ON COLLAPSIBLE BALL PAIRS[1]

BY
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One of the essential parts of Zeeman's proof [20], [21] to show that ball pairs B^q, B^s, $q - s \geq 3$, were unknotted was to show that B^q collapses to B^s. For $q - s = 2$, it is well known that there exist ball pairs B^q, B^{q-2}, such that B^q, B^{q-2} are knotted but B^q collapses to B^{q-2} for $q \geq 4$. For $q = 1, 2, 3$, it is known that B^q, B^{q-1} is unknotted and hence B^q collapses to B^{q-1} [5]. We say B^q, B^s is a *collapsible ball pair* if B^q collapses to B^s. In this paper we examine ball pairs B^q, B^{q-1} for $q \geq 4$ with regards to collapsibility. It is known that B^4, B^3 is unknotted iff B^4, B^3 is a collapsible ball pair; however, it is unknown whether there exist knotted B^q, B^{q-1} for $q \geq 4$. We show that for $q \geq 6$, every B^q, B^{q-1} is a collapsible ball pair and give some necessary and sufficient conditions that B^q collapses to B^{q-1} for $q = 4, 5$. We also characterize all ball pairs B^5, B^4.

Terminology and definitions will be as in [20] except as follow. By a manifold, we mean a locally Euclidean, separable metric space. When referring to combinatorial manifolds and piecewise linear maps we shall always use the adjectives combinatorial and piecewise linear. Let M be an orientable manifold; by bdry M we mean the boundary of M with the induced orientation; by int M, the interior of M; by M^- we mean M with its orientation reversed. By Cl X, we mean the closure of X.

THEOREM 1. *Let B^n, B^{n-1} be a ball pair with $n \geq 6$; then B^n collapses to B^{n-1}.*

1. Proof of Theorem 1 for $n \geq 7$

Let N be an admissible regular neighborhood of B^{n-1} in B^n [20; Chap. VII, p. 67]. Then $N \cap$ bdry B^n is a regular neighborhood of bdry B^{n-1} in bdry B^n. It was shown in [8] that

$$\mathrm{Cl}\,(\mathrm{bdry}\, B^n - (N \cap \mathrm{bdry}\, B^n))$$

is the union of two disjoint combinatorial $(n - 1)$-cells, say $S_1 \cup S_2$. Similarly, Cl (bdry N − ($N \cap$ bdry B^n)) is the union of two disjoint combinatorial $(n - 1)$-cells, say $T_1 \cup T_2$, indexed so that $S_i \cap T_i \neq \emptyset$, $i = 1, 2$. Then each $S_i \cup T_i$ is a combinatorial $(n - 1)$-sphere. Hence by considering the double of B^n, it follows from [4], [15] that each $S_i \cup T_i$ bounds a topological

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cell R_i in B^n. By Smale [17], [11], each R_i is a combinatorial n-cell. Hence each R_i collapses to T_i so that B^n collapses to N and hence B^n collapses to B^{n-1}.

2. Proof of Theorem 1 for $n = 6$

Two orientable combinatorial manifolds M and N are said to be *equivalent*, $M \sim N$, if there exists an orientation preserving, onto, piecewise linear homeomorphism taking M onto N. $\sim$ is clearly an equivalence relation and so if one considers the set $S(T)$ of all combinatorial manifolds which triangulate some fixed orientable manifold, $\sim$ induces a decomposition of $S(T)$ into equivalence classes each of which will be called a *combinatorial structure* on T. The set of combinatorial structures on T will be denoted by $CS(T)$. In general, we shall not distinguish between a combinatorial manifold and the combinatorial structure containing it, in fact, we often use the same symbol for both. The necessary details for making the transition from element to equivalence class and *vice versa* in the following are easily supplied.

