ALMOST PERIODIC SEMIGROUPS IN TRANSFORMATION GROUPS

BY
William Reddy

Introduction

If a measure preserving flow is defined on a measure space X of finite measure, then the flow is incompressible, and it follows that almost all the points of X are recurrent (Poincaré recurrence theorem; see [5, p. 10]). W. H. Gottschalk and G. A. Hedlund [3], [4, chapter 8], and C. W. Williams [10] have exploited similar ideas in topological dynamics, and J. D. Baum [1] has also done this in a more general setting. A. Khintchine [8] has shown that if m is a proability measure defined on X, then

$$\lim_{T-S\to\infty} \frac{1}{T-S} \int_S^T m(A \cap At)\, dt > m^2(A)$$

for a measure preserving flow, where A is any measurable set. (See also E. Hopf [7, p. 40].) If the measure is a Borel measure not vanishing on open sets and the flow is continuous, it follows that the flow is regionally almost periodic (under Gottschalk and Hedlund's definition [4]). The purpose of this paper is to "topologize" notions related to the measure preserving property and investigate their relationship to almost periodicity properties.

In the first section it is shown that in many phase spaces it is sufficient for some almost periodicity properties to establish the corresponding property with respect to a replete semigroup in the transformation group, but that this is not true for regional almost periodicity. The following section investigates boundedness, incompressibility, and dissipative properties.

1. S-almost periodicity

1.1 *Notation.* Let (X, T, π) or (X, T) denote a transformation group. We assume that T is generative, and we let S denote a replete semigroup in T (see [4] for definitions). We assume that X is a Hausdorff space. Whenever we assume that X is a uniform space, we assume that it has the uniform topology and write small Greek letters for elements of the uniformity.

1.2 Definition. A subset B of T is called *S-syndetic* if there exists a compact subset K of T for which $BK \supset S$. This definition differs from that of Gottschalk and Hedlund [4, p. 63]. All S-almost periodicity properties which we will study are now defined as in [4, 3.13] where "S-syndetic" replaces "admissible."

1.3 *Remark.* The following statements are pairwise equivalent:

(1) The transformation group (X, T) is S-almost periodic.

Received May 13, 1964; received in revised form March 28, 1967.

(2-3-4-5) To each index α of X there exists a compact subset K of T such that to each $s \in S$ there corresponds a $k \in K$ such that, for each $x \in X$

$$(xs \in x\alpha k) \qquad (xsk \in x\alpha) \qquad (xs \in xk\alpha) \qquad (xk \in xs\alpha).$$

1.4 Inheritance Theorem. *Let P be a closed syndetic subgroup of T. If (X, T) is S-almost periodic, then (X, P) is $P \cap S$-almost periodic.*

1.5 *Remark.* The following remarks, leading to Theorem 1.10, can be replaced by using Baum's equicontinuity condition (John D. Baum, *An equicontinuity condition in topological dynamics*, Proc. Amer. Math. Soc., vol. 12 (1961), pp. 30–32).

1.6 Lemma. *If A is an S-syndetic set, then so is $A \cap S$.*

Proof. There exists a compact set K such that, for each $p \in S$, $pK \cap A \neq \emptyset$. Further, there exists an element $p_1 \in S$ such that $p_1 K = K' \subset S$. Now, to each $p \in S$, we have

$$\begin{aligned} pK' \cap (A \cap S) &= (pK' \cap S) \cap (pK' \cap A) \\ &= pK' \cap A \\ &= pp_1 K \cap A \neq \emptyset. \end{aligned}$$

Therefore $A \cap S$ is an S-syndetic set.

1.7 Lemma. *If A is an S-syndetic set and B is an S^{-1} syndetic set, then AB is a syndetic set in T.*

Proof. There exist compact sets K' and K'' with $AK' \supset S$ and $BK'' \supset S'$. Using [4, 6.04], we infer

$$T = SS^{-1} = AK'BK'' = (AB)K'K''.$$

1.8 Theorem. *If X is a compact uniform space, then the following statements are pairwise equivalent:*

(1) *(X, T) is S-almost periodic.*

(2) *(X, T) is discretely S-almost periodic.*

(3) *The set of transitions $\{\pi^s : s \in S\}$ is totally bounded subset of T in the space-index uniformity.*

(4) *The set of transitions $\{\pi^s : s \in S\}$ is uniformly equicontinuous.*

Proof. (1) is equivalent to (2) by an argument much like [4, 4.35]. (2) is equivalent to (3) by application of Remark 1.3, (4). (3) is equivalent to (4) by [4, 11.12].

1.9 Theorem. *If X is uniform and compact, then (X, T) is S-almost periodic if and only if it is S^{-1}-almost periodic.*

Proof. It suffices to show that the set of transitions $\{\pi^s : s \in S\}$ is a totally bounded subset of T in the space-index uniformity if and only if $\{\pi^s : s \in S^{-1}\}$ is such a set. But this follows from [4, 11.18(1)].

1.10 THEOREM. *Let X be a compact space. (X, T) is S-almost periodic if and only if it is almost periodic.*

Proof. It is obvious that almost periodicity implies S-almost periodicity. Suppose now that (X, T) is S-almost periodic. Then $S \cup S^{-1}$ is a uniformly equicontinuous subset of T. Let an index α of X be given. There exists an index β such that $\beta t \subset \alpha$ for all t in $S \cup S^{-1}$. Choose an S-syndetic set A and an S^{-1}-syndetic set B such that $x(A \cup B) \subset x\beta$ for all points x in X. Now let $a \in A$ and $b \in B$. We have $xab \in (x\beta)b \subset x\alpha$. Since AB is a syndetic subset of T, (X, T) is almost periodic.

