to the value $\lambda_j F$ at t_0, where $|\lambda_j| = 1$, then $f \circ w$ is a (single-valued) analytic function on U which is automorphic with character χ, where $\chi(T_j) = \lambda_j$. (The number λ_j is called the phase of f along α_j.) Hence, an automorphic function on U is equivalent to an analytic function on D whose modulus is single-valued. (Voichick has examined this relation in more detail in [8; §3].) This duality will be exploited later in this section to prove Theorem 1.

It is convenient at this point to give two simple lemmas which will be used to prove the first theorem.

LEMMA 1. *Let D be a bounded open set in $\mathbf{C}$ whose complement has a finite number of components $C_0, C_1, \cdots, C_n$ where C_0 is the unbounded component and no C_i is trivial. Then there are bounded harmonic functions $g_1, \cdots, g_n$ and $h_1, \cdots, h_n$ on D such that (a) $g_i \geq 0$ and $h_i \leq 0$ for $i = 1, \cdots, n$ and (b) the period of g_i^* and the period of h_i^* about C_j is δ_{ij} for $1 \leq i, j \leq n$ where u^* is the harmonic conjugate of u.*

Proof. It clearly suffices to prove the lemma in the case $n = 1$ and there is no loss in assuming that D is bounded by disjoint circles.

Let $C_r(\partial D)$ be the continuous real-valued functions on ∂D. Let L be the

linear functional on $C_r(\partial D)$ given by $L(u)$ = period of the harmonic conjugate of $\tilde{u}$ about C_1, where $\tilde{u}$ is the harmonic extension of u to D. Then L is continuous and is orthogonal to the continuous analytic functions. Hence, $L(u) = \int_{\partial D} ufds$ where f is real, by the F. and M. Riesz Theorem. Since $0 = L(1) = \int_{\partial D} f\,ds$, f can not be of constant sign. It is now clear that the desired functions exist.

LEMMA 2. *Suppose $f_1, f_2, \cdots$ are elements of $H^{\infty}(U)$ with $\| f_n \|_{\infty} \leq 1$ and $f_n(z) \to 1$ for each z in U.   Then some subsequence of the f_n's converges a.e. to 1 on Γ.*

Proof. On Γ we have $\| f_n \|_2 \leq \| f_n \|_{\infty} \leq 1$ and hence some subsequence, again denoted by $\{f_n\}$, converges weakly in L^2 to an element f of H^2. Since $f_n(z) \to 1$ for all z in U, we must have $f \equiv 1$.   Hence

$$1 = \| f \|_2 \leq \liminf \| f_n \|_2 \leq \limsup \| f_n \|_2 \leq 1.$$

Thus $\lim \| f_n \|_2 = 1 = \| f \|_2$ so that f_n converges strongly to 1 in L^2. A further subsequence converges to 1 a.e.

THEOREM 1. *Let $S_1, S_2, \cdots$ be disjoint closed discs in U with centers on the real axis whose radii decreased to zero and whose centers increase to 1. Suppose there is a point $p_i \epsilon S_i$ for each i such that $\sum_{i=1}^{\infty} [1 - |p_i|] < \infty$. Let $D = U - \bigcup_{i=1}^{\infty} S_i$. Then $H^{\infty}(D)$ is dense in $H^p(D)$.*

Proof. Let

$$b_N(z) = \prod_{i=N}^{\infty} \frac{p_i - z}{1 - \bar{p}_i z} \left(\frac{\bar{p}_i}{|p_i|} \right).$$

