HARMONIC FUNCTIONS ON THE UNIT DISC II[1]

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1. Introduction

This report continues a study begun in [3] of harmonic functions defined on the unit disc in the plane. In part I it was proved that f is harmonic on $\{r < 1\}$ if and only if

$$(1.1) \qquad f(r,\theta) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} P_r^{(n)}(\theta - t) g_n(t)\, dt$$

where $P_r(\theta - t)$ is the Poisson kernel and $\{g_n\}$ is a sequence of continuous functions satisfying $\lim\, (n!\, \| g_n \|)^{1/n} = 0$.

In this part we present three characterizations for functions which have a terminating series representation and two results concerning the lack of uniqueness of the representation.

In Section 2 two characterizations are obtained for those harmonic functions which have a representation

$$(1.2) \qquad f(r,\theta) = \sum_{n=0}^{N} \int_{\Gamma} P_r^{(n)}(\theta - t)\, d\mu_n(t)$$

where Γ denotes the unit circle and $\mu_0, \cdots, \mu_N$ are measures on Γ. Attention may be limited to those harmonic functions which are zero at the origin. The n-th integral F_n of f is the unique harmonic function such that $f = \partial^n F_n / \partial\theta^n$. Then (1.2) is equivalent to each of the following conditions.

$$(1.3) \qquad \int_0^{2\pi} | F_N(r,\theta) |\, d\theta \leqq B \qquad \text{for } 0 \leqq r < 1,$$

(1.4) F_{N+1} is bounded and $\lim_{r \nearrow 1} F_{N+1}(r, \theta) = \beta(\theta)$ for all θ where β is a function of bounded variation.

Turning to a consideration of radial growth another criterion for (1.2) is discussed in Section 3. These are the functions $f(r, \theta) = O((1 - r)^{-k})$ uniformly with respect to θ for some positive k. More rapid growth requires an infinite representation. For analytic functions the growth conditions are somewhat more precise than for harmonic functions in general.

The representation (1.2) should prove useful in the study of various special classes of functions. For example, if an analytic function $f(r, \theta) = O(\exp\, (1 - r)^{-1})$ has no zeros, then $\log f(r, \theta)$ has a representation (1.2) with $N = 3$. Such is the case for the elliptic modular function.

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Section 4 deals with a special case in which a unique extremal solution for the sequence $\{\mu_0, \cdots, \mu_N\}$ in (1.2) is possible. To obtain some insight into the character of the multiplicity of sequences $\{g_n\}$ in (1.1) and to illustrate the use of the Hardy H^p spaces in this theory we assume in Section 5 that $\{g_n\}$ is a sequence in H^p. A certain quotient space of the sequence space is linearly and topologically isomorphic to the space $\mathfrak{F}_a$ of analytic functions on $\{r < 1\}$ provided with the subuniform convergence topology.

2. Terminating series

Our problem is to characterize those harmonic functions which have a representation

$$f(r,\theta) = \sum_{n=0}^{N} \int_0^{2\pi} P_r^{(n)}(\theta - t)\, d\alpha_n(t) \tag{2.1}$$

where α_n is a function of bounded variation on $[0, 2\pi]$.

To relate this to a familiar result suppose first that $N = 0$; that is, f is a Poisson-Stieltjes integral.

$$f(r,\theta) = \int_0^{2\pi} P_r(\theta - t)\, d\alpha(t). \tag{2.2}$$

The function α may be normalized so that

$$\alpha(t) = (\tfrac{1}{2})[\alpha(t + 0) + \alpha(t - 0)] \quad \text{for } t \in (0, 1),$$

$\alpha(0) = 0$, and $\alpha(2\pi) = \alpha(0 + 0) + \alpha(2\pi - 0)$. Then (see [1])

$$\int_0^{2\pi} |f(r,t)|\, dt \leqq B \quad \text{for } 0 \leqq r < 1 \tag{2.3}$$

and

$$\int_0^{\theta} f(r,t)\, dt \to \alpha(\theta) \quad \text{for all } \theta \text{ as } r \nearrow 1. \tag{2.4}$$

Condition (2.3) is also sufficient in order that the harmonic function f be a Poisson-Stieltjes integral. That (2.4) is not sufficient is illustrated by the example $f(r, \theta) = P_r^{(2)}(\theta)$. The integral

$$P_r^{(1)}(\theta) = \int_0^{\theta} P_r^{(2)}(t)\, dt$$

converges to zero for all θ. This property of $P_r^{(1)}$ was brought to the author's attention by Lohwater [5].