1.11 *Remark.* Statements 1.14 through 1.31 are a proof that S-almost periodicity at a point implies almost periodicity there, provided that the phase group is generative and the phase space is a compact, metric space, or a locally compact, separable, metric space. The idea of the proof is to show that if a counter-example exists, then one also can be found in a subspace restriction of a transformation group like Bebutov's dynamical system [9, pp. 420–424], and then to show by ad hoc methods that this cannot be.

1.12 *Convention.* We will make free use of the notion of homomorphism and isomorphism of transformation groups, and any results of [2].

1.13 DEFINITION. Let (X, T) be a transformation group. If x is a point which is S-almost periodic for some replete semigroup in T and is not almost periodic, we say that x is a *partly almost periodic point.* We also say that (X, T) is *point partly almost periodic.*

1.14 *Remark.* If (X, T) possesses a partly almost periodic point, there exists a transformation group (X, T') possessing a partly almost periodic point, where T' has no proper compact subgroups.

1.15 *Remark.* It follows from remark 1.14 and [4, 3.36 and 6.14] that if there exists a transformation group (X, T) possessing a partly almost periodic point, then there exists a discrete n-parameter flow (for some n) with the same property.

1.17 LEMMA. *Let (X, T) be a point partly almost periodic discrete n-parameter flow for which X is a locally compact, separable, metric space. Then there exists a point partly almost periodic fix-point free continuous n-parameter flow (Y, R^n) where Y is a locally compact, separable, metric space.*

Proof. Let $f_1, \cdots, f_n$ be the generators of the discrete n-parameter flow. Let I denote the real line, and let $Z = X \times I \times \cdots \times I$, where there are n factors of I. Let H denote the set of points of $I \times \cdots \times I$ having integral coordinates. If $h = (h_1, \cdots, h_n)$ is in H, we associate to it the element g_h of T which has coordinates $(h_1, \cdots, h_n)$ with respect to the generators $f_1, \cdots, f_n$. To each point x in X, form the equivalence relation in Z which identifies (x, t) to $(xg_h, t + h)$ for each h in H. Let R denote the equivalence relation just defined, and let $Y = Z/R$. The space Y is clearly locally compact and Hausdorff, and, since the projection is open, completely separable, and hence metric. We now define an n-parameter flow on Y. If $t = (t_1, \cdots, t_n)$ is an element of R^n, and (x, τ) is an element of Z, define

$$(x, \tau)\cdot t = (x, \tau + t)$$

This flow clearly respects the relation, and hence it is a continuous n-parameter flow on Y. It is obviously totally fixed-point free. The subgroup of lattice points in R^n leaves fixed a subspace homeomorphic to X on which its action is topologically equivalent to that of T. Then by [4, 3.36], (Y, R^n) is a point partly almost periodic flow.

1.18 *Remark.* Lemma 1.17 remains valid if we assume that X is a compact metric space.

1.19 *Examples and notation.* Let F_n denote the set of continuous E_n-valued functions defined on Euclidean n-space, E_n. If M is a positive number, let

$$E(M) = \{P = (P_1, \cdots, P_n) \in E_n : |P_i| \leq M, \quad i = 1, \cdots, n\}$$

and

$$E(M, +) = \{P = (P_1, \cdots, P_n) : 0 \leq P_i < M, \quad i = 1, \cdots, n\}.$$

Consider the function on $F_n \times F_n$ to R

$$d(f, g) = \mathrm{Sup}_{M>0}\{\min [1/M, \quad \max_{p \in E(M)} |f(p) - g(p)|]\}.$$

This function is a metric for F_n and induces the topology of uniform convergence on compact sets. We define an action of R^n on F_n. Let

$$f\cdot(x_1, \cdots, x_n) \in F_n,$$

and let $t = (t_1, \cdots, t_n)$. We define

$$(ft)\cdot(x_1, \cdots, x_n) = f(x_1 + t_1, \cdots, x_n + t_n).$$

We will also write f_t for the function ft. We will call the transformation group (F_n, R^n) thus obtained the *n-dimensional Bebutov system.* We will not always attempt to distinguish explicitly in the notation between elements of E_n and R^n. We will use the fact [9, p. 420] that for f and g, elements of F_n, and $e > 0$, $d(f, g) \leq e$ if and only if $\max |f(x) - g(x)| \leq e$ for $|x| \leq 1/e$.

1.20 LEMMA. *Let (X, R^n) be a continuous n-parameter flow where X is a compact metric space having at most one fixed point. Then (X, R^n) is isomorphic to some subspace restriction of the n-dimensional Bebutov system.*

Proof. Let $\Phi(p)$ be a continuous E_n-valued function defined on the space X such that, if $p \neq q$ are points of X, there exists $t \in R^n$ such that $\Phi(pt) \neq \Phi(qt)$. The existence of such a function follows from the application of [9, pp. 445–452] to the coordinate continuous one-parameter subgroup restrictions of (X, R^n). Now to each p in X, we define a continuous function Φ_p on all of E_n. If $t \in E_n$, let

$$\Phi_p(t) = \Phi(pt).$$

It is easy to see that the mapping $p \rightarrow \Phi_p$ is a one-to-one continuous mapping of X into F_n, which respects the actions of R^n. Since X is compact, this is sufficient.

