FULL REFLECTIVE SUBCATEGORIES AND GENERALIZED COVERING SPACES

BY

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Introduction

Let $\mathfrak{A}$ be a full subcategory of $\mathfrak{C}$. A morphism $e : X \to X^*$ is said to be a *reflection map* which reflects X into the category $\mathfrak{A}$ iff $X^* \in \mathfrak{A}$ and every morphism $f : X \to A$ with $A \in \mathfrak{A}$ can be factored as $f = ge$ for a unique $g : X^* \to A$. The full subcategory $\mathfrak{A} \subseteq \mathfrak{C}$ is *reflective* if every object $X \in \mathfrak{C}$ admits a reflection map $e : X \to X^*$. Dually $\mathfrak{A} \subseteq \mathfrak{C}$ is *coreflective* if each $X \in \mathfrak{C}$ admits a *coreflection map* $e : X^* \to X$ such that $X^* \in \mathfrak{A}$ and each morphism $f : A \to X$ with $A \in \mathfrak{A}$ factors as $f = eg$ for a unique g.

In this paper we obtain necessary and sufficient conditions for a full subcategory to be reflective. Our methods also yield some information about the reflection maps and the full reflective subcategory generated by certain types of subcategories. Applying the dual of these results to the category of pointed topological spaces (i.e. topological spaces with base points), we show that the simply connected spaces in the sense of Hu [4] generate an interesting coreflective subcategory. The coreflection maps, $p : (X^*, x^*) \to (X, x)$ can be regarded as generalized universal coverings since p is the usual universal covering if such a covering exists. In general X^* is simply connected in the sense of Chevalley [2] and p is a fiber map with pathwise totally disconnected fibers. (An example shows that the fibers are not always discrete.) X^* appears to be related to the *universal procovering* of X obtained by Lubkin [11] for certain spaces. Our coreflection enables us to extend some of the Lubkin and Chevalley theory of covering spaces to the category of **all** topological spaces.

As is the case with many conditions for reflectivity our conditions are closely related to the Freyd adjoint functor theorem [3 p. 84] even though a knowledge of that theorem is not a prerequisite for reading this paper. In effect, we show that for full reflective subcategories, the "solution set" hypothesis of the adjoint functor theorem can always be satisfied in a convenient, canonical way which leads to simplifications; these simplifications are illustrated by the well known type of theorem that for a well-behaved category $\mathfrak{C}$, a full subcategory $\mathfrak{A} \subseteq \mathfrak{C}$ is reflective if $\mathfrak{A}$ is closed under the formation of products and subobjects (e.g. see [3, p. 87]). Isbell in [5] and [6] demonstrates that the notion of "subobject" is often best defined in the context of a bicategory structure (defined below). In [5, p. 1276], it is shown that if $\mathfrak{C}$ is well-behaved and has a suitable bicategory structure, then $\mathfrak{A} \subseteq \mathfrak{C}$ is reflective whenever $\mathfrak{A}$ is closed under the formation of products and subobjects in the bicategory sense. These condi-

Received July 30, 1966.

tions, moreover, imply that $\mathcal{A}$ is *epi-reflective* by which we mean a full reflective subcategory all of whose associated reflection maps are epimorphisms. If $\mathcal{C}$ is co-well-powered (which is defined below and in [3]), then Isbell's result can generally be used to obtain necessary and sufficient conditions for $\mathcal{A} \subseteq \mathcal{C}$ to be epi-reflective. By constructing a suitable bicategory structure, we show that this result can in effect be extended to obtain necessary and sufficient conditions for reflectivity in general. This bicategory structure is also used to investigate the reflection maps and the smallest reflective subcategory containing a category.

We must mention our indebtedness to Maranda's paper [13] from which we have borrowed an important definition. We are also indebted to the referee for suggesting much needed improvements in the organization of this paper.

1. Right and left bicategory structures

For technical reasons we need to investigate a generalization of Isbell's notion of bicategory. First we shall establish the following notation.

Notation. In general, our terminology is based on Freyd [3]. We shall also use the following special conventions:

(1) The class of all epimorphisms of $\mathcal{C}$ shall be denoted by $E_{\mathcal{C}}$ or simply E if there is no danger of confusion. Similarly $M_{\mathcal{C}}$ or M shall denote the class of monomorphisms of $\mathcal{C}$.