With the use of integrals of f, (2.4) may be reformulated as a necessary and sufficient condition for (2.2) and both conditions may be extended as criteria for (2.1). Before embarking on the description of a notational scheme for handling the extension let us look at the first integral of (2.2).

We suppose that f is zero at the origin which means $\alpha(2\pi) = \alpha(0) = 0$.

Taking the indefinite integral of (2.2) and integrating by parts

$$\int_0^\theta f(r,\tau)\, d\tau = \int_0^{2\pi} \int_0^\theta P_r(\tau - t)\, d\tau\, d\alpha(t)$$

$$= \int_0^{2\pi} \alpha(t)\, [P_r(\theta - t) - P_r(-t)]\, dt.$$

The function

$$\begin{aligned} F_1(r,\theta) &= \int_0^\theta f(r,\tau)\, d\tau + \int_0^{2\pi} P_r(-t)\alpha(t)\, dt - \int_0^{2\pi} \alpha(t)\, dt \\ &= \int_0^{2\pi} P_r(\theta - t)\alpha(t)\, dt - \int_0^{2\pi} \alpha(t)\, dt \\ &= \int_0^{2\pi} P_r(\theta - t) \left[\alpha(t) - \frac{1}{2\pi}\int_0^{2\pi} \alpha(\tau)\, d\tau\right] dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\beta(t)\, dt \end{aligned}$$

is harmonic and is zero at the origin. Moreover $f = \partial F_1/\partial\theta$ so that F_1 is the first integral of f.

In view of the representation of F_1 as the Poisson integral of β we may replace (2.4) with

(2.5) F_1 is bounded and $F_1(r, \theta) \to 2\pi\, \beta(\theta)$ for all θ as $r \nearrow 1$.

Conversely, using Fatou's theorem (2.5) implies that F_1 is the Poisson integral of β and (2.2) is a consequence.

We must consider the k-th ntegral $F_{n,k}$ of a function

$$f_n(r, \theta) = \int_0^{2\pi} P_r^{(n)}(\theta - t)\, d\alpha_n(t)$$

where α_n is normalized as in (2.2). For $k = 0, 1, \cdots, n - 1$ this is

$$F_{n,k}(r,\theta) = \int_0^{2\pi} P_r^{(n-k)}(\theta - t)\, d\alpha_n(t)$$

and for $k = n$

$$\begin{aligned} F_{n,n}(r, \theta) &= \int_0^{2\pi} P_r(\theta - t)\, d\alpha_n(t) - \int_0^{2\pi} d\alpha_n(t) \\ &= \int_0^{2\pi} P_r(\theta - t)\, d\beta_{n,n}(t) \end{aligned}$$

where $\beta_{n,n}(t) = \alpha_n(t) - \alpha_n(2\pi)t/2\pi$. Also for $k = n + 1, n + 2, \cdots$,

$$F_{n,k}(r, \theta) = \int_0^{2\pi} P_r(\theta - t)\beta_{n,k}(t)\, dt$$

where

$$\beta_{n,n+1}(t) = \beta_{n,n}(t) - \frac{1}{2\pi}\int_0^{2\pi} \beta_{n,n}(\tau)\, d\tau$$

and

$$\beta_{n,k}(t) = \int_0^t \beta_{n,k-1}(\tau)\, d\tau - \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\tau_2} \beta_{n,k-1}(\tau_1)\, d\tau_1\, d\tau_2$$

for $k = n + 2, n + 3, \cdots$.

From these expressions for $F_{n,k}$ we may conclude

$$\int_0^{2\pi} |F_{n,k}(r, t)|\, dt \leqq B \quad \text{for } 0 \leqq r < 1 \quad \text{if } n \leqq k \tag{2.6}$$

and

(2.7) $F_{n,k}$ is bounded and $F_{n,k}(r, \theta) \to 2\pi\, \beta_{n,k}(\theta)$ for all θ as $r \nearrow 1$ if $n + 1 \leqq k$.