1.21 Corollary. *Let (X, R^n) be a continuous n-parameter flow where X is a locally compact, completely separable, metric space. If X has no fixed points under the flow, then (X, R^n) is isomorphic to a subspace restriction of the n-dimensional Bebutov system.*

Proof. Let Y denote the one point compactification of X, $Y = X \cup p$. Let R^n fix p and let its action on the rest of Y be induced by its action on X. Then (X, R^n) is isomorphic with a subspace restriction of (Y, R^n) which is in turn, by Lemma 1.20, isomorphic with a subspace restriction of (F_n, R^n). This suffices.

1.22 *Remark.* Let (X, T) be transformation group for which X is a locally compact, separable, metric space, or a compact metric space. If (X, T) is point partly almost periodic, then there exists a positive integer n such that (F_n, R^n) is point partly almost periodic. To see this use, in turn, 1.15, 1.17 or 1.18, and 1.20 or 1.21.

1.23 Definition. Let f and g be points of F_n, let e be a positive number and let J_1 and J_2 be n-dimensional intervals. We say that $f \mid J_1$ *is e-contained in* g/J_2 if there exists t in E_n such that $J_1 + t \subset J_2$ and $\mid f(x) - g(x + t) \mid < e$ for all x in J_1. We say that $f \mid J_1$ *is contained in* $g \mid J_2$ if there exists an element t in E_n such that $J_1 + t \subset J_2$ and $f(x - t) = g(x)$ for all x in $J_1 + t$. We say that $f \mid J_1$ *is congruent to* $g \mid J_2$ if there exists t in E_n such that $J_1 + t = J_2$ and $f(x - t) = g(x)$ for all x in $J_1 + t$.

1.24 *Remark.* Let (X, T) be a transformation group, let S and S' be replete semigroups in T such that $S' \subset SK$ for some compact subset K of T. Then the following statements are equivalent:

(1) The point x is S-almost periodic.
(2) The point x is S'-almost periodic.

1.25 *Remark.* Let (X, R^n) be a continuous n-parameter flow. Suppose it is point partly almost periodic. Then by proper choice of generators, and by use of remark 1.24, it is possible to find a point x of X which is S-almost periodic, but not almost periodic, where $S = R^{n-m} \times P^m$, for some positive integer m, where P denotes the positive real numbers.

1.26 *Notation.* For the n-dimensional Bebutov system, we let Q represent a semigroup of the form $R^{n-m} \times P^m$, where P denotes the positive real numbers, and $0 < m < n$.

1.27 LEMMA. *A point f of Bebutov's n-dimensional system is an almost periodic point if and only if, to each positive number M there exists a positive number N such that $f \mid E(M)$ is $(1/M)$-contained in $f \mid [y + E(N, +)]$ for all y in R^n.*

Proof. Suppose f is an almost periodic point in Bebutov's n-dimensional system. Then, to each positive number M it is possible to find a set A, syndetic in R^n, such that t in A implies $d(f, f_t) < 1/M$. That is, t in A implies

$$|f(x + t) - f(x)| < 1/M$$

for x in $E(M)$. Since A is syndetic, there exists a positive number N_1 such that $y + E(N_1, +)$ meets A for each y in R^n. Choose N so large that there exists a point y' in $y + E(N, +)$ such that y'' in $y' + E(N_1, +)$ implies that

$$y'' + E(M) \subset y + E(N, +).$$

But for some point y'' in $y' + E(N_1, +)$, we have y'' in A. It follows that

$$f \mid [y + E(N, +)] \quad (1/M)\text{-contains } f \mid E(M).$$

Suppose, on the other hand, that the condition is valid. Then, given $M > 0$, to each y in R^n there exists a positive number N such that

$$f \mid [y + E(N, +)] \quad (1/M)\text{-contains } f \mid E(M).$$

That is, there exists a point t in $y + E(N, +)$ such that

$$E(M) + t \subset y + E(N, +) \quad \text{and} \quad |f(x + t) - f(x)| < 1/M \quad \text{for } x \text{ in } E(M).$$

That is, $d(f, f_t) < 1/M$. Let A denote the set of points for which this inequality holds. Then $A - E(N, +) = R^n$, whence A is syndetic and f is an almost periodic point.

1.28 *Remark.* A point f of Bebutov's n-dimensional system is Q-almost periodic if and only if to each positive number M there exists a positive number N such that to each y in Q, $f \mid [y + E(N, +)](1/M)$-contains $f \mid E(M)$.

