TOPOLOGICALLY UNKNOTTING TUBES IN EUCLIDEAN SPACE

BY

R. C. LACHER[1]

In this paper we consider closed, locally flat embedding of tubes $B^{k-1} \times R^1$ and $S^{k-1} \times R^1$ into R^n. In Part I we show that $B^{k-1} \times R^1$ knots in R^3 but unknots in R^n if $n \geq 4$. The situation with $S^{k-1} \times R^1$ is more complicated. In Parts II and III, we show that $S^{k-1} \times R^1$ can knot in R^{k+2} and in R^{2k} and in most R^n for $k + 2 \leq n \leq 2k$. Thus a general low-codimensional unknotting theorem is nonexistent. However, in Part IV we show that any closed, locally flat embedding of $S^{k-1} \times R^1$ in R^n, $k \leq n - 3$, is unknotted provided that it is "unlinked at infinity", a condition derived while proving that the examples in Part III actually knot. A corollary is that $S^{k-1} \times R^1$ unknots in R^n if $n \geq 2k + 1$, $k \geq 2$.

Embeddings of $S^{n-2} \times R^1$ into R^n are studied in Part V.

Several discussions with Joe Martin were helpful in the formulation of Parts II and III.

Added in Proof. Closed, locally flat embeddings of $S^{k-1} \times R^1$ in R^n are classified by the homotopy group $\pi_{k-1}(S^{n-k-1})$, provided $3(k + 1) < 2n$.

Definitions and Notation. We think of B^n as the closed unit ball in euclidean n-space R^n, and we identify R^k with $R^k \times 0$ in R^n. Also, S^n is the boundary of B^{n+1}. Thus $B^k \times R^{n-k} \subset R^n$ and $S^{k-1} \times R^{n-k} \subset R^n$. $\hat{R}^n$ is used to denote the one-point compactification of R^n. Of course, $\hat{R}^n$ is homeomorphic to S^n.

Let K be a (topological) k-manifold contained in the interior of the n-manifold N. K is *locally flat at the point* $x \in \text{Int } K$ (the *interior* of K) if x has a neighborhood U in N such that $(U, U \cap K)$ and (R^n, R^k) are homeomorphic as pairs. K is *locally flat at the point* $x \in \text{Bd } K$ (the *boundary* of K) if x has a neighborhood U in N such that $(U, U \cap K)$ and (R^n, R^k_+) are homeomorphic as pairs, where $R^k_+ = R^{k-1} \times [0, \infty) \subset R^k$.

An embedding f of a k-manifold K into the interior of the n-manifold N is *locally flat at the point* $x \in K$ if $f(K)$ is locally flat at x; f is called a *locally flat embedding* if f is locally flat at every point of K.

Finally, an embedding is *closed* if its image is a closed subset of its range.

Part I. Unknotting $B^{k-1} \times R^1$ in R^n for $n \geq 4$

Before stating the main unknotting theorem, we prove two propositions. The first says essentially that "setwise" unknotting implies "pointwise" unknotting. The second shows that knotting occurs in dimension three.

Received March 28, 1967.

[1] Partially supported by a National Science Foundation grant.

PROPOSITION 1.1. *Any homeomorphism of $B^{k-1} \times R^1$ onto itself can be extended to a homeomorphism of R^n onto itself, $k \leq n$.*

Proof. Let f be a homeomorphism of $B^{k-1} \times R^1$. Notice that the closure of the complement of $B^{k-1} \times R^1$ *in* R^k is homeomorphic to $S^{k-2} \times R^1 \times [0, \infty)$. Thus f can be extended to a homeomorphism F of R^k by transposing the formula

$$(x, t) \to (f(x), t), \qquad x \in S^{k-2} \times R^1, \qquad t \geq 0.$$

But then F can be extended to a homeomorphism of R^n by a standard method.

PROPOSITION 1.2. *For $k = 1, 2, 3$, there is a closed, locally flat copy X of $B^{k-1} \times R^1$ in R^3 such that the pairs (R^3, X) and $(R^3, B^{k-1} \times R^1)$ are not homeomorphic.*

Proof. First notice that there are locally flat, closed, copies Y of R^1 in R^3 such that the pairs (R^3, Y) and (R^3, R^1) are not homeomorphic. For example, one could take a simple trefoil knot (S^3, K) and remove a point p of K from the pair, letting

$$(R^3, Y) = (S^3 - \{p\}, K - \{p\}).$$

The proposition follows by modifying (R^3, Y) in obvious ways.