Let T be a closed orientable m-manifold with $CS(T) \neq \emptyset$. If $M, N \in CS(T)$, define the *connected sum* [9] of M and N, $M \# N$, as follows. Choose piecewise linear embeddings.

$$i_1 : B^m \to M, \qquad i_2 : B^m \to N,$$

where B^m is the oriented combinatorial m-cell, i_1 is orientation preserving and i_2 is orientation reversing. $M \# N$ is obtained from $\mathrm{Cl}\,(M - i_1 B^m) \cup \mathrm{Cl}\,(N - i_2 B^m)$ by identifying $i_i(t)$ with $i_2(t)$ for each $t \in \mathrm{bdry}\, B^m$. That the connected sum is a well defined operation follows from [5] and [14]. It is then easily seen that $CS(T)$ is a semigroup.

Let T be a compact orientable m-manifold with a non-empty connected boundary and $CS(T) \neq \emptyset$. If $M, N \in CS(T)$, define the *connected sum* of M and N, $M \# N$, as follow. Choose piecewise linear embeddings

$$i_1 : B^{m-1} \to \mathrm{bdry}\, M, \qquad i_2 : B^{m-1} \to \mathrm{bdry}\, N,$$

where bdry M, bdry N have orientations induced from M, N respectively and i_1 is orientation preserving, i_2 is orientation reversing. $M \# N$ is obtained from $M \cup N$ by identifying $i_1(t)$ with $i_2(t)$ for each $t \in B^{m-1}$. That this connected sum is well defined follows from [6] and the fact that combinatorial manifolds are combinatorially collared [16], [20]. Then it also follows easily that $CS(T)$ is a semigroup.

We shall be interested in the case when T is either the n-sphere S^n or the n-cell C^n. For $n = 1, 2$, it is a classical result that $CS(C^n)$ and $CS(S^n)$ are trivial [5]. Moise [12] and Bing [3] have shown that these semigroups are trivial for $n = 3$; Smale has shown this for $n \geq 6$ [17]. It is also known that $CS(S^5)$ is trivial [11]. That $CS(C^4)$ is trivial is equivalent to an affirmative answer to the Schoenflies Conjecture [10].

Consider the following maps:

$$\partial : CS(C^n) \rightarrow CS(S^{n-1})$$

defined by $\partial M = \text{bdry } M$ for each $M \in CS(C^n)$ and

$$\lambda : CS(S^n) \rightarrow CS(C^n)$$

defined by $\lambda M = \text{Cl}\,(M - N)$ for each $M \in CS(S^n)$ where N is a combinatorial n-cell embedded piecewise linearly in M. By [6], [14], λ is a well defined map. It is easily seen that both ∂ and λ are homomorphisms for each n.

Let $U(M)$ be the subset of $CS(M)$ of those elements which have inverses under $\#$.

LEMMA 1. *$U(C^4)$, $U(C^5)$, $U(S^4)$ are groups.*

LEMMA 2. *$\partial : CS(C^5) \rightarrow CS(S^4)$ is an epimorphism.*

Proof. Let $E \in CS(S^4)$; want to find $D \in CS(C^5)$ such that $\partial D = E$. By [13], E has a differentiable structure compatible with its combinatorial structure. From [9], Θ_4 is the trivial group, i.e., E is h-cobordant to S^4, the standard 4-sphere. Hence by [9; Lemma 2.3], [11, p. 110], $E = \partial D$ where D is a contractible differentiable manifold. Consider the double, $2D$, of D. It follows from the Mayer-Vietoris sequence and Van Kampen's theorem that $2D$ is a homotopy 5-sphere. Hence by Smale [17], $2D$ is diffeomorphic to the 5-sphere. By [4], [15], D is a topological 5-cell. By [18], it follows that D has the required combinatorial structure.

LEMMA 3. *The kernel of $\partial : CS(C^5) \rightarrow CS(S^4)$ contains only the trivial element of $CS(C^5)$.*

Proof. Let D be an element of the kernel of ∂ and give D a differentiable structure compatible with its combinatorial structure. By [11; p. 110], D is diffeomorphic to the standard 5-cell. The lemma then follows from the uniqueness of the compatible combinatorial structure [18].