1.29 *Remark.* A point f of Bebutov's n-dimensional system is partly almost periodic if and only if it is possible to choose parameters in R^n such that there exists a replete semigroup Q for which

(1) The criterion of remark 1.28 is valid, and

(2) There exists a positive number M_1 such that to each $M > M_1$ and each $N > 0$ there exists an element $y(M, N)$ of $R^n - Q$ such that

$$f \mid [y(M, N) + E(N, +)] \quad \text{does not} \quad (1/M)\text{-contain } f \mid E(M).$$

1.30 LEMMA. *Bebutov's n-dimensional system contains no partly almost periodic points.*

Proof. Suppose f were such a point, and adopt the notation of 1.29. Then there exist positive numbers M and N such that y in Q implies

$$f \mid [y + E(N, +)](1/2M)\text{-contains } f \mid E(M),$$

and there exists an element $y(M, N) = Z \in R^n - Q$ such that

$$f \mid [Z + E(N, +) \quad \text{does not} \quad (1/M)\text{-contain}\, f \mid E(M).$$

Suppose t is in $Q - Z$. Then $f_t \mid [Z + E(N, +)]$ is congruent to

$$f \mid [Z + t + E(N, +)]$$

which $(1/2M)$-contains $f \mid E(M)$, since $A + t$ is in Q. That is, there exists $\lambda \in R^n$ such that

(1) $\lambda + E(M) \subset Z + E(N, +)$ and
(2) $|f_t(\lambda + x) - f(x)| < 1/2M$ for x in $E(M)$.

Since $f \mid [Z + E(N, +)]$ does not $(1/M)$-contain $f \mid E(M)$, we know in particular

(3) there exists $\bar{x}$ in $E(M)$ such that $|f(\lambda + \bar{x}) - f(\bar{x})| > 1/M$.

Choose M' such that $Z + E(N, +) \subset E(M')$. Then there exists $\tau = \lambda + \bar{x} \in E(M')$ such that $|f_t(\tau) - f(\tau)| > 1/2M$. It follows that $d(f, f_t) \geq \min\{1/2M, 1/M'\}$, for all $t \in Q - Z$. Since each Q-syndetic set meets $Q - Z$, it follows that f is not Q-almost periodic contrary to assumption.

1.31 THEOREM. *If (X, T) is a transformation group for which X is a compact metric space, or a locally compact, separable metric space, then X contains no partly almost periodic points.*

Proof. Use 1.22 and 1.30.

1.32 *Remark.* We now construct an example of a transformation group with compact phase space possessing a point which is regionally S-almost periodic and not regionally almost periodic.

1.33 *Notation and definitions.* Let X denote the set of functions from $Z \times Z$ to $\{0, 1\}$, where Z denotes the set of integers. We will denote the two constant functions by 0 and 1.

If f and g are elements of X let

$$d(f, g) = 1/(1 + M) \quad \text{where} \quad m = \max\{M : f \mid Z(M) = g \mid Z(M)\}.$$

The function d is a metric which makes X into a compact metric space. We define an action of $Z \times Z$ on X to obtain a transformation group $(X, Z \times Z, \pi)$ by

$$[f, (i, j)]\pi \cdot (p, q) = f(p + i, q + j).$$

Let P denote the set of positive integers, let N denote the set of negative integers and let A be a $P \times P$-syndetic subset of $P \times P$ in $Z \times Z$. We write $A = \{a_1, a_2, a_3, \cdots\}$.

We define a sequence of elements of X as follows:

(1) Let $Z_1 \subset Z_2 \subset Z_3 \subset \cdots \subset Z_m \subset \cdots$ be a sequence of squares in

$Z \times Z$ centered at $(0, 0)$ such that

(α) $(Z_m - a_m) \cap Z_m = \emptyset$
(β) $\bigcup_1^\infty Z_m = Z \times Z$.

(2) (α) Let $g_0(i, j) = 0$ if and only if i and j are both even non-negative integers.

(β) If $m = 1, 2, 3, \cdots$ let b denote a point of $Z \times Z$ and define

$$g_m(b) = g_0(b) \qquad \text{if} \quad b \in Z_m$$
$$g_m(b) = g_0(b + a_m) \quad \text{if} \quad b \in Z_m - a_m$$
$$g_m(b) = 1 \qquad \text{otherwise.}$$

Now let $Y = \text{Cl} \bigcup_1^\infty 0(g_m)$. Then Y is a compact metric space invariant under the action of $Z \times Z$. If $t = (i, j)$ is a point of $Z \times Z$, we define $|t| = |i| + |j|$.

1.34 LEMMA. *The following statements are valid:*

(1) $\lim_{m\to\infty} g_m = g_0$.
(2) $\lim_{|t|\to\infty} g_m(t) = 1$ *if* $m \neq 0$.
(3) $\lim_{m\to\infty} g_m \cdot a_m = g_0$.
(4) *If* t_i *is a sequence of elements of* $Z \times Z$ *such that* $|t_i| \to \infty$ *and* $h = \lim_{i\to\infty} g_0 t_i$ *exists, then* $d(g_0, h) \geq 1/3$.
(5) *Let* g_i *be a subsequence of* g_m. *Let* $t_i \neq a_i$ *be a sequence of elements in* $Z \times Z$ *such that* $t_i \to \infty$. *If* $\lim_{i\to\infty} g_i t_i = h$ *exists, then* $d(g_0, h) \geq 1/3$.
(6) *If* t_i *is a bounded sequence of elements of* $Z \times Z$ *and* g_i *is a subsequence of* g_m, *and if* $\lim_{i\to\infty} g_i t_i = h$ *exists, then* $h \in \bigcup_0^\infty 0(g_m)$.

Proof. (1), (2), and (3) follow immediately from the definitions. We prove (4). Let $t_i = (x_i, y_i)$. Then there exists a subsequence (which we renumber and call t_i) such that one of the following statements is valid:

(i) $x_i \to +\infty$,
(ii) $x_i \to -\infty$,
(iii) $y_i \to +\infty$,
(iv) $y_i \to -\infty$.