THEOREM 1.3. *Let f be a closed, locally flat embedding of $B^{k-1} \times R^1$ into R^n. If $n \geq 4$ then there is a homeomorphism h of R^n onto itself such that hf is the identity on $B^{k-1} \times R^1$.*

Proof. We let $\hat{X} = X \cup \{\infty\}$ denote the one-point compactification of the space X. Set $\Delta^k = [f(B^{k-1} \times [0, \infty))]^\wedge$. Δ^k is a k-cell in $\hat{R}^n$, and Δ^k is locally flat at every point other than the point ∞, a boundary point of Δ^k. Corollary 2.4 of [7] says that, since $n \geq 4$, the pairs $(\hat{R}^n, \Delta^k)$ and $(\hat{R}^n, k\text{-simplex})$ are homeomorphic. Since this homeomorphism may be chosen to leave the ideal point fixed, we simply assume that $\Delta^k = [B^{k-1} \times [0, \infty)]^\wedge$. We think of Δ^k as a simplex of $\hat{R}^n$ having ∞ as a vertex.

Let b be an interior point of Δ^k. Let Δ_1^k be the join of b with the face of Δ^k opposite ∞, and let A be the line segment joining b and ∞. Denote by φ the homeomorphism of Δ_1^k onto Δ^k which "stretches" line segments parallel to A. That is, φ is the identity on the face of Δ_1^k opposite $b, \varphi(b) = \infty$, and φ is linear on Δ_1^k. It is easily seen that φ can be extended to a mapping (denoted again by φ) of $\hat{R}^n$ onto itself with the following properties:

The only non-degenerate inverse set of φ is $\varphi^{-1}(\infty) = A$, and

φ is the identity on $B^{k-1} \times (-\infty, 0]$ and on $f(B^{k-1} \times (-\infty, 0])$.

Now, let $Q = [f(B^{k-1} \times (-\infty, 0])]^\wedge \cup \Delta_1^k$. Q is a k-cell in $\hat{R}^n$ which is locally flat except possibly at the point ∞ of Bd Q. Again applying [7], there is a homeomorphism g_1 of $\hat{R}^n$ onto itself such that

$$g_1(Q) = [B^{k-1} \times (-\infty, 0]]^\wedge \cup \Delta_1^k.$$

It is a simple matter to modify g_1 so that

$$g_1(Q) = [B^{k-1} \times (-\infty, 0]]^\wedge \cup \Delta_1^k$$

$$g_1(\Delta_1^k) = \Delta_1^k,$$

$$g_1(b) = b \quad \text{and} \quad g_1(\infty) = \infty.$$

Moreover, using Corollary 3.2 of [8], we can find a homeomorphism g_2 of $\hat{R}^n$ onto itself such that

$$g_2 \text{ is the identity on } g_1(Q)$$

and

$$g_2 \text{ agrees with } g_1^{-1} \text{ on } g_1(A).$$

(Here, again, the restriction $n \geq 4$ is needed.) Notice that $g_2 g_1$ is a homeomorphism of $\hat{R}^n$ which agrees with g_1 on Q and is the identity on A.

Define g_3 by the formula $g_3 = \varphi g_2 g_1 \varphi^{-1}$. Even though φ^{-1} is not a function, g_3 is a well-defined homeomorphism of $\hat{R}^n$ onto itself. It follows immediately that

$$g_3 f(B^{k-1} \times R^1) = B^{k-1} \times R^1.$$

Finally, by applying Proposition 1.1, let g_4 be a homeomorphism of R^n onto itself which agrees with $(g_3 f)^{-1}$ on $B^{k-1} \times R^1$, and let $h = g_4 g_3$. This completes the proof.

COROLLARY 1.4. *Let f be a closed, locally flat embedding of $S^{k-1} \times R^1$ into R^n, $n \geq 4$. If f can be extended to a closed, locally flat embedding of $B^k \times R^1$ into R^n then there is a homeomorphism h of R^n onto itself such that hf is the identity on $S^{k-1} \times R^1$.*

Part II. Remark on links and cones

A. *Links.* We describe here a well-known procedure for constructing a pair of linked k-sphere in S^n whenever $\pi_k(S^{n-k-1}) \neq 0$, $1 \leq k \leq n - 2$. These and other constructions may be found in [12].