The following two lemmas are easily proved.

LEMMA 4. *If G, H are semigroups and if $f : G \rightarrow H$ is anepimorphismsuch that the kernel of f contains only the trivial element, then an element a of G has an inverse if and only if $f(a)$ has an inverse.*

LEMMA 5. $\partial^{-1}(U(S^4)) = U(D^5)$.

By using Alexander [1], one can prove easily:

LEMMA 6. *$\lambda : CS(S^4) \rightarrow CS(D^4)$ is an epimorphism.*

LEMMA 7. *The kernel of $\lambda : CS(S^4) \rightarrow CS(D^4)$ contains only the trivial element of $CS(S^4)$.*

LEMMA 8. $\lambda^{-1}(U(D^4)) = U(S^4)$.

Finally, we have

LEMMA 9. *The restricted maps*

$$\partial : U(C^5) \to U(S^4), \qquad \lambda : U(S^4) \to U(D^4)$$

are isomorphisms.

Proof of Theorem 1. Let B^6, B^5 be a ball pair and let N, S_1, S_2, T_1, T_2 be defined as in Section 1 with the additional stipulation that each of the sets be given the induced orientation. Our difficulty is that S_1, S_2, T_1, T_2 may not be combinatorial 5-cells.

By [15; Lemma 10] and the uniqueness of regular neighborhoods, $N \cap \text{bdry } B^6$ is homeomorphic to $S^4 \times [0, 1]$. Hence, by [15], S_1, S_2, T_1, T_2 are topological 5-cells and are therefore elements of $CS(D^5)$. We wish to show that they are elements of $U(D^5)$.

Let K_1 be a triangulation of bdry B^6 such that K_1 contains a subcomplex K_2 which triangulates bdry B^5. Let v be a vertex of K_2 such that $|\text{st}(v, K_1)|$, $|\text{st}(v, K_2)|$ is an unknotted ball pair. (For example, one could pick v to be a point in the interior of some 4-simplex in K_2 and consider the new triangulation formed from K_1 by coning from v.) Let K_3 be the subcomplex of K_1 which triangulates $\text{Cl}(\text{bdry } B^6 - |\text{st}(v, K_1)|)$; Let $K_4 = K_3 \cap K_2$. Hence by [1], $|K_3|$, $|K_4|$ is a ball pair. Let N_1 be a second derived neighborhood of K_4 in K_3. By [19], N_1 is a combinatorial 5-cell and bdry N_1 is a combinatorial 4-sphere. Then

$$\text{Cl}(\text{bdry } N_1 - \text{bdry } |K_3|) = L_1 \cup L_2$$

where $L_1, L_2 \in U(D^4)$.

However $|\text{lk}(v, K_1)|$, $|\text{lk}(v, K_2)|$ is an unknotted sphere pair and

$$\text{bdry } N_1 \cap |\text{lk}(v, K_1)|$$

is a regular neighborhood of $|\text{lk}(v, K_2)|$ in $|\text{lk}(v, K_1)|$. Hence

$$\text{Cl}(|\text{lk}(v, K_1)| - \text{bdry } N_1) = M_1 \cup M_2$$

which are disjoint combinatorial 4-cells. Note that $N_1 \cup |\text{st}(v, K_1)|$ is a regular neighborhood of bdry B^5 in bdry B^6 and

$$\text{bdry}(N_1 \cup |\text{st}(v, K_1)|) = L_1 \cup L_2 \cup M_1 \cup M_2,$$

which we may assume are so indexed that $L_i \cap M_i \neq \emptyset$.

Therefore $L_i \cup M_i \in U(S^4)$ for each i. Hence

$$\text{Cl}(\text{bdry } B^6 - (N_1 \cup |\text{st}(v, K_1)|)) = P_1 \cup P_2$$

where $\partial P_i = L_i \cup M_i$ and by Lemma 5, $P_i \in U(D_5)$ for each i. By the uniqueness theorem of regular neighborhoods it follows that each S_i is piecewise linearly homeomorphic to some P_j. Similar arguments also that the T_i's belong to $U(D_5)$. Since $\partial S_i = \partial T_i$, by Lemma 9, $S_i = T_i$ for each i.