No matter which possible combination occurs, one of the following statements is valid:

(α) $h(x, 0) = 1$ for all x,
(β) $h(0, y) = 1$ for all y, or
(γ) there exists $x < 0$ and y such that $h(x, y) = 0$, and $(x, y) \in Z(2)$.

In all these cases $d(g_n, h) \geq 1/3$.

The proof of (5) is similar to that of (4), and (6) is obvious.

1.35 *Remark.* In the compact metric space Y,

$$N(g_0, 1/M) = \bigcup_M^\infty \{g_n a_n^{-1}, g_n, g_n a_n\}, \qquad \text{if} \quad M \geq 3.$$

1.36 *Remark.* Let $(Y, Z \times Z, \pi)$ be the subspace restriction of $(X, Z \times Z, \pi)$. The point g_0 is regionally $P \times P$-almost periodic but not regionally almost periodic.

2. Boundedness, incompressibility, evanescence

2.1 *Notation.* Let (X, T) be a transformation group where T is generative. Let $\{S, K_1, K_2, K_3, \cdots\} = P$ denote a countable collection of subsets of T. Let $P^* = \{S^{-1}, K_1, K_2, K_3, \cdots\}$.

2.2 DEFINITION. A subset B of T is called *P-admissible* provided that there exists a positive integer N such that $p \geq N$ and $t \in S$ imply $B \cap tK_p \neq \emptyset$

2.3 DEFINITION. We now define P-recursion properties as in [4, Definition 3.13, p. 21] where "P-admissible" replaces "admissible."

2.4 *Remark.* The group T is generative and hence is the countable union of compact subsets.

2.5 LEMMA. *Let P be a replete semigroup in T. Let*

$$C_1 \subset C_2 \subset C_3 \subset \cdots \subset C_n \subset \cdots$$

be a countable collection of compact subsets of T whose union is T. A subset B of T is P-syndetic if and only if there exists a positive integer N such that $n \geq N$ and $t \in P$ imply $B \cap tC_n \neq \emptyset$.

Proof. Suppose that B is a P-syndetic set. Then there exists a compact set K such that $BK \supset P$. There exists a positive integer N with $K \subset C_n$, whence we infer that if $n \geq N$, then $BK_n \supset P$, let $t \in P$. Then there exist elements b in B and c in C_n, such that $bc = t$ or $b = tc^{-1} \in tC_n$, whence $b \in B \cap tC_n \neq \emptyset$. On the other hand, suppose there exists a positive integer N such that for each t in P, $B \cap tC_n \neq \emptyset$. Then there exist elements b in B and c in C_n with $bc = t$, whence $BC_n \supset P$, that is, B is P-syndetic.

2.6 *Remark.* Let S be a replete semigroup in T, and let $C_1, C_2, C_3, \cdots$ be a collection of compact sets as in Lemma 2.5. If $P = \{S, C_1, C_2, C_3, \cdots\}$, then the P-admissible sets are just the S-syndetic sets.

2.7 *Remark.* Let S be a replete semigroup in T. There exists a countable collection

$$P_1, P_2, P_3, \cdots P_n, \cdots$$

of replete semigroups lying inside P such that if Q is a replete semigroup lying inside S, $Q \supset P_n$ for some positive integer N. We can partially order the set $\{P_i\}$ by containment so that each set P_i contains some set P_j with $j > i$.

2.8 *Remark.* Let S be a replete semigroup in T. Let $P_1, P_2, \cdots$ be a collection of replete semigroups as in Remark 2.6. A set E is S-extensive (meets each replete semigroup contained in S), if and only if there exists a positive integer N such that $E \cap P_n \neq \emptyset$ for all $n \geq N$.

2.9 *Remark*. Let S be a replete semigroup in T and let $P_1, P_2, P_3, \cdots$ be a collection of replete semigroups as in Remark 2.6. If e denotes the identity element of T, and $P = [\{e\}, P_1, P_2, P_3, \cdots]$, then the P-admissible sets are just the S-extensive sets.

2.10 DEFINITION. Let M and N denote subsets of X. We say that N *has a countable* (M, P) *representation* if there exists a sequence $\{t_n : n = 1, 2, 3, \cdots\}$ of elements of S such that $N = \bigcup_{n=1}^{\infty} Mt_n K_n$.

2.11 DEFINITION. A set M is said to be *P-reducible* if there exists a set N having a countable (M, P) representation with Int $(M - N)$ not empty. If every subset of X is not P-reducible, we say that (X, T) *is P-irreducible.*

2.12 THEOREM. *The transformation group* (X, T) *is regionally P-recursive if and only if it is P-irreducible.*

Proof. Suppose that (X, T) is not regionally P-recursive. Then there exists a nonempty open set U such that if B is any P-admissible set, there exists an element b in B with $U \cap Ub = \emptyset$. Therefore $B = \{t \in T : U \cap Ut \neq \emptyset\}$ is not P-admissible, and, hence, to each positive integer n there exists an element t_n of S for which $t_n K_n \cap B = \emptyset$. Then $V = \bigcup_{n=1}^{\infty} Ut_n K_n$ has a countable (U, S) representation and Int $(U - V) = U \neq \emptyset$, and (X, T) is not P-irreducible.