(2) The statement "X is an object of $\mathcal{C}$" is sometimes denoted by "$X \in \mathcal{C}$." Similarly "$\mathcal{B} \subseteq \mathcal{C}$" indicates that $\mathcal{B}$ is a full subcategory of $\mathcal{C}$.

The following definition is equivalent to Isbell's.

DEFINITION Let $\mathcal{C}$ be a category. Let I and P be classes of morphisms on $\mathcal{C}$. Then (I, P) is a *bicategory structure* on $\mathcal{C}$ provided that:

B_0. Every equivalence is in $I \cap P$.

B_1. I and P are closed under the composition of morphisms.

B_2. Every morphism f can be factored as $f = f_1 f_0$ with $f_1 \in I$ and $f_0 \in P$. Moreover this factorization is unique to within an equivalence in the sense that if $f = gh$ and $g \in I$ and $h \in P$ then there exists an equivalence e for which $ef_0 = h$ and $ge = f_1$.

B_3. $P \subseteq E$.

B_4. $I \subseteq M$.

We shall consider the following generalization:

DEFINITION. (I, P) is a *right bicategory structure* on $\mathcal{C}$ if it satisfies B_0, B_1, B_2 and B_3.

Dually, (I, P) is a *left bicategory structure* on $\mathcal{C}$ if it satisfies B_0, B_1, B_2 and B_4.

DEFINITION. Let $\mathcal{C}$ be a category and let I and P be classes of morphisms of $\mathcal{C}$. Let $X \in \mathcal{C}$. Then A is an *I-subobject* of X if $I \cap \operatorname{Hom}(A, X) \neq \emptyset$.

The category $\mathcal{C}$ is *I-well-powered* if each $X \in \mathcal{C}$ has a representative set of I-subobjects (such that every I-subobject of X is equivalent to a member of the representative set).

The terms *P-quotient* and *P-co-well-powered* are defined dually.

Note that *well-powered* as defined in [3] means the same thing as *M-well-powered* and, dually, *co-well-powered* is synonomous with *E-co-well-powered.*

DEFINITION. A bicategory structure (I, P) on $\mathcal{C}$ is *well-founded* if $\mathcal{C}$ is I-well-powered and P-co-well-powered.

Notation. (1) In the context of a right (or left) bicategory structure (I, P) on $\mathcal{C}$, the notation "$f = f_1 f_0$" shall always be understood to indicate the factorization mentioned in B_2 (that is, for which $f_1 \in I$ and $f_0 \in P$).

(2) Let $\{f : X_i \to Y_i\}$ be an indexed set of morphisms of a category $\mathcal{C}$ which has products. Then $\prod \{f_i\}$ denotes the morphism $f : \prod \{X_i\} \to \prod \{Y_i\}$ for which $\bar{p}_i f = f_i p_i$ for all i (where p_i and $\bar{p}_i$ are projection maps). If $\mathcal{C}$ has coproducts then $\sum \{f_i\}$ is defined dually.

The following proposition shows that many of the properties of bicategory structures readily generalize to right bicategory structures.

PROPOSITION 1.1. *Let (I, P) be a right bicategory structure on $\mathcal{C}$. Then:*

(1) *$f \in I$ iff f_0 is an equivalence and $f \in P$ iff f_1 is an equivalence. Thus $I \cap P$ is precisely the class of all equivalences.*

(2) *$fe \in I$ implies $e \in I$.*

(3) *$gh \in P$ and $h \in E$ implies $g \in P$.*

(4) *I is uniquely determined by P. In fact*

$$I = \{g \mid g = fe \quad \text{and} \quad e \in P \quad \text{imply} \quad e \quad \text{is an equivalence}\}.$$

(5) *P is uniquely determined by I. In fact*

$$P = \{f \mid f = gh, \quad h \in E \quad \text{and} \quad g \in I \quad \text{imply} \quad g \quad \text{is an equivalence}\}.$$