Each $S_i \cup T_i^-$ is a combinatorial 5-sphere and each bounds a combinatorial 6-cell R_i in B^6 [11], [20]. What we want to show now is that each R_i collapses to T_i^-. Consider $R_i' = S_i \times [0, 1]$; R_i' is clearly a topological 6-cell and hence a combinatorial 6-cell. Clearly R_i' collapses to $S_i \times 1$ which is piecewise linearly homeomorphic to S_i^-. By using [20; Lemma 10], we have then that R_i collapses to T_i^- for each i.

3. Theorem 2 for $n = 4, 5$

THEOREM 2. *Every ball pair B^5, B^4 is collapsible if and only if every ball pair B^4, B^3 is collapsible.*

Proof. The "if" part is well known [10]. Suppose there exists a ball pair B^4, B^3 which is not collapsible. $B^4 = B_1 \cup B_2$ where $B_1 \cap B_2 = B^3$ and each $B_i \in U(D^4)$. Let $N_i \in U(D^5)$ such that $B_i = \lambda\partial N_i$ for each i. By Lemma 9, each N_i is not a combinatorial 5-cell. Let $N^5 = N_1 \# N_2$, $N^4 = N_1 \cap N_2$. Claim that N^5, N^4 is not a collapsible ball pair. If N^5 collapses to N^4, then each N_i collapses to N^4. Hence by [19], each N_i is a combinatorial 5-cell.

4. Theorem 3 for $n = 5$

Let B^5, B^4 be a ball pair. B^4 separates B^5 into two components, the closure of which will be designated as B^5_+, B^5_- where B^5_+, B^5_-, B^4 have their orientation induced from B^5 and the orientation on B^4 agrees with the orientation induced from B^5_+. The set of points $\{v_i\}_{i=1}^{n}$ of B^4 at which B^4 could fail to be locally unknotted [20] is clearly finite. Let K be a triangulation of B^5 such that $\{v_i\} \subset K$ and $K \mid B^4 = L$. Define the *knot type* ${}_iK_+$ at v_i with respect to B^5_+ to be the element

$$|\operatorname{lk}(v_i, K)| \cap B^5_+ \in CS(C^4)$$

where the orientation of ${}_iK_+$ is induced by the orientation of $|\overline{\operatorname{st}}(v_i, K)|$ which, in turn, is oriented coherently with B^5. Similarly define

$${}_iK_- = |\operatorname{lk}(v_i, K| \cap B^5_- .$$

From [6], we have that $\{{}_iK_+, {}_iK_-\}$ is independent of the triangulation chosen. Let $S_+ = B^5_+ \cap \operatorname{bdry} B^5$, $S_- = B^5_- \cap \operatorname{bdry} B^5$ have the orientations induced by B^5_+, B^5_- respectively.

THEOREM 3. *B^5, B^4 is a collapsible ball pair iff*

$$S_+ = {}_1K_+ \# {}_2K_+ \# \cdots \# {}_nK_+ .$$

If B^5, B^4 is locally unknotted at each point of B^4, then B^5, B^4 is a collapsible ball pair iff $S^4_+ = 0$, i.e. iff S^4_+ is a combinatorial 4-cell.

Proof. The proof of the second statement is straightforward, so we only give a proof of the first statement. Suppose B^5, B^4 is a collapsible ball pair.