Let $\varphi : S^k \to S^{n-k-1}$ be a piecewise linear, essential, mapping, and let $g : S^k \to S^k \times S^{n-k-1}$ be the graph of φ, given by $g(x) = (x, \varphi(x))$.

We regard S^k and S^{n-k-1} as spheres in general position in a high-dimensional euclidean space, so that $S^k * S^{n-k-1}$, the join of S^k and S^{n-k-1}, is a piecewise linear copy of S^n. Moreover, $S^k \times S^{n-k-1}$ is embedded in a natural way in S^n as the set of midpoints of segments joining S^k to S^{n-k-1}.

$g : S^k \to S^n$ is a piecewise linear, locally flat embedding.

Clearly g is piecewise linear. To see that g is locally flat, let V be an open set in S^{n-k-1} and $h : V \approx R^n$ a homeomorphism, and let $U = \varphi^{-1}(V)$. We have a homeomorphism $H : U \times V \approx U \times V$ by the rule

$$H(x, y) = (x, h^{-1}[h(y) - h\varphi(x)]).$$

Since $Hg(x) = (x, h^{-1}(0))$, $x \in U$, g is locally flat in $S^k \times S^{n-k-1}$ and hence in S^n.

$g : S^k \to (S^n - S^k)$ is not homotopic to a constant map.

This last statement is clear, since there is a retraction of $S^n - S^k$ onto S^{n-k-1} which takes $g(x)$ to $\varphi(x)$, $x \in S^k$.

B. *Cones.* Now we describe a procedure for "local" linking of two cells in S^n. Suppose that S_1 and S_2 are locally flat $(k - 1)$-spheres in S^{n-1} such that S_1 is not contractible in $S^{n-1} - S_2$. Write S^n as the join $S^{n-1} * \{p, q\}$ of S^{n-1} with two points, and let $D_i = S_i * q$, $i = 1, 2$.

Let Σ_1 and Σ_2 be the respective boundaries of disjoint k-simplexes in S^{n-1}, and let $\Delta_i = \Sigma_i * q$, $i = 1, 2$.

There is no homeomorphism of S^n which takes $D_1 \cup D_2$ onto $\Delta_1 \cup \Delta_2$.

In fact, suppose such a homeomorphism exists. Then there is an isotopy of of S^n which moves points only in a neighborhood of q and which pushes D_1 onto a k-cell $\tilde{D}_1$ such that $\operatorname{Bd} \tilde{D}_1 = S_1$, $\tilde{D}_1 \subset S^{n-1} * q$, and $\tilde{D}_1 \cap D_2 = \emptyset$. But then retraction of $(S^{n-1} * q) - \{q\}$ onto S^{n-1} along join lines maps $\tilde{D}_1$ into $S^{n-1} - S_2$, and the fact that S_1 is not contractible in $S^{n-1} - S_2$ is contradicted.

COROLLARY 2.1. *Let K be the cone over the disjoint union of two $(k - 1)$-spheres. If $\pi_{k-1}(S^{n-k-1}) \neq 0$ then K knots in S^n, $2 \leq k \leq n - 2$. In particular, K knots in S^{2k} for $k \geq 2$.*

Added in proof. Using [4], [9], [11] and [13], one can prove: Equivalence classes of embeddings of K into S^n, locally flat on each simplex of K, are in one-one correspondence with $\pi_{k-1}(S^{n-k-1})$ provided $3(k + 1) < 2n$.

Part III. Knotting $S^{k-1} \times R^1$ in R^n

A. *Codimension two.* Knotting occurs in codimension two simply as a reflection of the knotting of codimension two sphere pairs, as follows. If S is a locally flat $(k - 1)$-sphere in R^{k+1}, let $(R^{k+2}, Y) = (R^{k+1} \times R^1, S \times R^1)$. Clearly (R^{k+2}, Y) deforms onto (R^{k+1}, S), so that, in particular, the homotopy groups $\pi_q(R^{k+2} - Y)$ and $\pi_q(R^{k+1} - S)$ are isomorphic.

COROLLARY 3.1. *If $k \geq 1$ there exists a closed, piecewise linear, locally flat copy Y of $S^{k-1} \times R^1$ in R^{k+2} such that the pairs (R^{k+2}, Y) and $[R^{k+2}, S^{k-1} \times R^1)$ are not homeomorphic.*

(This follows from the above discussion if $k \geq 2$. The case $k = 1$ is well known.)