Then b_N is bounded by 1 on U and converges uniformly on compact subsets in U, and hence in D, to 1 as $N \to \infty$. If $f \epsilon H^p(D)$, then $fb_N \epsilon H^p(D)$ also. I claim that some subsequence of $\{fb_N\}$ converges in $H^p(D)$ to f. To see this let w be the uniformizer of D and put $F = f \circ w$ and $B_N = b_N \circ w$. Then $F \epsilon H^p(U)$, and $B_N \epsilon H^{\infty}(U)$, $\| B_N \|_{\infty} \leq 1$. Furthermore, $B_N(z) \to 1$ for each z in U. Finally, the $H^p(D)$ norm of $(f - fb_N)$ is equal to $[\int_{\Gamma} |F - FB_N|^p d\sigma]^{1/p}$. By Lemma 2, some subsequence of the B_N's converges a.e. σ to 1. Hence, by the dominated convergence theorem, FB_N converges in L^p to F. Equivalently, fb_N converges in $H^p(D)$ to f, as desired. Consequently, to prove Theorem 1 it suffices to prove that for each N, fb_N lies in the $H^p(D)$-closure of $H^{\infty}(D)$.

Consider $F = f \circ w$. Since $F \epsilon H^p(U)$, F has an inner-outer factorization, $F = I \exp(V + iV^*)$ where V^* is the harmonic conjugate of V. I is not necessarily invariant under the group G but it is automorphic with a certain character χ, because the inner-outer factorization is unique up to constants of modulus one. Similarly, $\exp[V + iV^*]$ is automorphic with the character $\bar{\chi}$.
Let

$$V_n = V \quad \text{if} \quad |V| \leq n$$

$$= n \quad \text{otherwise}$$

and let $F_n = I \exp (V_n + iV_n^*)$. Since V_n is invariant under G, $\exp (V_n + iV_n^*)$ is automorphic with character ψ_n and because V_n converges in L^1 to V, $\psi_n(T) \to \bar{\chi}(T)$ for each $T \in G$. Hence, F_n is in $H^\infty(U)$, F_n is automorphic with character $\chi\psi_n$, and $(\chi\psi_n)(T) \to 1$ for each $T \in G$.

Let $f_n = F_n \circ w^{-1}$. Then f_n is a bounded analytic function on D which is not necessarily single-valued but its modulus is.

Let α_j be a circle in D such that S_j is interior to α_j and S_k is exterior to α_j for $k \neq j$. If we choose a branch, $\tilde{f}_n$, of f_n in a neighborhood of $\alpha_j(0)$ and continue this branch along the circle α_j, we end up with the value $\lambda_{nj}\tilde{f}_n$ at the point $\alpha_j(1) = \alpha_j(0)$, where $|\lambda_{nj}| = 1$. (See the comments at the beginning of this section.) Because F_n converges uniformly on compact subsets of U to $F = f \circ w$, we must have $\lambda_{nj} \to 1$ as $n \to \infty$ for each fixed j. Let s_{nj} be the unique number in $(-\frac{1}{2}, \frac{1}{2}]$ such that $\exp[2\pi i s_{nj}] = \bar{\lambda}_{nj}$ for each n and each j. Hence, $s_{nj} \to 0$ as $n \to \infty$ for each fixed j.

Let $g_1, \cdots, g_N$ and $h_1, \cdots, h_N$ be the harmonic functions given by Lemma 1 for the open set $U - S_1 \cup \cdots \cup S_N$. Let $u_n = \sum_{j=1}^{N} 2\pi s_{nj} H_j$ where

$$H_j = g_j \quad \text{if} \quad s_{nj} < 0$$

$$\quad = h_j \quad \text{if} \quad s_{nj} \geq 0.$$

Then u_n is negative and harmonic on D and the period of u_n^* about S_j is $2\pi s_{nj}$, $1 \leq j \leq N$, $n = 1, 2, \cdots$.