On the other hand, if (X, T) is not P-irreducible, then there exist sets M and N with Int $(M - N) \neq \emptyset$ such that there exists a sequence t_n ; $n = 1, 2, 3, \cdots$; of elements of S for which $N = \bigcup_{n=1}^{\infty} Mt_n K_n$. Then

$$U = \text{Int } (M - N)$$

is a nonempty open set for which $U \cap [\bigcup_{n=1}^{\infty} Ut_n K_n] = \emptyset$. Therefore, if $B = \{t \in T : U \cap Ut \neq \emptyset\}$, to each positive integer n there exists an element t_n of S such that $t_n K_n$ does not meet B, whence B is not P-admissible, and (X, T) is not regionally P-recursive.

2.13 DEFINITION. A set M is said to be *strictly P-irreducible* provided that if N has a countable (M, P) representation, then $N \supset M$.

2.14 THEOREM. *The transformation group* (X, T) *is pointwise P-recursive if and only if each open set is strictly* P^**-irreducible.*

Proof. If U is an open set which is not strictly P^*-irreducible there exists a sequence of elements $\{t_n^{-1}\}$ of S^{-1} such that the set $V = U - \bigcup_{n=1}^{\infty} Ut_n^{-1}K_n$ is not empty. Choose a point x in V. Then for each positive integer n we have $x \notin Ut_n^{-1}K_n$ or $xt_n K_n \cap U$ is empty. Now let $B = \{b \in T : xb \in U\}$. Since B meets no $t_n K_n$, B is not P-admissible whence x is not S-almost periodic.

On the other hand, if x is not an S-recursive point, there exists a neighborhood U of x such that $B = \{b \in T : xb \in U\}$ is not P-admissible. Then, to each positive integer n there exists an element t_n of S for which $B \cap t_n K_n$ is empty. Therefore $x \in U - \bigcup_{n=1}^{\infty} Ut_n^{-1}K_n$, whence U is a nonempty open set which is not strictly P^*-irreducible.

2.15 DEFINITION. A point x of X is called S-periodic, if there exists a P-admissible set B for which $xB = x$.

2.16 THEOREM. *The transformation group (X, T) is pointwise P-periodic if and only if each subset of X is strictly P-irreducible.*

Proof. Suppose that not every subset of X is strictly P-irreducible. Then there exists a set M and a sequence of elements $\{t_n\}$ of S such that

$$M - \bigcup_{n=1}^{\infty} Mt_n K_n$$

is not empty. Let x be a point of this set, and let $B = \{b \in T : xb = b\}$. The set B meets no $t_n K_n$ and hence, is not P-admissible whence it follows that x is not a P-periodic point.

On the other hand, suppose that there exists a point x which is not P-periodic; that is, suppose that the set B defined above is not P-admissible. Then there exist elements t_n of S such that $x \in \bigcup_{n=1}^{\infty} xt_n K_n$, and the set $\{x\}$ is not strictly P-irreducible.

2.17 DEFINITION. If P is chosen as in 2.6 so that the P-admissible sets are just the S-syndetic sets, we replace the term "P-irreducible" by "S-bounded" and the term "P-recursive" by "S-almost periodic." This definition is consistent with Definition 1.2.

2.18 DEFINITION. If P is chosen as in 2.9 so that P-admissible sets are just the S-extensive sets, we replace the term "P-irreducible" by "S-incompressible" and the term "P-recursive" by "S-recurrent." This definition is consistent with J. D. Baum's [1] usage.

2.19 THEOREM. *The following statements are pairwise equivalent:*

(1) *(X, T) is pointwise periodic.*

(2) *Each subset of X is strictly T-bounded.*

(3) *If M is a subset of X, and S is a replete semigroup on T, then $MS \subset M$ implies $MS = M$.*

Proof. Use Theorem 2.16 and [3, Theorem 8].

2.20 DEFINITION. A subset M of X is called *strictly S-incompressible*, where S is a replete semigroup in T if $MS \subset M$ implies that $MS = M$.

2.21 THEOREM. *Let (X, T) be a transformation group. If each open set of X is strictly S^{-1}-bounded, then each closed set of X is strictly sS-incompressible, and each open set is $s^{-1}S^{-1}$-incompressible, for each element s of S.*

Proof. By Theorem 2.14 and [1, Theorems 12 and 13], we can reduce this theorem to the assertion that if T is S-almost periodic, then it is S-recurrent, which is obviously true.

2.22 *Remark.* In order to establish another theorem like Theorem 2.21, we will prove Theorem 2.24, which is very like the theorems proven in Baum's paper [1].

2.23 DEFINITION. The subset M of X is called *S-incompressible* where S is a replete semigroup in T, if $MS \subset M$ implies that Int $(M - MS) = \emptyset$.

2.24 THEOREM. *The following statements are pairwise equivalent:*

(1) *(X, T) is regionally S-recurrent.*

(2) *If U is an open subset of X, there exists an S-extensive set B such that $b \in B$ implies that $U \cap Ub \neq \emptyset$.*

(3) *If M is an open set of X, then M is sS-incompressible for each element s of S.*

Proof. (1) and (2) are equivalent by definition. To show that (3) may be added to the list, we need only employ techniques like those used in Baum's paper [1]. For completeness, we include the details.

Suppose that (X, T) is not S-regionally recurrent. Then by [1, Lemma 3] we know there exists an element s of S such that for some nonempty open set U in X, $U \cap UsS = \emptyset$. The set $M = U \cup UsS$ is obviously open and not sS-incompressible.