B. *Codimension three or more.* Suppose that S_1 and S_2 are locally flat $(k-1)$-spheres in S^{n-1} with the following two properties: S_1 is not contractible in $S^{n-1} - S_2$, and there is a nice piecewise linear annulus A properly embedded in $S^{n-1} * p$ such that $\text{Bd}\, A = S_1 \cup S_2$. Then we can construct a knotted embedding of $S^{k-1} \times R^1$ in R^n as follows. Write $S^n = S^{n-1} * \{p, q\}$, let $K = (S_1 \cup S_2) * q$, and let

$$(R^n, Z) = (S^n - \{q\}, A \cup K - \{q\}).$$

It follows from IIB that (R^n, Z) and $(R^n, S^{k-1} \times R^1)$ are not homeomorphic.

THEOREM 3.2. *If* $\pi_{k-1}(S^{n-k-1}) \neq 0$, *there is a closed, piecewise linear, (locally flat) copy* Z *of* $S^{k-1} \times R^1$ *in* R^n *such that the pairs* (R^n, Z) *and* $(R^n, S^{k-1} \times R^1)$ *are not homeomorphic.*

Proof. This follows from the above discussion, except for the existence of the annulus, which follows from Theorem 1.1 of [6].

Remarks. 1. If $n \geq 4$, any "non-standard" embedding of $S^{k-1} \times R^1$ into R^n provides an example of and embedding which cannot be nicely extended over $B^{k-1} \times R^1$. See Corollary 1.4.

2. Corollary 3.1 and Theorem 3.2 illustrate the fact that closed, locally flat embeddings f of $S^{k-1} \times R^1$ into R^n may knot for two reasons: the spheres $f(S^{k-1} \times t)$ may be knotted in cross-sectional hyperplanes, or the spheres $f(S^{k-1} \times t)$ and $f(S^{k-1} \times (-t))$ may be linked in cross-sectional hyperplanes for large t. In Part IV we show that, if $k \leq n - 3$ (so that $(k-1)$-spheres cannot knot in R^{n-1}) and if the spheres $f(S^{k-1} \times t)$ and $f(S^{k-1} \times (-t))$ are topologically unlinked for large t, then f is unknotted. See Theorems 4.3 and 4.4.

3. The example in Theorem 3.2, $n = 2k$, is the non-compact version of Hudson's example of a knotted $S^{k-1} \times S^1$ in S^{2k}. (See a description of Hudson's example in [11].)

Added in proof. Using [4] and [13] it follows that closed, locally flat embeddings of $S^{k-1} \times R^1$ in R^n are classified by $\pi_{k-1}(S^{n-k-1})$ provided $3(k+1) < 2n$.

Part IV. Unknotting $S^{k-1} \times R^1$ in Codimension Three

As in Part I, the "pointwise" and "setwise" unknotting problems are equivalent. This fact is stated explicitly in the corollary following the next proposition.

PROPOSITION 4.1. *Any homeomorphism of* $S^{k-1} \times R^1$ *onto itself,* $k \geq 2$, *can be extended to a homeomorphism of* $B^k \times R^1$ *onto itself.*

Proof. Let f be a homeomorphism of $S^{k-1} \times R^1$ onto itself. For each $t \in R^1$, let $S_t = S^{k-1} \times t$ and $\Sigma_t = f(S_t)$. Since Σ_t separates $S^{k-1} \times R^1$ for each t, we can define $\Sigma_t < \Sigma_s$ if Σ_s lies in the complementary domain of Σ_t which contains S_u for arbitrarily large values of u. The following is an easy exercise:

The function $t \to \Sigma_t$ is either order-preserving or order-reversing, and consequently the ordering $\Sigma_t < \Sigma_s$ is a linear ordering. Since the homeomorphism $(x, t) \to (x, -t)$ of $S^{k-1} \times R^1$ can obviously be extended to a homeomorphism of $B^k \times R^1$, we may, and henceforth do, assume that the function $t \to \Sigma_t$ is order-preserving.

Now we need the following

SUBLEMMA. *Suppose that Σ_{t_0} lies interior to $S^{k-1} \times [a, b]$ for some $a < b$. Then there is a k-cell Δ_0 in $B^k \times (a, b)$ with the following properties*:

(i) $\Delta_0 \cap (S^{k-1} \times R^1) = \mathrm{Bd}\, \Delta_0 = \Sigma_{t_0}$,

(ii) Int Δ_0 *is locally flat in* $B^k \times R^1$, *and*

(iii) Δ_0 *is "locally topologically perpendicular" to* $S^{k-1} \times R^1$ *at each point of* Σ_{t_0}.