Finally, let

$$y_n = \sum_{j=N+1}^{\infty} [s_{nj} + 1] \log \left| \frac{z - p_j}{1 - \bar{p}_j z} \right|.$$

Then y_n is a negative harmonic function on D and the period of y_n^* about S_j is

$$2\pi(s_{nj} + 1) \equiv 2\pi s_{nj} \pmod{2\pi} \quad \text{for} \quad N + 1 \leq j < \infty \quad \text{and} \quad n = 1, 2, \cdots.$$

Furthermore, as $n \to \infty$, u_n converges uniformly to 0 on D and y_n converges uniformly on compact subsets of D to

$$y = \sum_{N+1}^{\infty} \log \left| \frac{z - p_j}{1 - \bar{p}_j z} \right|.$$

Let $g_n = f_n \exp[u_n + y_n + i(u_n^* + y_n^*)]$. Then g_n is a bounded, single-valued analytic function on D and g_n converges to fb_N in $H^p(D)$ as $n \to \infty$. To see the latter fact, let $G_n = g_n \circ w$, where w is the uniformizer of D. Then

$$G_n = F_n \circ \exp[u_n \circ w + y_n \circ w + i(u_n \circ w)^* + i(y_n \circ w)^*].$$

Now $u_n \circ w$ converges uniformly to 0 on Γ and $y_n \circ w$ converges in L^1 to $y \circ w$ by the dominated convergence theorem. Hence, at least some subsequence of $(y_n \circ w)^*$ converges a.e. σ to $(y \circ w)^*$. Thus the dominated convergence theorem implies that (some subsequence of) G_n converges in $L^p(\Gamma, \sigma)$ to $(F)(b_N \circ w) = (fb_N) \circ w$. Hence, g_n converges in $H^p(D)$ to

fb_N as desired.   This implies that fb_N is in the $H^p(D)$-closure of $H^\infty(D)$ and by the remarks at the start of the proof, this in turn implies that $H^\infty(D)$ is dense in $H^p(D)$.

This theorem actually holds in a much more general class of regions.   The proof given here did not depend in any way on the assumption that the deleted sets $\{S_i\}$ were discs; the only important fact was that the sum

$$\sum_1^\infty \log \left| \frac{z - p_i}{1 - \bar{p}_i z} \right|$$

converged.   The more general case can be stated as follows.

THEOREM 2.   *Let R be a domain bounded by a finite number of disjoint circles and let $g(\,\cdot\,, p)$ be the Green's function for R with singularity at p.   Let D be the subdomain of R: $D = R - \bigcup_{i=1}^\infty S_i$ where the S_i are a sequence of disjoint, compact, connected subsets of R all of whose accumulation points lie on ∂R and such that for each i there is a point $p_i \epsilon S_i$ satisfying $\sum_{i=1}^\infty g(z, p_i) < \infty$.   Then $H^\infty(D)$ is dense in $H^p(D)$, $1 \leq p < \infty$.*

Note that if $R = U$ the condition $\sum g(z, p_i) < \infty$ is equivalent to $\sum (1 - |p_i|) < \infty$.

Proof.   The proof follows along exactly the same lines as the proof of Theorem 1, with a few added complications.   Note that if some S_i is a point, then it is a removable singularity for H^p; hence it may be assumed that each S_i is non-trivial.

The (multiple-valued) Blaschke product

$$b_N(z) = \exp \left[\sum_{j=N}^\infty g(z, p_j) + i \left(\sum_N^\infty g(z, p_j) \right)^* \right]$$

replaces the Blaschke product used in the proof of Theorem 1.   Since R is not simply-connected the automorphic functions on U correspond to functions on D which may have non-trivial phases around the bounded components in the complement of R.   These phases and the multiple-valuedness of b_N may be corrected by using the bounded harmonic functions on R given by Lemma 1.

Finally, in a paper dealing with another aspect of H^∞ on such an infinitely-connected domain Voichick [7; Lemma 1] showed that there are simple closed curves $\alpha_1 , \alpha_2 , \cdots$ in D such that S_j is interior to α_j and S_k is exterior to α_j when $j \neq k$; also that there are simple closed curves $\beta_1 , \cdots , \beta_n$ in D which are an homology basis for R (where n is one less than the number of boundary components of R) and the totality of these curves $\{\beta_1 , \cdots , \beta_n , \alpha_1 , \alpha_2 , \cdots\}$ form an homology basis for D.   These facts and the technique of the proof of Theorem 1 provide a proof of Theorem 2.