THEOREM 1. *If f is harmonic on the unit disc then the possibility of the representation*

$$f(r, \theta) = \sum_{n=0}^{N} \int_0^{2\pi} P_r^{(n)}(\theta - t)\, d\alpha_n(t) \tag{2.1}$$

is equivalent to the validity of either of the conditions

$$\int_0^{2\pi} |F_N(r, t)|\, dt \leqq B \quad \text{for } 0 \leqq r < 1 \tag{2.8}$$

(2.9) *F_{N+1} is bounded and $F_{N+1}(r, \theta) \to 2\pi\, \beta(\theta)$ for all θ as $r \nearrow 1$ where β is a function of bounded variation.*

F_N and F_{N+1} are integrals of $f - f(0)$.

Proof. We may suppose that $f(0) = 0$. If (2.1) holds we write it in the form $f = f_0 + \cdots + f_N$ and using the notation previously established

$$F_N = F_{0,N} + \cdots + F_{N,N} \quad \text{and} \quad F_{N+1} = F_{0,N+1} + \cdots + F_{N,N+1}.$$

Then (2.8) and (2.9) are consequences of (2.6) and (2.7).

Conversely (2.8) implies

$$F_N(r, \theta) = \int_0^{2\pi} P_r(\theta - t)\, d\alpha(t)$$

$$f(r, \theta) = \frac{\partial^n}{\partial\theta^n} F_n(r, \theta) = \int_0^{2\pi} P_r^{(n)}(\theta - t)\, d\alpha(t)$$

and (2.9) implies

$$F_{N+1}(r, \theta) = \int_0^{2\pi} P_r(\theta - t)\beta(t)\, dt$$

$$F_N(r, \theta) = \int_0^{2\pi} P_r^{(1)}(\theta - t)\beta(t)\, dt$$

$$= \int_0^{2\pi} P_r(\theta - t) d\beta(t)$$

and the proof is completed with N differentiations as above.

Property (2.7) will find application in a uniqueness discussion for an extremal representation in Section 4. Another criterion for a terminating series seems appropriate for consideration now. It follows from (2.4) in [3] that any harmonic function f with a representation (2.1) satisfies a growth condition $f(r, \theta) = O((1 - r)^{-N-1})$ uniformly with respect to θ as $r \nearrow 1$. That this growth rate is also sufficient for a terminating series representation is the subject of the next section.

3. Growth conditions

If $f(r, \theta) = O((1 - r)^{-N+1})$, $N > 1$, and N is even then f has a representation (2.1). For an analytic function the result is valid for any $N > 1$ and if $f(r, \theta) = O(\log (1 - r)^{-1})$ then the representation is possible with $N = 1$. Some remarks on the sharpness of the latter statement are made at the end of the section.

It will be convenient to consider a real-valued harmonic function u which is zero at the origin. Let v denote its conjugate harmonic function also zero at the origin and set $f = u + iv$. The n-th integrals F_n, U_n, V_n satisfy $F_n = U_n + iV_n$ and F_n is analytic.

From the Cauchy-Riemann formula $r\partial u/\partial r = \partial v/\partial \theta$ we obtain

$$u(r, \theta) = \int_0^r \rho^{-1} \frac{\partial v(\rho, \theta)}{\partial \theta}\, d\rho$$

and

$$\int_0^\theta u(r, t)\, dt = \int_0^r \rho^{-1}[v(\rho, \theta) - v(\rho, 0)]\, d\rho.$$

Then

$$U_1(r, \theta) = \int_0^\theta u(r, t)\, dt + \int_0^r \rho^{-1} v(\rho, 0)\, d\rho = \int_0^r \rho^{-1} v(\rho, \theta)\, d\rho \tag{3.1}$$

is the first integral of u. Since $-u$ is the conjugate of v

$$V_1(\rho, \theta) = -\int_0^r \rho^{-1} u(r, \theta)\, d\rho \tag{3.2}$$

is the first integral of v.

PROPOSITION 1. *If v is the harmonic conjugate of u then*

(3.3) $u(r,\theta) = O((1-r)^{-n})$ *implies* $V_1(r,\theta) = O((1-r)^{-n+1})$ *if* $n > 1$.

(3.4) $u(r,\theta) = O((1-r)^{-1})$ *implies* $V_1(r,\theta) = O(\log(1-r)^{-1})$

(3.5) $u(r,\theta) = O(\log(1-r)^{-1})$ *implies* $V_1(r,\theta) = O(1)$.

All growth conditions are uniform with respect to θ.