Proof of sublemma. $B^k \times [a, b]$ is a $(k + 1)$-cell, and Σ_{t_0} is a bicollared' hence flat, $(k - 1)$-sphere in the boundary of $B^k \times [a, b]$. The existence of Δ_0 follows immediately. (See [1].) Thanks to the referee for pointing out this short proof of the sublemma.

We can now extend f as follows. Construct a sequence $\{t_i\}_{i=-\infty}^{\infty}$ of numbers, with $t_i < t_{i+1}$, such that

for each i, there is a number t with Σ_{t_i} separated from $\Sigma_{t_{i+1}}$ by S_t,

$$t_i \to \infty \text{ as } i \to \infty \quad \text{and} \quad t_i \to -\infty \text{ as } i \to -\infty.$$

Then, using the sublemma, construct cells Δ_i, pairwise disjoint, and let Γ_i be the $(k + 1)$-cell in $B^k \times R^1$ bounded by

$$\Delta_i \cup \Delta_{i+1} \cup f(S^{k-1} \times [t_i, t_{i+1}]),$$

set $D_i = B^k \times t_i$ and $C_i = B^k \times [t_i, t_{i+1}]$. Extend f radially to a homeomorphism of D_i onto Δ_i for each i, and then extend radially to a homeomorphism of C_i onto Γ_i for each i.

COROLLARY 4.2. *Any homeomorphism of $S^{k-2} \times R^1$ onto itself can be extended to a homeomorphism of R^n onto itself*, $3 \leq k \leq n$.

Proof. Apply Propositions 4.1 and 1.1.

THEOREM 4.3. *Let f be a closed, locally flat embedding of $S^{k-1} \times R^1$ into R^n, $k \leq n - 3$. If f can be extended to a closed, locally flat embedding of $S^{k-1} \times R^1 \cup B^k \times [b, \infty)$ into R^n, then there is a homeomorphism h of R^n onto itself such that hf is the identity on $S^{k-1} \times R^1$.*

Proof. As in Theorem 1.3, we work in the one-point compactification

$\hat{R}^n$ of R^n. We may assume that the embedding f can be extended to an embedding F of $(S^{k-1} \times R^1) \cup (B^k \times [0, \infty))$ into $\hat{R}^n$ in such a way that $[F(B^k \times [0, \infty))]^\wedge$ is a locally flat $(k+1)$-cell in $\hat{R}^n$ (see [7]). We assume, therefore, that F is actually the identity on $B^k \times [0, \infty)$.

The proof proceeds now following the same idea as the proof of Theorem 1.3. We consider the k-sphere $S = [f(S^{k-1} \times (-\infty, 0])]^\wedge \cup B^k \times 0$. This sphere is locally flat except possibly at the ideal point, so [10] there is a homeomorphism g of $\hat{R}^n$ onto itself taking S onto $[S^{k-1} \times (-\infty, 0]]^\wedge \cup B^k \times 0$; here is where we use the hypothesis $k \leq n-3$. It is easy to modify g so that, n addition,

$$g(B^k \times 0) = B^k \times 0, \quad g(\infty) = \infty \quad \text{and} \quad g(0) = 0.$$

Using Corollary 3.2 of [8], we may assume that

$$g \text{ is the identity on } [0 \times [0, \infty)]^\wedge = A.$$

Now, let φ be a mapping of $\hat{R}^n$ onto itself with the following properties:

The only non-degenerate inverse set under φ is $\varphi^{-1}(\infty) = A$.
φ is the identity on $S^{k-1} \times (-\infty, 0]$ and on $f(S^{k-1} \times (-\infty, 0])$, and
φ maps $B^k \times 0$ homeomorphically onto $[S^{k-1} \times [0, \infty)]^\wedge$.

Define h by $h = \varphi g \varphi^{-1}$. Clearly h is a homeomorphism of $\hat{R}^n$, and

$$hf(S^{k-1} \times R^1) = S^{k-1} \times R^1.$$

An application of Proposition 4.1 completes the proof, provided $k \geq 2$. The case $k = 1$ may be handled separately altogether using trivial range techniques.

