CRITERIA FOR POSITIVE GREEN'S FUNCTIONS

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If L is an elliptic operator defined by

$$Lv = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial v}{\partial x_i}\right) - 2\sum_{i=1}^{n} b_i \frac{\partial v}{\partial x_i} + cv \tag{1}$$

and $c(x) \geq 0$ in a bounded domain $D \subset R^n$, then the Green's function $G(x, \xi)$ associated with L on D is known to be positive in $D \times D$ whenever it exists. This fact follows readily from the characteristic properties of $G(x, \xi)$ and the Hopf maximum principle which applies to solutions of $Lv = 0$ when $c \geq 0$.

In this paper we shall use recently proved comparison theorems for elliptic equations to establish more general conditions under which fundamental solutions of (1) are non-negative in $D \times D$. Such conditions will be seen to involve *all* the coefficients of L and, in the case L is self-adjoint, will reduce to the assumption that the smallest eigenvalue of

$$\begin{aligned} Lu &= \lambda u \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D \end{aligned} \tag{2}$$

be positive.

Comparison theorems for elliptic equations deal with solutions v of $Lv = 0$ and u of $\Lambda u = 0$ where L is given by (1) and

$$\Lambda u = -\sum \frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial u}{\partial x_i}\right) - 2\sum_i \beta_i \frac{\partial u}{\partial x_i} + \gamma u. \tag{3}$$

If there exists a non-trivial solution u of $\Lambda u = 0$ in a domain $D_0 \subset D$ which vanishes on ∂D_0, and if the operator L is "smaller" than Λ in an appropriate sense to be made precise below, then such comparison theorems assert that every solution of $Lv = 0$ has a zero in $\bar{D}_0$.

For non self-adjoint equations of the form $Lv = 0$ where, L is given by (1), we make use of a comparison theorem due to Swanson [1].

In order to make the matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} & -b_1 \\ \vdots & & & \vdots \\ a_{n1} & \cdots & a_{nn} & -b_n \\ -b_1 & \cdots & -b_n & g \end{pmatrix}$$

positive semidefinite, Swanson formulates the condition

$$g \det(a_{ij}) \geq -\textstyle\sum_{i=1}^{n} b_i B_i\, , \tag{4}$$

Received March 20, 1967.

where B_i is the cofactor of $-b_i$. The following, then, is a special case of the principal result of [1].

THEOREM 1. *Suppose $g(x)$ satisfies (4) in D and that for some $D_0 \subset D$ there exists a non-trivial solution of $Lu = 0$ such that $u = 0$ on ∂D_0. If $g_0 > g(x)$ in D_0 then every solution of $(L - g_0 I)v = 0$ has a zero in $\bar{D}_0$.*

In order to derive criteria for the non-negativeness of $G(x, \xi)$ from this theorem, we recall that $G(x, \xi)$ is zero for $x \in \partial D$ or $\xi \in \partial D$, and that for $x \neq \xi$, G is a solution of $LG = 0$ in the variable x and a solution of $L^*G = 0$ in the variable ξ, where

$$L^*v = -\sum \frac{\partial}{\partial \xi_i}\left(a_{ij}\frac{\partial v}{\partial \xi_j}\right) + 2\sum b_i \frac{\partial v}{\partial \xi_i} + \left(c + 2\sum \frac{\partial b_i}{\partial \xi_i}\right)v.$$

Furthermore, $\lim x \to \xi_0\, G(x, \xi_0) = \lim \xi \to x_0\, G(x_0, \xi) = +\infty$ for $(x_0, \xi_0) \in D \times D$.

THEOREM 2. *Suppose $g(x)$ satisfies (4) in D and that for some $g_0 > g$ either $Lv = g_0 v$ or $L^*v = g_0 v$ has a solution v which is non-zero in $\bar{D}$. If $G(x, \xi)$ exists, then $G(x, \xi)$ is non-negative in $D \times D$.*

Proof. Suppose, to the contrary, that $G(x_0, \xi_0) < 0$ and, to be specific, that $Lv = g_0 v$ has a solution v which is non-zero in $\bar{D}$. Since

$$\lim_{x \to \xi_0} G(x, \xi_0) = +\infty,$$

there exists a proper sub-domain $D_0 \subset D$ (not containing ξ_0) such that $G(x, \xi_0) < 0$ for $x \in D_0$, $G(x, \xi_0) = 0$ for $x \in \partial D_0$, and (since $\xi_0 \in D - \bar{D}_0$) in which $G(x, \xi_0)$ is a regular solution of $LG = 0$. By Theorem 1 we obtain the contradiction that $v(x)$ has a zero in $\bar{D}_0$. A similar argument applies in case L^*v has a non-zero solution in $\bar{D}$.