Proof. If $n \geqq 1$, $u(r,\theta) = O((1-r)^{-n})$, and $u(0) = 0$, then there is a constant B such that $|u(r,\theta)| \leqq Br(1-r)^{-n}$ for $0 \leqq r < 1$. Using (3.2)

$$|V_1(r,\theta)| \leqq \int_0^r \rho^{-1}|u(\rho,\theta)|\,d\rho \leqq B\int_0^\rho (1-\rho)^{-n}\,d\rho$$

from which follow (3.3) and (3.4). A similar argument gives (3.5)

It may be remarked that each result stated in the proposition is the best possible. This assertion is verified by consideration of the example $f(z) = z/(1-z)^n$. For $n > 1$

$$F_1(z) = \frac{-i}{n-1}\left[\frac{1}{(1-z)^{n-1}} - 1\right]$$

and for $n = 1$

$$F_1(z) = i\log(1-z).$$

THEOREM 2. *Let u be harmonic on $\{|z| < 1\}$. If $N > 1$, N is even and $u(r,\theta) = O((1-r)^{-N+1})$, then u has a representation*

$$u(r,\theta) = \sum_{n=0}^{N}\int_0^{2\pi} P_r^{(n)}(\theta - t)\,d\alpha_n(t). \tag{3.6}$$

Proof. Suppose u is zero at the origin and let us consider the case $N > 2$. According to (3.3) $V_1(r,\theta) = O((1-r)^{-N+2})$. Applied to the function V_1, (3.3) yields $U_2(r,\theta) = O((1-r)^{-N+3})$. Inductively one finds $U_{N-2}(r,\theta) = O((1-r)^{-1})$. Now using (3.4) and (3.5) one obtains successively $V_{N-1}(r,\theta) = O(\log(1-r)^{-1})$ and $U_N(r,\theta) = O(1)$. For $N = 2$ only the last two steps are required. The proof is completed by an application of Theorem 1.

If u grows as an even power of $(1-r)^{-1}$, $u(r,\theta) = O((1-r)^{-N+2})$, then $u(r,\theta) = O((1-r)^{-N+1})$ and the above representation applies. If $u(r,\theta) = O(\log(1-r)^{-1})$ then the representation is possible with $N = 2$. When both functions u and v satisfy the growth condition, i.e. when the analytic function $f = u + iv$ satisfies the growth condition, the result for these latter cases may be sharpened.

THEOREM 3. *Let f be analytic on $\{|z| < 1\}$. If $N > 1$ and $f(r, \theta) = O((1 - r)^{-N+1})$, then f has a representation*

$$(3.1) \qquad f(r, \theta) = \sum_{n=0}^{N} \int_0^{2\pi} P_r^{(n)}(\theta - t)\, d\alpha_n(t).$$

If $f(r, \theta) = O(\log (1 - r)^{-1})$, then the representation is possible with $N = 1$.

Proof. Reasoning as in the proof of Theorem 2 with u and v simultaneously we find that F_N is bounded. The result follows from Theorem 1.

There are a number of questions lurking behind the results of this section· Only the question of sharpness of the second part of Theorem 4 will be mentioned. A theorem of MacLane [6] proves that this result is the best possible. He proves that for any function $\mu(r)$, $0 < \mu(r) \nearrow \infty$ as $r \nearrow 1$, there is a function $f = u + iv$ analytic on $\{|z| < 1\}$ such that $f(r, \theta) = O(\mu(r))$ uniformly with respect to θ,

$$\limsup_{r \nearrow 1} u(r, \theta) = \infty \quad \text{and} \quad \liminf_{r \nearrow 1} u(r, \theta) = -\infty$$

for each $\theta \in [0, 2\pi)$. The function f is not a Poisson-Stieltjes integral. For if it were, it would have radial limits for almost all θ by Fatou's theorem.

A MacLane function having growth $O(\log (1 - r)^{-1})$ is an example for which Theorem 3 cannot be improved. With regard to boundary limit questions, it is interesting to notice that even in the case $N = 1$ a function with the representation (2.1) may have no radial limits.

The function $\exp [1/(1 - z)]$ is an example of an analytic function which has no terminating series representation. We now turn to the extremal problem mentioned in the introduction.