Remark. Intuitively, Theorem 4.3 says than an embedding f unknots if, for sufficiently large t, $f(S^{k-1} \times t)$ is geometrically unlinked from $f(S^{k-1} \times s)$ for all s. We can refine this idea slightly, making use of the following definition.

In the light of the proof given in Part IIB, it seems reasonable to say that a closed embedding f of $S^{k-1} \times R^1$ into R^n is *topologically unlinked at infinity* if there is a locally flat $(n-1)$-cell Q in $\hat{R}^n$ such that the following conditions are satisfied.

(i) The ideal point ∞ is an interior point of Q,
(ii) Q does not intersect the image of f, and
(iii) There is an open set U in $\hat{R}^n$ containing ∞ which is separated by Q such that $U \cap f(S^{k-1} \times (-\infty, -1])$ and $U \cap f(S^{k-1} \times [1, \infty))$ lie in different components of $U - Q$. That is, $f(S^{k-1} \times (-\infty, -1])$ and $f(S^{k-1} \times [1, \infty))$ approach ∞ from opposite sides of Q.

THEOREM 4.4. *If f is a closed, locally flat embedding of $S^{k-1} \times R^1$ into R^n, $k \leq n-3$, which is topologically unlinked at infinity, then there is a homeomorphism h of R^n onto itself such that hf is the identity on $S^{k-1} \times R^1$.*

Proof. It suffices to show that f can be extended to a closed, locally flat embedding of $S^{k-1} \times R^1 \cup B^k \times [b, \infty)$ into R^n for some b.

Let $D_1 = f(S^{k-1} \times (-\infty, -1])$ and $D_2 = f(S^{k-1} \times [1, \infty))$. Let Q be a locally flat $(n-1)$ − cell in $\hat{R}^n$ such that $\infty \in \operatorname{Int} Q$ and such that D_1 and D_2 approach ∞ from opposite sides of Q. By a collaring argument [1], we can find a locally flat embedding φ of $B^{n-1} * q$ into $\hat{R}^n$ such that

$$\varphi(B^{n-1}) = Q \quad \text{and} \quad \varphi(B^{n-1} * q) \cap D_2 = \{\infty\}.$$

Let A be the arc $\varphi(0 * q)$.

Since φ can be extended to a homeomorphism of $\hat{R}^n$, there is a mapping ψ of $\hat{R}^n$ onto itself whose nondegenerate inverse sets are precisely the sets $\varphi(S_t * tq)$, $0 < t \leq 1$, S_t being the sphere of radius t in R^{n-1}. ψ maps the $(n-1)$-cell $\varphi(S_t * tq)$ onto $\varphi(tq) \in A$. We may take ψ to be the identity on D_2.

Now, D_2 is a locally flat k-cell in $\hat{R}^n$ by Corollary 5.3 of [3], since we have $n \geq 4$. Also, since $n \geq 4$, there is a homeomorphism g of $\hat{R}^n$ such that

$$g(D_2) = [S^{k-1} \times [1, \infty)]^\wedge$$

and $g(A)$ is a straight line segment. (See Theorem 3.1 of [8].) Since there is a neighborhood U of ∞ such that $U \cap \psi f(S^{k-1} \times R^1)$ lies in $A \cup D_2$, it is clear that there is a locally flat $(k+1)$-cell E, containing $g(D_2)$ as a locally flat face, such that

$$E \cap g\psi f(S^{k-1} \times R^1) = g(D_2) \quad \text{and} \quad E \cap g(A) = \{\infty\}.$$

Thus ψ^{-1} is defined and continuous on $g^{-1}(E)$, as well as on a neighborhood of $g^{-1}(E) - \{\infty\}$. Therefore f can be extended to a closed, locally flat embedding of

$$S^{k-1} \times R^1 \cup B^k \times [b, \infty)$$

into R^n for some b by mapping $B^k \times [b, \infty)$ onto $\psi^{-1}g^{-1}(E) - \{\infty\}$.

An application of Theorem 4.3 completes the proof.

