NORMAL COMPLEMENTS OF CARTER SUBGROUPS

BY

SURINDER K. SEHGAL AND WILLIAM A. MCWORTER, JR.

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Carter [3] has shown that if U is a subgroup of a finite group G that satisfies

(a) U is nilpotent,
(b) $N_G(U) = U$,
(c) U is a Hall subgroup of G,
(d) the Sylow subgroups of U are regular;

then G has a normal subgroup N satisfying $UN = G$ and $U \cap N = \{1\}$, (i.e., U has a normal complement in G).

In case G is solvable, a nilpotent, self-normalizing subgroup U is called a Carter subgroup of G. It follows easily from Lemma 1 below that if U is a Carter subgroup of a solvable group G and U has a normal complement in G, then G has property P (for a definition of property P see Carter [1]). We will show in this paper that if U is a subgroup of a finite solvable group G having property P and satisfying properties (a), (b), and (c) of Carter's theorem stated above, then U has a normal complement in G.

We will show by means of an example that condition (c) in the statement of our theorem is necessary. We will also show that there exists groups G having property P where the Sylow subgroups of Carter subgroups are not regular. The last construction also shows the existence of infinitely many finite solvable groups containing a given nilpotent group as a Carter subgroup.

The following lemma is proved in [5].

LEMMA. *A finite solvable group G has property P if and only if it has a subgroup U satisfying*

(a) *U is maximal nilpotent,*
(b) *there exists a normal subgroup $N \neq \{1\}$ of G with $U \cap N \subseteq Z(G)$ or $U = G$,*
(c) *property* (b) *is satisfied by the image of U in any homomorphic image. Furthermore U is a Carter subgroup of G.*

THEOREM. *If U is a subgroup of a finite solvable group G having property P such that*

(a) *U is nilpotent,*
(b) *$N_G(U) = U$,*
(c) *U is a Hall subgroup of G,*

then U has a normal complement in G.

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Proof. Suppose the result is not true. Let G be a group of least order for which the result is false. By the lemma there exists a normal subgroup N satisfying $U \cap N \subseteq Z(G)$. Pick N minimal normal in G and satisfying $U \cap N \subseteq Z(G)$. In case $U \cap N = \{1\}$, we have UN/N satisfies properties (a), (b) and (c) in G/N which has property P by [1]. Therefore UN/N has a normal complement M/N i.e. we have $U \cdot M = G$ and $U \cap M = \{1\}$. So we can assume $1 \neq U \cap N \subseteq Z(G)$. In which case $N \subseteq U$.

Now U/N satisfies properties (a), (b) and (c) in G/N which has property P by [1]. Therefore U/N has a normal complement M/N. Thus we have $U \cdot M = G$ and $U \cap M = N$. Since $| G | = | U | \cdot | M |/| N |$ and U is a Hall subgroup of G and $N \subseteq U$, we have $(| N |, | M/N |) = 1$. By Hall's Theorem there exists a subgroup N' of M such that $| N' | = | M/N |$. But $N \subseteq Z(G)$. Therefore N' is a normal, Hall subgroup of M. Hence N' is normal in G. Now we have $U \cdot N' = G$ and $U \cap N' = \{1\}$. i.e. N' is a normal complement of U in G. Which gives us a contradiction and proves our result.

Our first example is of a solvable group G having property P in which the Carter subgroup U is not a Hall subgroup and does not have a normal complement in G. Let A be the non-abelian group of exponent 3 and order 27. A has two generators a and b. Define the mapping α on A by $\alpha(a) = a^2(a, b)^{-1}$ and $\alpha(b) = b^2(a, b)$. It is easily checked that the mapping α extends to an automorphism of A of order 2 centralizing the center of A. Let G be the semidirect produce of A by α. Then $U = \{\alpha, (a, b)\}$ is a Carter subgroup of G which is not a Hall subgroup of G and does not have a normal complement in G. Since the group G has nil length 2 it follows that G has property P by [1].

Our next example exhibits a solvable group G with property P whose Carter subgroup contains a non-regular Sylow subgroup. The Carter subgroup has a normal complement in G. Let U be any non-regular p-group of order p^n. Let V be the group algebra of U over the field of q elements, q a prime different from p. Considering V as a vector space let W' be the subspace on which U acts trivially. Clearly, W' is a proper subspace of V. By Maschke's Theorem W' has a U-invariant complement W in V. Let G be the semidirect product of W by U. Clearly, G is solvable. We claim that U is a Carter subgroup of G. Since W is a normal complement of U in G, any element of W which normalizes U also centralizes U. Hence $N_G(U) = U$. Since G is of nil length 2, G has property P.

Clearly, this last construction does not depend on the structure of the group U. Hence, given any nilpotent group U, we may, by varying the choice of the prime q, construct infinitely many solvable groups G having U as their Carter subgroup.

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The Ohio State University
Columbus, Ohio