Let the v_i's be ordered such that $v_1, v_2, \cdots, v_q \in \text{bdry } B^4$, $v_{q+1}, \cdots, v_n \in \text{int } B^4$: let A_1 be a polygonal arc in bdry B^4 such that bdry $A_1 = \{v_i, v_q\}$ and $\{v_i\}_{i=1}^{q} \subseteq A_1$ and let A_2 be a polygonal arc in B^4 such that $A_2 \cap \text{bdry } B^4 = \{v_q\}$, bdry $A_2 = \{v_q, v_n\}$, and $\{v_i\}_{i=q+1}^{n} \subseteq A_2$. Let $A = A_1 \cup A_2$ and let the usual ordering $<$ be given on A and suppose that the v_i's are so indexed that $v_i < v_{i+1}$ for each i. Claim $B^4 \searrow A$.

Let N_1 be a regular neighborhood of A_1 mod (bdry A_1) $\cup A_2$ in B^4 meeting bdry B^4 regularly [7]. Hence Cl $(B^4 - N_1)$ is a combinatorial 4-cell [1]. Let N_2 be an admissible regular neighborhood of A_2 in Cl $(B^4 - N_1)$ and again Cl (Cl $(B^4 - N_1) - N_2$) is a combinatorial 4-cell so that

$$\text{Cl}\,(\text{Cl}\,(B^4 - N_1) - N_2) \searrow \text{Cl}\,(\text{Cl}\,(B^4 - N_1) - N_2) \cap N_2 .$$

Hence Cl $(B^4 - N_1) \searrow N_2 \searrow A_2$, so that $B^4 \searrow N_1 \cup N_2 \searrow A_1 \cup A_2 = A$.

Let K^* be a subdivision of K such that $L^* \searrow^s L^* \mid A$, [19] [20]. By Whitehead [19], the second-derived neighborhood $N(B^4, K^{*\prime\prime})$, it follows that there exists an orientation preserving piecewise linear homeomorphism between $N(B^4, K^{*\prime\prime})$ and $N(A, K^{*\prime\prime})$. It follows then from the properties of dual complexes [2] that

$$N(A, K^{*\prime\prime}) \cap B^5_+ = N(B^4, K^{*\prime\prime}) \cap B^5_+ = {}_1K_+ \# {}_2K_+ \# \cdots \# {}_nK_+ ,$$

where the orientation of $N(B^4, K^{*\prime\prime})$ is the one induced from B^5. Since B^5 is also a regular neighborhood of B^4 in B^5, the conclusion follows from [7].

Conversely if we let $M_+ = \text{Cl}\,(B^5_+ - N(B^4, K^{*\prime\prime}))$, then

$$\text{bdry } M_+ = S_+ \cup (M_+ \cap N(B^4, K^{*\prime\prime}))^-$$

where M_+ has the orientation induced from M_+. Hence

$$\text{bdry } M_+ = ({}_1K_+ \# \cdots \# {}_nK_+) \cup ({}_1K_+ \# \cdots \# {}_nK_+)^-,$$

mplying $M_+ = ({}_1K_+ \# \cdots \# {}_nK_+) \times I$ as in proof of Theorem 1, $n = 6$; so as in that proof $M_+ \searrow {}_1K_+ \# \cdots \# {}_nK_+$. Define

$$M_- = \text{Cl}\,(B^5_- - N(B^4, K^{*\prime\prime}))$$

and by noting that $S_- = S_+^-$,

$$(\text{bdry } N(B^4, K^{*\prime\prime})) \cap B^5_- = [(\text{bdry } N(B^4, K^{*\prime\prime})) \cap B^5_+]^-$$

by arguments of Theorem 1, $n = 6$, we get similarly

$$M_- \searrow [{}_1K_+ \# \cdots \# {}_nK_+]^-.$$

Therefore $B^5 \searrow N(B^4, K^{*\prime\prime}) \searrow B^4$.

COROLLARY. *There exists a* 1-1 *correspondence between ball pairs* B^5, B^4 *and ordered triples* $(\{K_+\}, \{L_+\}, M_+)$ *where* $\{K_+\}$, $\{L_+\}$ *are two finite unordered collections of knot types,* K_+*'s occurring at vertices of* bdry B^4 *and* L_+*'s occurring at vertices in* int B^4, *and where* $M_+ \in U(C^4)$.

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