4. An extremal representation

The object of discussion in this section is a harmonic function f which has a representation

$$(4.1) \qquad f(r, \theta) = \sum_{n=0}^{N} \int_\Gamma P_r^{(n)}(\theta - t)\, d\mu_n(t)$$

where $\mu_0, \cdots, \mu_N$ are Radon measures on $\Gamma = \{|z| = 1\}$. There seems to be no useful extremal representation for all such f. However, if there is an extremal of the type to be described below, it is unique and takes a rather special form.

Let $\mathfrak{M}$ be the space of Radon measures on Γ and denote by $\mu = (\mu_0, \cdots, \mu_N)$ an arbitrary element of the $(N + 1)$-fold direct product $\mathfrak{M} \times \cdots \times \mathfrak{M}$. A linear mapping T of this space into the space of harmonic functions is defined by (4.1). For fixed N and certain harmonic functions f, the set $G = \{\mu : f = T\mu\}$ will have an extremal which minimizes $\| \mu_N \|$. An element $\mu \in G$ is extremal in this sense if and only if μ_N is singular with respect to Lebesque measure. The measure μ_N is the same for all extremal μ.

Among these extremal elements it may be possible to minimize $\| \mu_{N-1} \|$ and again μ_{N-1} is singular and uniquely determined. One thus obtains a decreasing sequence of extremal classes. If the process continues to include a minimal $\| \mu_1 \|$, then the only arbitrary term remaining in the extremal elements μ is the measure μ_0. But then μ_0 is the unique measure appearing in the Poisson-Stieltjes representation of a fixed harmonic function.

Roughly speaking, if there is a $\mu \in G$ in which the mass of its terms has been shifted to the left as far as possible, then there is no more than one and $\mu_1, \cdots, \mu_N$ are singular measures. The latter property for μ insures that it is the desired extremal element of G.

In proving these assertions we note first that the norm of a nonsingular measure can always be reduced by an integration by parts to the left. For suppose $n \geqq 1$ and $d\mu_n = f\,dt + d\mu_s$ where $\int f\,dt$ is the absolutely continuous part of μ_n and μ_s is the singular part. Then $\| \mu_n \| = \int_0^{2\pi} |f|\,dt + \| \mu_s \|$. It is possible to choose a periodic function $g \in C^1$ such that $\int_0^{2\pi} |f - g|\,dt < \int_0^{2\pi} |f|\,dt$. Then

$$\begin{aligned}\int_\Gamma P_r^{(n)}(\theta - t)\,d\mu_n(t) &= \int_0^{2\pi} P_r^{(n)}(\theta - t)[f(t) - g(t)]\,dt \\ &\quad + \int_\Gamma P_r^{(n)}(\theta - t)\,d\mu_s(t) + \int_0^{2\pi} P_r^{(n)}(\theta - t)g(t)\,dt \\ &= \int_\Gamma P_r^{(n)}(\theta - t)\,d\nu_n(t) + \int_0^{2\pi} P_r^{(n-1)}(\theta - t)\,dg(t)\end{aligned}$$

where $\| \nu_n \| = \int_0^{2\pi} |f - g|\,dt + \| \mu_s \| < \| \mu_n \|$. Thus the extremal μ_N mentioned above, if it exists, is singular.

PROPOSITION 2. *Let* $\inf_{\mu \in G} \| \mu_N \| = b$. *Suppose there is a constant* B *and a sequence* $\{\mu^k\} \subset G$ *such that* $\| \mu_n^k \| \leqq B$, $n = 0, 1, \cdots, N - 1$, $k = 1, 2, \cdots$, *and* $\| \mu_N^k \| \to b$. *Then there exists* $\mu \in G$ *such that* $\| \mu_N \| = b$.

Proof. The set $\{\nu \in \mathfrak{M} : \| \nu \| \leqq B\}$ is compact in the w^*-topology. Choosing a subsequence if necessary we may suppose that $\mu_n^k \to \mu_n \in \mathfrak{M}$ in the w^*-topology for $n = 0, \cdots, N$. If $\mu = (\mu_0, \cdots, \mu_N)$ it follows that $f = T\mu$ and $\mu \in G$. Since the norm $\| \cdot \|$ is lower semicontinuous on $\mathfrak{M}$ relative to the w^*-topology, $\| \mu_N \| \leqq \lim \| \mu_N^k \| = b$ and consequently $\| \mu_N \| = b$. This completes the proof.