On the other hand, suppose that (3) is not true. Then there exists an open set M and an element s of S such that $U = \text{Int}\,(M - MsS)$ is a nonempty open set of X, and it is clear that $U \cap UsS = \emptyset$, whence, by [1, Lemma 3], it follows that (X, T) is not S-regionally recurrent.

2.25 *Remark.* It is an easy consequence of (2) above that if (X, T) is S-regionally almost periodic, it is S-regionally recurrent.

2.26 THEOREM. *Let (X, T) be a transformation group. If (X, T) is S-bounded, then every open set in X is sS-incompressible for each s in A.*

Proof. By Theorem 2.12 and Theorem 2.24, this theorem can be reduced to the statement that if (X, T) is S-regionally almost periodic, it is S-regionally recurrent.

2.27 *Remark.* In the following sections we will assume that X is a compact metric space whose metric is d. It is the purpose of this section to consider a property analogous to the dissipative property for measure preserving flows. Notice that if (X, f) is a dissipative measure preserving transformation group, and F is a "dissipative" set, then $\lim_{n \to \infty} m[f^n(F)] = 0$, and hence, if that limit is positive for each set F of positive measure, then the flow must be incompressible. We use this remark to suggest a version suitable to metric spaces.

2.28 DEFINITION. If B is a subset of X, the *girth of B*, written $g(B)$, is defined by

$$g(B) = \sup \{r : \text{ there exists a point } x \text{ with } N(x; r) \subset B\}.$$

2.29 Definition. Let T be a topological group, and let f be a real-valued function defined on T. A real number z is called an *extensive* (*resp. syndetic*) *S-limit of f*, where S is a replete semigroup in T, provided that to each positive number e there exists an S-extensive (resp. S-syndetic) set E such that if $q \epsilon E$, then $|f(q) - z| < 0$. We also define two symbols:

$$\text{Lim sup ext}_S f(t) = \sup \{z : z \text{ is an extensive } S\text{-limit of } f\},$$

and

$$\lim \sup \text{syn}_S f(t) = \sup \{z : z \text{ is a syndetic } S\text{-limit of } f\}.$$

If no ambiguity is to be feared, we will omit the subscript naming the semigroup.

2.30 Corollary. *If $T = I^m \times R^n$, then, with respect to a replete semigroup S, we have for each real-valued function f,*

$$\lim \sup \text{syn} f(t) \leq \lim \sup \text{ext} f(t).$$

2.31 Definition. Let (X, T) be a transformation group. A subset M of X is called *extensively* (*resp. syndetically*) *S-evanescent*, where S is a replete semigroup in T, if

$$\lim \sup \text{ext}\, g(Mt) = 0 \qquad (\text{resp. } \lim \sup \text{syn}\, g(Mt) = 0).$$

We say the flow (X, T) is *extensively* (*resp. syndetically*) *S-evanescent* if there exists a nonempty open subset of X which is extensively (resp. syndetically) S-evanescent.

2.32 Theorem. *If (X, T) is not extensively S-evanescent, then it is S-regionally recurrent.*

Proof. Let M be any non-empty open subset of X. Then

$$\lim \sup \text{ext}\, g(Mt) = 2e > 0.$$

It follows that there exists an S-extensive set E such that $t \epsilon E$ implies that $g(Mt) < 0$. Since X is compact, there exists a finite collection $\{M_1, \cdots, Mt_n\}$ of pairwise disjoint images of M under elements of E such that if q is any element of E, then Mq meets one of these sets. For $j = 1, \cdots, N$, let

$$B_j = \{t \epsilon E : Mt \cap Mt_j \neq \emptyset\}.$$

Consider the set $F = \bigcup_{j=1}^{N} B_j t_j^{-1}$. We show that F is S-extensive. Let any element s of S be given. It is clear that there exists an element s' of S such that for each $K = 1, \cdots, N$, we have

$$sS \supset (\textstyle\prod_{j \neq k} t_j) s'S.$$

Since E is S-extensive there exists an element

$$q \epsilon E \cap (\textstyle\prod_{j=1}^{N} t_j) s'S.$$

Since $E = \bigcup_1^N B_j$, there exists some integer K between one and N with $q \in B_k$. It follows that $qt_k^{-1} \in F$. Also,

$$qt_k^{-1} \in (\textstyle\prod_{j \neq k} t_j)s'S \subset sS.$$

Therefore, F meets each sS, and so is S-extensive. Finally, if q is any element of F, there exists an integer k for which $q \in B_k t_k^{-1}$. Then $q = bt_k^{-1}$ where $Mb \cap Mt_k \neq \emptyset$, whence $Mq \cap M \neq \emptyset$, and (X, T) is S-regionally recurrent.

2.33. THEOREM. *If (X, T) is not syndetically S-evanescent, then it is S-regionally almost periodic.*

Proof. Let M be a nonempty open subset of X. Just as in the proof of Theorem 28, we find an S-syndetic set E and a finite set of integers J such that

(1) $E = \bigcup_J B_j$ and
(2) If $t \in B_j$, then $Mt \cap Mt_j \neq \emptyset$.