In case L is self-adjoint (i.e., $b_i \equiv 0$ for $i = 1, \cdots, n$) we can choose $g(x) \equiv 0$. If the first eigenvalue of (2) satisfies $\lambda_1 > 0$, then there exists a slightly larger domain $D' \supset \bar{D}$ for which the first eigenvalue of

$$Lu = \lambda' u \quad \text{in } D'$$

$$u = 0 \qquad \text{on } \partial D'$$

also satisfies $\lambda_1' > 0$ and the corresponding eigenfunction v_1' is positive in $\bar{D}$. Setting $g_0 = \lambda_1'$ we obtain

COROLLARY 1. *If L is self-adjoint and the first eigenvalue of (2) is positive, then $G(x, \xi)$ is non-negative in $D \times D$.*

It is of interest to note that for self-adjoint operators the hypotheses of Corollary 1 are satisfied whenever $c(x) \geq 0$ in D and that these hypotheses also imply the existence of $G(x, \xi)$. In case L is self-adjoint one can also derive similar conclusions about the Robin's function $R_\sigma(x, \xi)$ associated with L and the mixed boundary conditions

$$(5) \qquad \frac{\partial v}{\partial \nu} + \sigma v = 0 \quad \text{on} \quad \partial D; \qquad -\infty < \sigma(x) \leq +\infty$$

(where $\sigma(x) = +\infty$ denotes the boundary condition $v(x) = 0$). This fact follows from the characteristic properties of $R\sigma(x, \xi)$ and the following special case of a comparison theorem proved by the author [2].

THEOREM 3. *Suppose $u(x)$ and $v(x)$ are, respectively, non-trivial solutions of*

$$-\sum \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + cu = 0; \qquad \frac{\partial u}{\partial \nu} + \sigma(x)u = 0 \quad on \quad \partial D_0,$$

$$-\sum \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_i}\right) + cv = \lambda v; \qquad \frac{\partial v}{\partial \nu} + \tau(x)v = 0 \quad on \quad \partial D_0.$$

If $\lambda \geq 0$ and $-\infty < \tau(x) \leq \sigma(x) \leq +\infty$ on ∂D_0, then either $v(x)$ has a zero in the interior of D_0 or else $\lambda = 0$ and v is a constant multiple of u.

THEOREM 4. *Let $R_\sigma(x, \xi)$ be the Robin's function associated with the self-adjoint operator L and the boundary conditions $\partial v/\partial \nu + \sigma v = 0$ on ∂D. If the first eigenvalue λ_1 of*

$$Lv = \lambda v \text{ in } D$$

$$\partial v/\partial \nu + \sigma v = 0 \text{ on } \partial D; \qquad -\infty < \sigma(x) \leq \infty$$

is positive, then $R_\sigma(x, \xi)$ is non-negative in $D \times D$.

Proof. Suppose to the contrary that $R_\sigma(x_0, \xi_0) < 0$ for some

$$(x_0, \xi_0) \in D \times D.$$

Then there exists a proper sub-domain $D_0 \subset D$ in which $R_\sigma(x, \xi_0)$ is negative, R_σ satisfies $LR\sigma = 0$, and such that

$$R_\sigma(x, \xi_0) = 0 \quad \text{for } x \in \partial D_0 \cap D,$$

$$\partial R_\sigma/\partial \nu + \sigma R = 0 \quad \text{for } x \in \partial D_0 \cap \partial D.$$

Setting $u(x) = R_\sigma(x, \xi_0)$ in Theorem 3, it follows that every solution of $(L - \lambda_1 I)v = 0$ which satisfies $\partial v/\partial \nu + \sigma v = 0$ on $\partial D_0 \cap \partial D$ has a zero in D_0 and we obtain the contradiction that the first eigenfunction of (6) has a zero in D.

The fact that the non-negative Green's and Robin's functions considered above are actually positive in $D \times D$ follows from an application of the Hopf maximum principle. If $\lim_{x \to x_0} G(x, \xi_0) = 0$ for some interior point $x_0 \in D$ and if $c(x_0) > 0$, then the Hopf maximum principle implies that $G(x, \xi_0)$ changes sign at x_0. However even if $c(x_0) \leq 0$ the Hopf maximum principle can be applied *locally* to yield the same conclusions, as is shown in [3]. Thus the fact that $G(x, \xi)$ or $R_\sigma(x, \xi)$ is non-negative in $D \times D$ is actually equivalent to the positivity of the fundamental solution.

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