That μ_N singular implies $\| \mu_N \|$ is minimal and that there is a unique singular μ_N is a consequence of

THEOREM 4. *If* $T\mu = 0$, *then* μ_N *is absolutely continuous.*

Proof. Using the notation of Section 2 and taking the $(N + 1)$-st integral, $0 = F_{0,N+1} + \cdots + F_{N,N+1}$. Letting $r \nearrow 1$ and applying (2.7), $0 = \beta_{0,N+1} + \cdots + \beta_{N,N+1}$. Recalling that $\beta_{n,k}$ is absolutely continuous for $n + 2 \leqq k$, we see that $\beta_{N,N+1}$ is absolutely continuous.

Continuing to use the formulation in Section 2 let α_N be the normalized function of bounded variation associated with the measure μ_N. Then

$$\beta_{N,N+1}(t) = \beta_{N,N}(t) - \frac{1}{2\pi}\int_0^{2\pi} \beta_{N,N}(\tau)\,d\tau$$

$$= \alpha_N(t) - \frac{\alpha_N(2\pi)}{2\pi}t - \frac{1}{2\pi}\int_0^{2\pi} \beta_{N,N}(\tau)\,d\tau$$

from which the absolute continuity of α_N is apparent. This is the assertion of the theorem.

To verify the first statement preceding the theorem suppose that $\mu, \nu \in G$ and μ_N is singular. Then $T(\mu - \nu) = 0$ and $\mu_N - \nu_N$ is absolutely continuous. Hence μ_N is the singular part of ν_N and consequently $\| \mu_N \| \leqq \| \nu_N \|$. As for the second assertion, if μ_N and ν_N are both singular then $\mu_N - \nu_N$ is both singular and absolutely continuous which implies $\mu_N - \nu_N = 0$.

The remaining statements in the introduction to this section are immediate consequences of what we have proved.

If $\mu = (\mu_0, \mu_1, \cdots)$ is an infinite sequence in $\mathfrak{M}$ satisfying

$$\lim\, (n!\, \| \mu_n \|)^{1/n} = 0$$

and the measures $\mu_1, \mu_2, \cdots$ are singular then it is not possible to reduce the mass of any term in favor of lower order terms in the representation

$$f(r, \theta) = \sum_{n=0}^{\infty} \int_\Gamma P_r^{(n)}(\theta - t)\, d\mu_n(t).$$

It should be possible to show that μ is the only sequence with singular terms which yields the function f but a proof has eluded the author.

5. An isomorphic representation space for the analytic functions

The results of Section 4 have indicated to some extent the character of the multiplicity of the representation in special cases. The object of this section is to provide a general setting in which one may consider the problem for all cases.

One of the elegant theories related to the Poisson integral concerns the Hardy H^p speces. Denoting $f_r(\theta) = f(r, \theta)$, H^p is the space of analytic functions f for which the L^p norms $\| f_r \|_p$ are uniformly bounded for $0 \leqq r < 1$. If $1 \leqq p \leqq \infty$, f is the Poisson integral of a boundary function $\tilde{f} \in L^p(\Gamma)$. In the case $p = 1$ this is an important theorem of F. and M. Riesz. (See [2].) With the norm $\| f \| = \lim_{r \nearrow 1} \| f_r \|_p = \| \tilde{f} \|_p$, H^p is a Banach space isomorphic to a closed subspace of $L^p(\Gamma)$ which is also denoted H^p. The subspace is characterized by the condition $\int_0^{2\pi} f(\theta) e^{in\theta}\, d\theta = 0$, $n = 1, 2, \cdots$.

The space $\mathfrak{F}_a$ of all analytic functions on $\{|z| < 1\}$ with the topology of subuniform convergence (uniform convergence on compact subsets) is a Frechet space; that is, a metrizable complete locally convex space. An isomorphism will be established with a certain space of sequences in H^p considered as the space of boundary functions $\tilde{f}$ described above.

Let $\mathcal{G}^p$ denote the linear subspace of the countable complete direct product $\times H^p$ consisting of sequences $g = (g_0, g_1, \cdots)$ satisfying

$$\lim (n! \, \| g_n \|_p)^{1/n} = 0. \tag{5.1}$$

We define the translation invariant metric

$$| g - h | = \sup_{n \geqq 0} (n! \, \| g_n - h_n \|_p)^{1/(n+1)} \tag{5.2}$$

on $\mathcal{G}^p$. Providing $\mathcal{G}^p$ with the metric topology we have

PROPOSITION 3. *$\mathcal{G}^p$ is a Frechet space.*

Proof. The continuity of the mapping $(g, h) \to g + h$ of $\mathcal{G}^p \times \mathcal{G}^p$ into $\mathcal{G}^p$ is an immediate consequence of the metric properties. We must prove also that the mapping $(\lambda, g) \to \lambda g$ of $C \times \mathcal{G}^p$ into $\mathcal{G}^p$, where C denotes the complex plane, is continuous.