COROLLARY 4.5. *If f is a closed, locally flat embedding of $S^{k-1} \times R^1$ into R^n, $k \geq 2$, $n \geq 2k+1$, then there is a homeomorphism h of R^n such that hf is the identity on $S^{k-1} \times R^1$.*

Proof. First, it follows that $k < 2n/3 - 1$. Therefore, by Theorem 1 of [4] the embedding $\hat{f}$ of $[S^{k-1} \times R^1]^\wedge$ into $\hat{R}^n$ is locally tame at the point ∞. That is, there is a homeomorphism g of $\hat{R}^n$ such that $g\hat{f}$ is piecewise linear on

$$[S^{k-1} \times (-\infty, -b] \cup S^{k-1} \times [b, \infty)]^\wedge$$

for some $b > 0$. Since k-dimensional cones unknot piecewise linearly in S^n for $n \geq 2k+1$, it is clear that gf is topologically unlinked at infinity, and the result follows from Theorem 4.4.

The fact that k-dimensional cones unknot piecewise linearly in S^n for $n \geq 2k+1$ follows by combining [5] and [9].

Part V. Unknotting $S^{n-2} \times R^1$ in R^n for $n \geq 4$

We begin by showing that $S^1 \times R^1$ knots in R^3 in the worst possible way, as follows. (Compare with Proposition 1.2.)

PROPOSITION 5.1. *There exists a closed, locally flat embedding f of $S^1 \times R^1$ onto R^3 with the following properties:*
(i) *f cannot be extended to a closed embedding of $B^2 \times R^1$ into R^3, and*
(ii) *f cannot be extended to a closed embedding of $S^1 \times R^1 \times [0, \infty)$ into R^3.*

Proof. Let g be a closed, locally flat embedding of $B^2 \times [0, \infty)$ into R^3 which embeds $0 \times [0, \infty)$ as a wild ray in R^3. Now let h be an embedding of $B^2 \times (-\infty, 0]$ into $B^2 \times [0, \infty)$ such that h is the identity on $B^2 \times 0$,

$$h(B^2 \times (-\infty, 0)) \subset (\text{Int } B^2 \times (0, \infty),$$

and h ties a trefoil knot in $0 \times (-\infty, 0]$. Then define f by

$$f \mid S^1 \times (-\infty, 0] = gh \mid S^1 \times (-\infty, 0] \quad \text{and} \quad f \mid S^1 \times [0, \infty) = g \mid S^1 \times [0, \infty).$$

We have the following criterion for unknottedness when $n \geq 4$.

THEOREM 5.2. *Let f be a closed, locally flat embedding of $S^{n-2} \times R^1$ into $R^n, n \geq 4$. If there are numbers $a < b$ such that f can be extended to an embedding of $(S^{n-2} \times R^1) \cup (B^{n-1} \times (a, b))$ into R^n, then there is a homeomorphism h of R^n onto itself such that hf is the identity on $S^{n-2} \times R^1$.*

Proof. Consider the induced embedding $\hat{f}$ of $(S^{n-2} \times R^1)^\wedge$ into $\hat{N}$. By the hypothesis, $\hat{f}$ can be extended to an embedding F of $(S^{n-2} \times R^1)^\wedge \cup (B^{n-1} \times c)$ into $\hat{R}^n$ in such a way that the spheres

$$S_+ = F([S^{n-2} \times [c, \infty)]^\wedge \cup B^{n-1} \times c)$$

and

$$S_- = F([S^{n-2} \times (-\infty, c]]^\wedge \cup B^{n-1} \times c)$$

are locally flat in $\hat{R}^n$ except possibly at the ideal point. Therefore [2], since $n \geq 4$, S_+ and S_- are locally flat, and [1] bound n-cells Q_+ and Q_- in $\hat{R}^n$ such that $Q_+ \cap Q_- = \{\infty\} \cup F(B^{n-1} \times c)$. Hence F can be extended to an embedding $(B^{n-1} \times R^1)^\wedge$ into $\hat{R}^n$ by radial projection. An application of Corollary 1.4 completes the proof.

In order to pinpoint the unknotting problem for $S^{n-2} \times R^1$ in R^n, $n \geq 4$, we consider the following conjectures.

$\sigma(n)$. Let M be an $(n-1)$-manifold in the interior of the n-manifold N, and let p be an interior point of M. If p has a neighborhood U in M such that $U - \{p\}$ is locally flat in N, then M is locally flat at p.

$\tau(n)$. Let f be a closed, locally flat embedding of $S^{n-2} \times R^1$ into R^n. Then f can be extended to a closed embedding of $B^{n-1} \times R^1$ into R^n.