If the deleted sets S_i satisfy the further conditions that the interior of each S_i has only a finite number of components and is dense in S_i, and the

set of accumulation points of the S_i on ∂R has arc-length zero, then a modification of the proof of Theorem 2 combined with some results in [4; §3] shows that the rational functions with poles off $\bar{D}$ are dense in $H^p(D)$, $1 \le p < \infty$.

2.

There is another type of infinitely connected domain on which H^∞ is dense in H^p. This domain is obtained by deleting from U a sequence of disjoint closed discs S_1, S_2, $\cdots$ centered on the positive real axis which converge to the origin and whose centers and radii satisfy certain restrictions. These numerical conditions will be discussed later when they are relevant. Actually, we will show that R, the set of rational functions with poles off $\bar{D}$, is dense in H^p.

In this section μ will always denote harmonic measure on ∂D for the particular point t_0 in D for which the $H^p(D)$-norm is defined, and ds will denote arc-length on ∂D.

LEMMA 3. *Let D be obtained by deleting from U the origin and a sequence S_1, S_2, $\cdots$ of disjoint closed discs centered on the positive real axis whose centers and radii decrease to 0. Let $f \epsilon H^p(D)$, $1 \le p \le \infty$. Then f has boundary-values f^* a.e. ds on ∂D and*
  (i) $f^* \epsilon L^p(\partial D, \mu)$,
  (ii) $\| f \|_{H^p} = [\int_{\partial D} | f^* |^p d\mu]^{1/p}$,
  (iii) $f(z) = \int_{\partial D} f^* d\mu_z$ *for all z in D, where μ_z is harmonic measure on ∂D for z.*

Proof. Let d_n be a small disc centered at 0 with boundary γ_n such that $S_1 \cup \cdots \cup S_n$ is exterior to γ_n and $S_{n+1} \cup \cdots$ is interior to γ_n. Let $D_n = D - d_n$. Then D_n is a finite circle domain and $f \epsilon H^p(D_n)$. Hence $f^* \epsilon L^p(\partial D_n, \mu_n)$, the norm of $f \mid D_n$ is equal to $[\int_{\partial D_n} | f^* |^p d\mu_n]^{1/p}$, and $f(z) = \int_{\partial D_n} f^* d\mu_n^z$ for all $z \epsilon D_n$, where μ_n^z is harmonic measure on ∂D_n for z. Since μ_k^z increases to μ_z on $\partial D \cap \partial D_n$ as $k \to \infty$ for each n and each $z \epsilon D$ and since $\bigcup_{n=1}^\infty [\partial D \cap \partial D_n] = \partial D - \{0\}$, assertion (i) follows immediately; in addition, this also gives

$$\int_{\partial D} | f^* |^p d\mu \le \| f \|_{H^p}^p .$$

Since $\mu_n(\gamma_n) \to 0$ as $n \to \infty$, we have $\int_{\gamma_n} f d\mu_n \to 0$ as $n \to \infty$ for each f in $H^p(D)$ when $1 < p \le \infty$ by Hölder's inequality; hence, (iii) holds for $f \epsilon H^p(D)$, $1 < p \le \infty$. This in turn implies that

$$| f(z) |^p \le \int | f^* |^p d\mu_z \qquad\qquad \text{for all } z \text{ in } D$$

and hence that $\| f \|_{H^p}^p \le \int | f^* |^p d\mu$. Thus (ii) and (iii) are established for $f \epsilon H^p$ when $1 < p \le \infty$.