The inequality $| g - \overset{0}{g} | \leqq \varepsilon$ implies $n! \, \| g_n - \overset{0}{g}_n \|_p \leqq \varepsilon^{n+1}$ for all n. From (5.1) we obtain

$$\begin{aligned} n! \, \| \overset{0}{g}_n \|_p &\leqq a^{n+1} \quad \text{for all } n \\ &\leqq \varepsilon^{n+1} \quad \text{for } n \geqq n(\varepsilon) \end{aligned}$$

where a is a positive constant. Making use of

$$\| \lambda g_n - \lambda_0 \overset{0}{g}_n \|_p \leqq | \lambda | \, \| g_n - \overset{0}{g}_n \|_p + | \lambda - \lambda_0 | \, \| \overset{0}{g}_n \|_p$$

one can show that

$$n! \, \| \lambda g_n - \lambda_0 \overset{0}{g}_n \|_p \leqq (2 + | \lambda_0 |) \varepsilon^{n+1}$$

for $| \lambda - \lambda_0 |$ sufficiently small and for $n \geqq 0$. Thus the image of a neighborhood $\{(\lambda, g) : | \lambda - \lambda_0 | \leqq \eta, | g - \overset{0}{g} | \leqq \varepsilon\}$ is contained in

$$\{\lambda g : | \lambda g - \lambda_0 \overset{0}{g} | \leqq (2 + | \lambda_0 |) \varepsilon\}.$$

Therefore $\mathcal{G}^p$ is a linear topological space. That it is locally convex follows from the convexity of $\{g : | g | \leqq \varepsilon\}$ for each $\varepsilon > 0$. It remains to be proved that $\mathcal{G}^p$ is complete. It is sufficient to prove that $\mathcal{G}^p$ is sequentially complete relative to the metric. This follows from a standard type of argument using (5.1) and (5.2).

Consider the linear mapping $f = Tg$ defined by

$$f(r, \theta) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} P_r^{(n)}(\theta - t) g_n(t) \, dt \tag{5.3}$$

of $\mathcal{G}^p$ onto $\mathfrak{F}_a$. T is continuous. For if $\rho < 1$ and $a > 0$ is chosen, as in (2.5) of [3], such that $| P_r^{(n)}(\theta - t) | \leqq n! \, a^n$ on $\{r \leqq \rho\}$, then

$$| f(r, \theta) | \leqq \sum_{n=0}^{\infty} n! \, a^n \, \| g_n \|_p \leqq \sum_{n=0}^{\infty} (a \, | g |)^{n+1} = \frac{a \, | g |}{1 - a \, | g |}$$

on $\{r \leqq \rho\}$ for $|g| < 1/a$. That is, the image of $\{g \in \mathcal{G}^p : |g| \leqq \varepsilon/a\}$ is contained in the neighborhood $\{f \in \mathfrak{F}_a : |f(r, \theta)| \leqq \varepsilon \text{ for } r \leqq \rho\}$.

The null space $T^{-1}0$ is a closed linear subspace of $\mathcal{G}^p$ and the quotient space $\mathcal{H}^p = \mathcal{G}^p/T^{-1}0$ is a Frechet space. If Q denotes the quotient mapping of $\mathcal{G}^p$ onto $\mathcal{H}^p$, then $|h| = \inf\{|g| : h = Qg\}$ defines an invariant metric on $\mathcal{H}^p$ which determines its topology.

There is a continuous isomorphism U of $\mathcal{H}^p$ onto $\mathfrak{F}_a$ such that $T = U \circ Q$. By the theorem on inverse maps U is a toplogical isomorphism. (See [4, p. 98]) The author is indebted to M. G. Arsove for suggesting the applicability of this theorem. We have therefore established

THEOREM 6. *The spaces $\mathcal{H}^p$ and $\mathfrak{F}_a$ are linearly and topologically isomorphic.*

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