THEOREM 5.3. *$\sigma(n) \Leftrightarrow \tau(n)$ for $n \geq 4$.*

Proof. First suppose that $\sigma(n)$ is true, and let f be a closed, locally flat embedding of $S^{n-2} \times R^1$ into R^n. Consider $D = [f(S^{n-2} \times [0, \infty))]^\wedge$. D is an $(n - 1)$-cell in $\hat{R}^n$ which is locally flat except possibly at the ideal point. By $\sigma(n)$, D is locally flat, and hence we can construct an extension of f over $(S^{n-2} \times R^1) \cup (B^{n-1} \times [0, \infty))$, making use of a collar for D on the side "away" from $f(S^{n-2} \times (-\infty, 0])$. Then f can be extended over all of $B^{n-1} \times R^1$ by Theorem 5.2.

Now suppose that $\tau(n)$ is true, and let D be an $(n - 1)$-cell in $\hat{R}^n$ which is locally flat except possibly at ∞, an interior point of D. By a collaring argument [1], we can find a closed embedding G of $S^{n-2} \times R^1 \times [0, \infty)$ into R^n such that

$$G(S^{n-2} \times 0 \times [0, \infty)) = D - \{\infty\},$$

and

$$G(S^{n-2} \times R^1 \times 0) \text{ is locally flat in } R^n.$$

Let f be G restricted to $S^{n-2} \times R^1$. By $\tau(n)$, f can be extended to a closed embedding F of $B^{n-1} \times R^1$ into R^n. Since the complementary domain of $f(S^{n-2} \times R^1)$ which intersects D is not homeomorphic to R^n, it follows that $F(B^{n-1} \times R^1)$ and $G(S^{n-2} \times R^1 \times [0, \infty))$ intersect in $f(S^{n-2} \times R^1)$. Since $B^{n-1} \times R^1$ and $S^{n-2} \times R^1 \times [0, \infty)$ intersect in $S^{n-2} \times R^1$ and fill up R^n in a natural way, we have constructed a homeomorphism $H = F \cup G$ of R^n onto itself which takes $S^{n-2} \times 0 \times [0, \infty)$ onto $D - \{\infty\}$. Thus $\hat{H}$ takes a standard cell onto D, and the proof is complete.

Remark. Both $\sigma(3)$ and $\tau(3)$ are false. See Proposition 5.1.

Added in Proof. R. C. Kirby has proved $\sigma(n)$ for $n \geq 4$.

References

1. M. Brown, *Locally flat embeddings of topological manifolds*, Ann. of Math. (2), vol. 75 (1962), pp. 331–341.
2. J. C. Cantrell, *Almost locally flat embeddings of S^{n-1} in S^n*, Bull. Amer. Math. Soc., vol. 69 (1963), pp. 716–718.
3. J. C. Cantrell and R. C. Lacher, *Local flattening of a submanifold*, Quart. J. Math., to appear.
4. A. V. Chernavskii, *On topological imbeddings of polyhedra into Euclidean spaces*, Soviet Math. Dokl. (English), vol. 6 (1965), pp. 1601–1610.
5. V. K. A. M. Gugenheim, *Piecewise linear isotopy and embeddings of elements and spheres*, Proc. London Math. Soc., vol. 3 (1953), pp. 29–53.
6. M. C. Irwin, *Embeddings of polyhedral manifolds*, Ann. of Math (2), vol. 82 (1965), pp. 1–14.
7. R. C. Lacher, *Locally flat strings and half-strings*, Proc. Amer. Math. Soc., vol. 18 (1967), pp. 299–304.
8. ———, *Some conditions for manifolds to be locally flat*, Trans. Amer. Math. Soc., vol. 126 (1967), pp. 119–130.
9. W. B. R. Lickorish, *The piecewise linear unknotting of cones*, Topology, vol. 4 (1965), pp. 67–91.
10. J. Stallings, *On topologically unknotted spheres*, Ann. of Math. (2), vol. 77 (1963), pp. 490–503.

11. E. C. ZEEMAN, *Isotopies and knots in manifolds*, Topology of 3-manifolds and related topics, Prentice-Hall, Englewood Cliffs, 1962, pp. 187–193.
12. ———, *Linking spheres*, Abh. Math. Sem. Hamburg, vol. 24 (1960), pp. 149–153.
13. C. WEBER, *Plongemonts de polyèdres dans le domaine métastable*, Comm. Math. Helv., vol. 42 (1967), pp. 1–27.

UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA