ON THE EXTENSIONS OF THE INFINITE CYCLIC GROUP BY A 2-MANIFOLD GROUP

BY
Peter Orlik

In this note we shall study certain group extensions

$$E : 0 \to F \xrightarrow{i} M \xrightarrow{\pi} B \to 1$$

where F is Abelian (written additively), M and B are non-Abelian (written multiplicatively) and all groups are assumed to have finite presentations. Following [2] we denote by $\varphi : B \to \operatorname{Aut} F$ "conjugation by elements of B" determining the B-module structure of F. A morphism $\Gamma : E \to E'$ is a triple $\Gamma = (f, g, h)$ of commuting homomorphisms:

$$(*) \qquad \begin{array}{ccccccccc} E : 0 & \to & F & \xrightarrow{i} & M & \xrightarrow{\pi} & B & \to & 1 \\ & & f\downarrow & & g\downarrow & & h\downarrow & & \\ E' : 0 & \to & F' & \xrightarrow{i'} & M' & \xrightarrow{\pi'} & B' & \to & 1 \end{array}$$

The classical theory defines a *congruence* ($E \equiv E'$) as a morphism $\Gamma : E \to E'$ such that $F = F'$, $B = B'$ and $\Gamma = (1_F, g, 1_B)$. It follows that g is an isomorphism and $\varphi = \varphi'$. The main result is that for given φ the congruence classes are in one-to-one correspondence with $H^2_\varphi(B; F)$.

Definition. An *equivalence* of extensions ($E \sim E'$) is a morphism $\Gamma : E \to E'$ where f and h are isomorphisms.

For convenience we shall assume $F = F'$, $B = B'$ and supress i and π. The non-commutative 5-lemma implies that g is an isomorphism. The following are standard or easily verified:

Proposition 1.

(i) *"$\sim$" is an equivalence relation*
(ii) $E \equiv E' \Rightarrow E \sim E'$.
(iii) $E \sim E'$ *gives rise to a commutative diagram*

$$(**) \qquad \begin{array}{ccc} B & \xrightarrow{\varphi} & \operatorname{Aut} F \\ h\downarrow\approx & & \downarrow\approx f^* \\ B & \xrightarrow{\varphi'} & \operatorname{Aut} F \end{array}$$

where f^* is conjugation by f.

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(iv) *A morphism* $\Gamma = (f, g, h)$ *factors through* $\Gamma' = (1_F, g', h)$ *and also through* $\Gamma'' = (f, g'', 1_B)$.

Thus in an equivalence we can always assume either f or h to be the identity automorphism.

PROPOSITION 2. *Given isomorphisms f and h, a commutative diagram* $(**)$ *and a φ-extension E there exists a φ'-extension E' and a commutative diagram* $(*)$ *inducing* $(**)$ *such that* $E \sim E'$.

The proof is immediate.

COROLLARY. *Given a commutative diagram* $(**)$ *the cohomology groups* $H^2_\varphi(B; F)$ and $H^2_{\varphi'}(B; F)$ *are isomorphic.*

Remark. In particular if $\varphi = \varphi'$ in $(**)$ then using (iv) and the functorial property of H^* we conclude that f_* (or h_*) induces a non-trivial automorphism of $H^2_\varphi(B; F)$ when non-congruent extensions are equivalent.

Let $F = (s)$ be the infinite cyclic group and let

$$B = (u_1, \cdots, u_n \mid r)$$

with either $r = [u_1, u_2] \cdots [u_{n-1}, u_n]$ (n even), or $r = u_1^2 \cdots u_n^2$ be the fundamental group of a closed 2-manifold. For $n > 1$ the homotopy exact sequence of a circle bundle over a closed 2-manifold reduces to our extension E and the equivariant classification of such bundles with group $O(2)$ coincides with the equivalence classes of extensions, because all isomorphisms involved are induced by homeomorphisms. The geometric construction is due to Seifert [4].

Clearly $\operatorname{Aut} F \approx C_2 = \{1, -1\}$. A homomorphism $\varphi : B \to C_2$ is determined by its values on the u_i; on the other hand any assignment of ± 1 to the generators lifts to a homomorphism since the exponent sum of r is even. Due to Lyndon [1] we can compute the cohmology groups for each φ.

B orientable. (1) $H^2_\varphi(B; F) = F \approx Z$ if $\varphi(u_i) = 1$ for all i,
(2) $H^2_\varphi(B; F) = F/2F \approx Z_2$ all other φ.