Since E is S-syndetic, there exists a set K_n such that sK_n meets E for each element s of S. Now let $F = \bigcup_J t_j^{-1}B_j$. We show that F is S-syndetic. First notice that there exists a set K_m for which $K_m \supset \bigcup_J t_j^{-1}K_n$. Let an element s of S be given and choose $s' \in sK_n E$. There exists an element k of J such that $s' \in B_k$, whence

$$s't_k^{-1} \in t_k^{-1}B_k \subset B_k^{-1} \cap st_k^{-1}K_n \subset F \cap sK_m,$$

and F is S-syndetic. Proceeding as in the previous theorem, we infer the S-regional almost periodicity of (X, T).

2.34 *Remark.* We now obtain a very weak converse to the two previous theorems.

2.35 DEFINITION. Let (X, T) be a transformation group, and let S be a replete semigroup in T. We say that (X, T) is *S-compactly evanescent* if there exists an open subset M of X such that for each positive integer n,

$$\limsup \operatorname{ext} g(MtK_n) = 0.$$

2.36 THEOREM. *If (X, T) is S-pointwise almost periodic, then it is not S-compactly evanescent.*

Proof. Suppose that (S, T) is S-compactly evanescent. Let M be the set whose existence is guaranteed by the definition. There exists a point x of X and a positive number r such that $N(x; r) \subset M$. Let $B = \operatorname{Cl} N(x; r/2)$. It follows that $\limsup \operatorname{ext} g(BtK_n) = 0$, for $n = 1, 2, 3, \cdots$ Then there exists an element t_1 of S such that $g(BK_1 t_1) < r/2$, when $B = Bt_1 K_1 \neq \emptyset$. Since this latter set is open in B, we see $M_1 = \operatorname{Int}(B - Bt_1 K_1)$ is not empty. Therefore we can find a point x and a positive number r_1 such that

$$B_1 = \operatorname{Cl} N(x_1; r_1) \subset M_1 \subset B - Bt_1 K_1.$$

It is now easy to show by induction that there exists a collection of closed subsets of X, namely $B_1, B_2, B_3, \cdots$ and a collection of distinct elements $t_1, t_2, t_3, \cdots$ of S, such that

(1) $B_j \subset B_{j-1} - Bt_j K_j$ and
(2) $\operatorname{Int} B_j \neq \emptyset$.

Therefore $B - \bigcap_{j=1}^{\infty} Bt_j K_j \supset \bigcap_1^{\infty} B_{j+1} \neq \emptyset$, whence, by Theorem 2.14 (X, T) is not S-pointwise almost periodic.

2.37 *Example.* It is the purpose of this example to show that the converses of Theorems 2.31 and 2.33 are not valid. Let X denote the space of bisequences of zeros and ones with the usual metric, d, and let σ be the left shift transformation. It is known [4, Chapter 12] that σ generates over the integers a transformation group which is regionally recurrent and regionally almost periodic, hence regionally positively recurrent and regionally positively almost periodic. We now show that is positively evanescent. Let $\bar{o}$ denote the bisequence in which only zeros appear, and let $M = n(\bar{o}, \frac{1}{2})$. If p is a point of M, it is of the form $p = (\cdots p(-2)0\dot{0}0\, p(2) \cdots)$. For each $p \in M$, define a point $p^* \notin M$ by $p^*(j) = p(j)$ if $j \neq 0$, and $p^*(0) = 1$. For any positive integer n,

$$d(p^*\sigma^n, p\sigma^n) = (n+1)^{-1}$$

whence $g(M\sigma^n) \leq (n+1)^{-1}$, and

$$\limsup \operatorname{syn} g(M\sigma^n) = \limsup \operatorname{ext} g(M\sigma^n) = \lim_{n\to\infty} g(M\sigma^n) = 0.$$

Therefore the transformation group generated by σ is positively evanescent.

Acknowledgement. The author is grateful to the referee for helpful remarks, in particular for pointing out that the notions of 2.1–2.14 are valid in a more general setting than the author originally thought likely.

Added in proof. The author has found that there is considerable overlap between this paper and the 1967 Yale dissertation of Ethan Coven, who has (among other things) a completely different proof of 1.31.

References

1. John D. Baum, *P-recurrence in topological dynamics*, Trans. Amer. Math. Soc., vol. 77 (1954), pp. 506–519.
2. Robert Ellis and Walter H. Gottschalk, *Homomorphisms of transformation groups*, Trans. Amer. Math. Soc., vol. 94 (1960), pp. 258–271.
3. Walter H. Gottschalk and Gustav A. Hedlund, *The dynamics of transformation groups*, Trans. Amer. Math. Soc., vol. 65 (1949), pp. 348–359.
4. ———, *Topological dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 36, 1955.
5. Paul R. Halmos, *Lectures on ergodic theory*, Chelsea, New York, 1956.
6. Eberhard Hopf, *Theory of measure and invariant integrals*, Trans. Amer. Math. Soc., vol. 34 (1932), pp. 373–393.
7. ———, *Ergodentheorie*, Chelsea, New York, 1948.

8. A. KHINTCHINE, *Eine Versharfung der Poincaréschen "Wiederkehrsatzes,"* Compositio Math., vol. 1 (1934), pp. 177–179.
9. V. V. NEMYCKII, *Topological problems of the theory of dynamical systems*, Amer. Math. Soc. Translations no. 103, 1954.
10. CHARLES W. WILLIAMS, *Recurrence and incompressibility*, Proc. Amer. Math. Soc., vol. 2 (1951), pp. 798–806.

WEBSTER COLLEGE,
ST. LOUIS, MISSOURI
SYRACUSE UNIVERSITY
SYRACUSE, NEW YORK