When $f \epsilon H^1(D)$, conclusions (ii) and (iii) are more subtle. As above, once assertion (iii) is established we will also have (ii), so we concentrate

on (iii).   To establish (iii) for $f \in H^1(D)$, it is sufficient (and necessary) to show that $\int_{\gamma_n} |f| \, d\mu_n \to 0$ as $n \to \infty$.

To do this, we consider the space S consisting of all functions u harmonic on D with the property that $\int_{\partial \Delta} |u|^2 \, d\mu_\Delta$ is bounded independent of Δ for all subdomains Δ of D which contain the point t_0 and which have a finite number of disjoint circles for boundary, where μ_Δ is harmonic measure on $\partial \Delta$ for t_0.   Note that if $u \in S$, then $\int_{\gamma_n} |u| \, d\mu_n \to 0$ as $n \to \infty$.

By using the uniformizer, w, from the unit disc U onto D it is not difficult to show that S is isometrically isomorphic to L^2/G, the (closed) subspace of $L^2(\Gamma, \sigma)$ consisting of functions invariant under the group G associated with w and D (see Section 1 for more details on w and G).   It is also the case that L^2/G is dense in L^1/G (which is the corresponding subspace of $L^1(\Gamma, \sigma)$; see Forelli, *Bounded analytic functions and projections*, this journal, vol. 10 (1966), pp. 367–380).

Now $F = f \circ w$ is in $H^1(U)/G$ and $F^* \in L^1/G$.   Hence, given $\varepsilon > 0$ there is an element H of L^2/G such that $\int_\Gamma |H - F^*| \, d\sigma < \varepsilon$.   If h denotes the harmonic function on D such that $(h \circ w)(z) = \int_\Gamma H P_z \, d\sigma$ for z in U, then $h \in S$ and $\int_{\partial \Delta} |h - f| \, d\mu_\Delta < \varepsilon$ for all subdomains Δ of D bounded by a finite number of disjoint circles.   Hence, $\int_{\gamma_n} |h - f| \, d\mu_n < \varepsilon$ for each n.   However, $\int_{\gamma_n} |h| \, d\mu_n \to 0$ as $n \to \infty$ since $h \in S$.   Thus $\lim \sup_{n \to \infty} \int_{\gamma_n} |f| \, d\mu_n < \varepsilon$ for each $\varepsilon > 0$.   Therefore, $\lim_{n \to \infty} \int_{\gamma_n} |f| \, d\mu_n = 0$, as desired.

LEMMA 4.   *Let D be a domain satisfying the hypotheses of Lemma 3 and let $g \in L^p(\partial D, ds)$, $1 \le p \le \infty$, and suppose that $\int_{\partial D} g(z)\phi(z) \, dz = 0$ for all $\phi \in R(\bar{D})$.   Let γ be a circle in D about 0 and let $E = \{z \in D \mid z \text{ is exterior to } \gamma\}$. Then*

$$G(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{g(w)}{w - z} \, dw$$

is in $H^p(E)$ and $G^* = g$ a.e. ds on $\partial E \cap \partial D$.

Proof.   Note that $\partial E = \gamma \cup [\partial E \cap \partial D]$.   Define a function h on ∂E by

$$h = g \quad \text{on} \quad \partial E \cap \partial D$$

$$= G \quad \text{on} \quad \gamma$$

where G is given above.   Since G is bounded on γ, $h \in L^p(\partial E, ds)$.   It is easy to verify that $\int_{\partial E} h(z)\phi(z) \, dz = 0$ for all rational functions ϕ with poles off $\bar{E}$.   Hence, there is an $H \in H^p(E)$ such that $H^* = h$ a.e. ds and

$$H(z) = \frac{1}{2\pi i} \int_{\partial E} \frac{h(w)}{w - z} \, dw \qquad \text{[6; Theorem 3.2]}.$$

But it follows immediately by integration that

$$H(z) = \frac{1}{2\pi i} \int_{\partial E} \frac{h(w)}{w - z} \, dw = \frac{1}{2\pi i} \int_{\partial D} \frac{g(w)}{w - z} \, dw = G(z).$$

The next lemma seems to be known by a number of people but apparently has not appeared in print; it was communicated to me by Lawrence Zalcman.