Any two maps of (2) are connected by $(**)$. Suppose for i, j $\varphi(u_i) = -1$, $\varphi(u_j) = 1$. We can assume $i = 1$. Let j be the first generator (of the given presentation) such that $\varphi(u_j) = 1$. Let $f = 1_F$ and if

(i) j is even: let $h(u_{j-1}) = u_{j-1}u_j$; $h(u_j) = u_{j-1}^{-1}$; $h(u_k) = u_k$, $k \neq j-1, j$.
(ii) j is odd $(j \geq 3)$:
(a) $\varphi(u_{j+1}) = 1$; let

$$h(u_{j-1}) = u_j^{-1}u_{j-1}; \quad h(u_j) = u_j^{-1}u_{j-1}u_{j-2}u_{j-1}^{-1}u_j u_{j+1}^{-1}; \quad h(u_{j+1}) = u_{j+1}u_j u_{j+1}^{-1};$$

$$h(u_k) = u_k, \quad k \neq j-1, j, j+1.$$

(b) $\varphi(u_{j+1}) = -1$; let $h(u_j) = u_j u_{j+1}$; $h(u_k) = u_k$, $k \neq j$.

Repeated application of these maps connect φ with $\bar{\varphi}$, where $\bar{\varphi}(u_i) = -1$ for all i. By Proposition 2, $\bar{\varphi}$ represents all φ of (2). The two non-congruent extensions of $H^2_{\bar{\varphi}}(B; F)$ are not equivalent by the remark, however there are equivalent ones among the central extensions (1). The connecting morphism $\Gamma = (f, g, 1_B)$ has $f(s) = -s$, the non-trivial automorphism of F inducing the non-trivial automorphism of $H^2_{\varphi}(B; F)$. Thus we have the

THEOREM 1. *For orientable B the equivalence classes are as follows*
(1) *one for each non-negative integer for the central extensions;*
(2) *two for non-central extensions (e.g. the congruence classes of $\bar{\varphi}$).*

B non-orientable. (3) $H^2_{\varphi}(B; F) = F \approx Z$ if $\varphi(u_i) = -1$ for all i
(4) $H^2_{\varphi}(B; F) = F/2F \approx Z_2$ all other φ.

Clearly (3) is analogous to (1) and has one equivalence class for each non-negative integer. On the other hand (4) is more complicated than (2). We shall show that any φ in (4) is connected by (**) to one of the following:

(4.1) $\varphi(u_i) = 1$ for all i
(4.2) $\varphi(u_i) = -1$ for $i = 1, \cdots, n-1$, $\varphi(u_n) = 1, n \geq 2$
(4.3) $\varphi(u_i) = -1$ for $i = 1, \cdots, n-2$, $\varphi(u_{n-1}) = \varphi(u_n) = 1, n \geq 3$.

Assume that we have $\varphi(u_{i_1}) = -1$ and $\varphi(u_{i_2}) = \varphi(u_{i_3}) = \varphi(u_{i_4}) = 1$. Without loss of generality we may assume i_1, i_2, i_3, i_4 to be 1, 2, 3, 4. Consider $f = 1_f$ and

$$h(u_1) = u_1 u_2 u_3 ;$$

$$h(u_2) = u_3^{-1} u_2^{-1} u_1^{-1} u_3^{-1} u_2^{-1} u_3 u_4^{-1} u_3^{-2} u_2^{-1} u_3 ;$$

$$h(u_3) = u_3^{-1} u_2 u_3^2 u_4 ;$$

$$h(u_4) = u_4^{-1} u_3^{-1} u_1 u_2^2 u_3^2 u_4^2 ;$$

$$h(u_i) = u_i , \qquad i > 4.$$

This map reduces the number of generators mapped into $+1$ by two, hence φ is eventually connected with (4.2) or (4.3). Clearly (4.1) is different from (4.2) and (4.3) since its image in C_2 is trivial. To show that (4.2) and (4.3) are not connected by (**) abelianize B; $B/[B, B] \approx C_\infty^{n-1} \times C_2$ where $u = u_1 \cdots u_n$ is the only element of finite order. Now observe that s commutes with u (in M) only in (4.2) for odd n and only in (4.3) for even n. This property is preserved by an isomorphism, hence extensions of (4.2) and (4.3) cannot be equivalent.

THEOREM 2. *For non-orientable B the equivalence classes are as follows:*
(3) *one for each non-negative integer when $\varphi(u_i) = -1$ for all i;*
(4.1) *two central extensions;*
(4.2) *two for the map $\varphi(u_i) = -1$ for $i = 1, \cdots, n-1$, $\varphi(u_n) = 1$, $n \geq 2$;*

(4.3) *two for the map* $\varphi(u_i) = -1$ *for* $i = 1, \cdots, n-2$, $\varphi(u_{n-1}) = \varphi(u_n) = 1$, $n \geq 3$.

In [3] it is shown that in most cases inequivalent extensions yield non-isomorphic groups, i.e. $M \approx M'$ if and only if $E \sim E'$.

If B is the fundamental group of a 2-manifold *with boundary*, then all cohomology groups vanish and there is one extension, the semidirect product, in each class above.

References

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University of Wisconsin
Madison, Wisconsin