LEMMA 5. *Let S_1, S_2, $\cdots$ be a sequence of disjoint closed discs in U where S_n is centered at c_n and has radius r_n. Suppose that $c_n > 0$ for all n and that c_n and r_n decrease to 0. Let $D = U - \{0\} \cup \bigcup_{n=1}^{\infty} S_n$. Suppose further that there is a $\delta > 0$ such that $r_n/c_n \geq \delta$ for all n. Then there is an element ϕ of the uniform closure R such that $\phi(0) = 1$ and $|\phi| < 1$ on $\bar{D} - \{0\}$.*

THEOREM 3. *Let S_1, S_2, $\cdots$ be a sequence of disjoint closed discs in U where S_n is centered at c_n and has radius r_n. Suppose that $c_n > 0$ for all n and that c_n and r_n decrease to zero as $n \to \infty$. Further, suppose that*

    (a)   *there is a $\delta > 0$ such that $r_n/c_n \geq \delta$ for all n;*

    (b)   *there is an integer N and a constant C such that*

$$\frac{(c_n - r_n)^N}{(c_n - r_n) - (c_{n+1} + r_{n+1})} \leq C \qquad \text{for all } n.$$

Let $D = U - \{0\} \cup \bigcup_{n=1}^{\infty} S_n$. Then $H^{\infty}(D)$ is dense in $H^p(D)$, $1 \leq p < \infty$.

Proof. We will, in fact, show that R is dense in $H^p(D)$, where R is the set of rational functions with poles off $\bar{D}$.

Let p be fixed, $1 \leq p < \infty$ and let q be the conjugate exponent of p. To show that R is dense in $H^p(D)$ it is sufficient by Lemma 3 to show that if $g \in L^q(\partial D, \mu)$ and $\int_{\partial D} g\phi \, d\mu = 0$ for all $\phi \in R$, then $\int_{\partial D} gf \, d\mu = 0$ for all $f \in H^p$.

On ∂D it is easy to see that the finite measure dz and $d\mu$ are mutually absolutely continuous; let $d\mu = F \, dz$. Then $gF \in L^1(\partial D, ds)$ and $\int_{\partial D} gF\phi \, dz = 0$ for all $\phi \in R$. Let γ_n be the circle in D centered at 0 of radius

$$\tfrac{1}{2}[(c_{n+1} + r_{n+1}) + (c_n - r_n)]$$

and let D_n consist of those points of D that are exterior to γ_n. Then D_n is bounded by the $n + 2$ disjoint circles $\Gamma_0, \Gamma_1, \cdots, \Gamma_n, \gamma_n$ where Γ_0 is the unit circle and $\Gamma_j = \partial S_j$ for $j = 1, \cdots, n$. By Lemma 4, the analytic function

$$G(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{g(w)F(w)}{w - z} \, dw = \frac{1}{2\pi i} \int_{\partial D} \frac{g(w)}{w - z} \, d\mu(w)$$

is in $H^1(D_n)$ for each n. Let $h(z) = z^N G(z)$ for $z \in D$. Then it is easy to check that

$$h(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{w^N g(w)}{w - z} \, d\mu(w).$$

Let μ_n be harmonic measure on ∂D_n for $t_0 \in D_n$, where t_0 is the point used for determining the norm in $H^p(D_n)$. Then the $H^1(D_n)$-norm of h equals

(*) $$\int_X |w|^N |g(w)| |F(w)| \, d\mu_n(w) + \int_{\gamma_n} |h| \, d\mu_n$$

where $X = \Gamma_0 \cup \cdots \cup \Gamma_n$. In order to estimate these and other terms, it will be useful at this point to derive some inequalities involving the various harmonic measures that have appeared.

On $\Gamma_0 \cup \cdots \cup \Gamma_n$, elementary arguments show that $\mu(E) \geq \mu_n(E) \geq 0$ for all Borel sets E in $\Gamma_0 \cup \cdots \cup \Gamma_n$. It is also true that for a fixed set E in $\Gamma_0 \cup \cdots \Gamma_n$, $\lim_{k\to\infty} \mu_k(E) = \mu(E)$ and hence $\mu_n(\Gamma_0 \cup \cdots \cup \Gamma_n)$ increases to $\mu(\partial D) = 1$ as $n \to \infty$. Thus $\mu_n(\gamma_n)$ decreases to 0 as $n \to \infty$. On Γ_j μ is dominated by the Poisson kernel for the domain exterior to Γ_j for $j \geq 1$ or by the Poisson kernel for the unit disc if $j = 0$. This kernel, for $j \geq 1$ and for the point t_0, is equal to

$$\frac{|t_0 - c_j|^2 - r_j^2}{|e^{i\theta}(t_0 - c_j) - r_j|^2}$$

where Γ_j is centered at c_j and of radius r_j. Hence, the kernel is less than or equal to

$$\frac{|t_0 - c_j| + r_j}{|t_0 - c_j| - r_j}.$$

Since $d\mu = |F| \, ds$ we have on Γ_j,

$$|F| \leq \left(\frac{1}{r_j}\right)\left(\frac{|t_0 - c_j| + r_j}{|t_0 - c_j| - r_j}\right) \leq \left(\frac{1}{r_j}\right) C_1$$

where C_1 depends on t_0 but not on j. Now $r_j/c_j \geq \delta > 0$ so that $c_j/r_j \leq 1/\delta$ and this implies that on Γ_j,

$$|z|^N |F| \leq \frac{(c_j + r_j)^N}{r_j} C_1 \leq \frac{1}{\delta} C_2$$

where C_2 is independent of j. Thus $|z|^N |F|$ is bounded on ∂D. This implies that the first term in $(*)$ is bounded by

$$\frac{1}{\delta} C_2 \int_X |g| \, d\mu_n \leq \frac{C_2}{\delta} \int_X |g| \, d\mu \leq \frac{C_2}{\delta} \int_{\partial D} |g| \, d\mu \leq \frac{C_2}{\delta} \|g\|_q.$$

Furthermore, if $z \in \gamma_n$ and $w \in \partial D$, then

$$\left|\frac{z^N}{w - z}\right| \leq (c_n - r_n)^N (\text{distance of } \gamma_n \text{ to } \partial D)^{-1} \leq 2C$$

by hypothesis (b) and the fact that the radius of γ_n is

$$\tfrac{1}{2}[(r_n - c_n) + (r_{n+1} + c_{n+1})].$$

Hence, we may estimate the second term in $(*)$ by

$$\int_{\gamma_n} \left|\frac{1}{2\pi i} \int_{\partial D} \frac{w^N g(w)}{w - z} \, d\mu(w)\right| d\mu_n(z)$$

$$= \int_{\gamma_n} \left|\frac{1}{2\pi i} z^N \int_{\partial D} \frac{g(w)}{w - z} \, d\mu(w)\right| d\mu_n(z)$$

$$\leq \frac{1}{2\pi} \int |g(w)| \left[\int_{\gamma_n} \left| \frac{z^N}{w-z} \right| d\mu_n(z) \right] d\mu(w)$$

$$\leq \frac{1}{2\pi} \|g\|_q \cdot 2C \cdot \mu_n(\gamma_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus the $H^1(D_n)$-norm of h is bounded independent of n and hence $h \in H^1(D)$. Actually, since $h = z^N Fg$ on ∂D where $g \in L^q(\partial D, \mu)$ and $z^N F$ is bounded on ∂D, we have $h \in H^q(D)$.

We need only one other estimate now. Let μ_z be harmonic measure on ∂D for $z \in \gamma_n$. Then we know that $d u_z = H_z \, ds$ on ∂D and

$$H_z(t) \leq \left(\frac{1}{r_j} \right) \left(\frac{|z - c_j| + r_j}{|z - c_j| - r_j} \right) \qquad \text{for } t \in \Gamma_j$$

by our previous estimates. Thus

$$|z|^N H_z(t) \leq \frac{1}{r_j} 4C \qquad \text{on } \Gamma_j.$$

Note that if $f \in H^p(D)$, then $fh \in H^1(D)$ and, in fact, since $h = z^N Fg$, $(fh)^* \in L^1(\partial D, ds)$. This gives

$$\int_{\gamma_n} |z|^{2N} |fh| \, d|z|$$

$$= \int_{\gamma_n} |z|^N \left| \int_{\partial D} t^N f(t) h(t) \, d\mu_z(t) \right| d|z|$$

$$\leq \int_{\partial D} |f(t) h(t)| \, |t|^N \left[\int_{\gamma_n} |z|^N H_z(t) \, d|z| \right] ds_t$$

$$\leq \sum_{j=0}^{\infty} \int_{\Gamma_j} |f(t) h(t)| \, |t|^N \left[\frac{1}{r_j} (4C) \, \text{length} \, (\gamma_n) \right] ds_t$$

$$\leq \left(\sum_{j=0}^{\infty} \int_{\Gamma_j} |f(t)| \, |h(t)| \, ds_t \right) (C_3 \, \text{length} \, (\gamma_n)) \to 0 \quad \text{as} \quad n \to \infty$$

where C_3 is a constant. We have used here the fact that $|t^N/r_j|$ is bounded for $t \in \Gamma_j$ independent of j. Since $fh \in H^1(D_n)$ we have

$$0 = \int_{\partial D_n} z^{2N} f(z) h(z) \, dz \qquad \text{for all } n.$$

Since $fh \in L^1(\partial D, ds)$ and since we've just shown that $\int_{\gamma_n} |z^{2N} fh| \, ds \to 0$ as $n \to \infty$, we have

$$0 = \int_{\partial D} z^{2N} f(z) h(z) \, dz = \int_{\partial D} z^{2N} f(z) z^N g(z) F(z) \, dz$$

$$= \int_{\partial D} z^{3N} f(z) g(z) \, d\mu(z) \qquad \text{for all } f \in H^p(D).$$

This is almost what we want; all that remains to be done is to eliminate the factor z^{3N}.

Hypothesis (a) implies by Lemma 5 that 0 is a peak point for the continuous analytic functions. Let ϕ peak at 0; i.e. $\phi(0) = 1$ and $|\phi| < 1$ elsewhere on $\bar{D}$. Fix n and consider the continuous analytic function $1 - \phi^n$. $1 - \phi^n$ vanishes at 0 and hence by a theorem of Arens [2] there are continuous analytic functions a_k on $\bar{D}$ such that $a_k z$ converges to $1 - \phi^n$ uniformly on ∂D. Now $0 = \int_{\partial D} z^{3N} a_k fg\, d\mu$ for all k since $a_k f \in H^p(D)$. Let $k \to \infty$; we get $0 = \int_{\partial D} z^{3N-1}(1 - \phi^n)fg\, d\mu$. Since this holds for all n, letting $n \to \infty$ we get $0 = \int_{\partial D} z^{3N-1} fg\, d\mu$ for all $f \in H^p(D)$. Continue this process; at the $3N$th step we find that $0 = \int_{\partial D} fg\, d\mu$, as desired.

It is worthwhile pointing out that the hypotheses of Theorem 3 are fulfilled, for example, when $r_n = \frac{1}{2}c_n$ and $c_{n+1} \leq (\frac{1}{8})c_n$ for all n, with $N = 1$.

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University of Wisconsin
Madison, Wisconsin