(6) *Every commutative diagram of the form indicated by the figure can be filled in at r with commutativity preserved, provided $f \in P$ and $g \in I$.*

$$\begin{tikzcd}
{} \arrow[r, "f \in P"] \arrow[d, "m"'] & {} \arrow[d, "n"] \arrow[dl, dashed, "r"'] \\
{} \arrow[r, "g \in I"'] & {}
\end{tikzcd}$$

(7) *$f_i \in P$ for all i implies $\sum \{f_i\} \in P$ (if $\mathcal{C}$ has coproducts).*

(8) *$f_i \in I$ for all i implies $\prod \{f_i\} \in I$ (if $\mathcal{C}$ has coproducts).*

(9) *Let e, f, g and h form a pushout diagram with $fe = hg$. Then $e \in P$ implies $h \in P$.*

Proof. (1) If f_0 is an equivalence then $f_0 \in I$ and so $f = f_1 f_0$ is in I as I is composition closed. Conversely if $f \in I$ then f_0 is equivalence in view of B_2 and the factorizations $f = f_1 f_0 = f \cdot 1$. Similarly $f \in P$ iff f_1 is an equivalence.

(2) The factorization $fe = (fe_1)_1(fe_1)_0 e_0$ implies $(fe_1)_0 e_0$ is an equivalence. Since $e_0 \in E$ it readily follows that e_0 is also an equivalence. Thus $e \in I$.

(3) Apply B_2 to the factorization $1 \cdot (gh) = g_1(g_0 h)_1(g_0 h)_0$. Thus there exists an equivalence e for which $e(gh) = (g_0 h)_0$ and $g_1(g_0 h)_1 e = 1$. We claim $(g_0h)_1 eg_1 = 1$ thus proving $(g_0 h)_1 e$ is the inverse of g_1 and so g_1 is an equivalence. This last equation follows since $(g_0 h)_1 eg_1(g_0 h) = g_0 h$ and $g_0 h \in E$.

(4) and (5) are now obvious.

As for (6), observe that $n_1(n_0 f) = (gm_1)m_0$. Let e be the equivalence mentioned in B_2. Choose $r = m_1 en_0$.

As for (7), let $f = \sum \{f_i\}$ and let c and $\bar{c}_i$ be the coprojections such that $fc_i = \bar{c}_i f_i$ for all i. In view of (6) there exists r_i such that $r_i f_i = f_0 c_i$ and $f_1 r_i = \bar{c}_i$ for each i. Let r be determined by $r\bar{c}_i = r_i$. Then $f_1 r\bar{c}_i = \bar{c}_i$ and so $f_1 r$ is an identity morphism. Similarly $(rf_1)(f_0 c_i) = f_0 c_i$ for all i, which implies $rf_1 f_0 = f_0$ and so rf_1 is also an identity as $f_0 \in E$. Thus f_1 is an equivalence and $f = f_1 f_0 \in P$. An analogous proof works for (8).

As for (9), observe that, in view of (6), there exists r such that $re = h_0 g$ and $h_1 r = f$. Since e, f, g and h form a pushout diagram, there exists s such that $sf = r$ and $sh = h_0$. It is readily verified that $s = h_1^{-1}$ (since sh_1 is an identity as $sh_1 h_0 = h_0$ and $h_1 s$ is an identity as $h_1 sh = h$ and $h_1 sf = f$). Thus h_1 is an equivalence and so $h = h_1 h_0 \in P$.

Definition. A subcategory $\mathcal{B} \subseteq \mathcal{C}$ is *replete* if $B \in \mathcal{B}$ and X equivalent to B imply $X \in \mathcal{B}$.

Definition. Let P be a class of morphisms of $\mathcal{C}$. Then $\mathcal{B} \subseteq \mathcal{C}$ is *P-reflective* if $\mathcal{B}$ is replete, reflective and if every reflection map is in P.

Theorem 1.2 (Freyd-Isbell). *Let $\mathcal{C}$ be a category with products. Let (I, P) be a right bicategory structure on $\mathcal{C}$ such that $\mathcal{C}$ is P-co-well-powered.*

Then $\mathcal{A} \subseteq \mathcal{C}$ is P-reflective iff $\mathcal{A}$ is closed under the formation of products and I-subobjects.

Proof. Assume that $\mathcal{A}$ is closed under the formation of products and I-subjects. $\mathcal{A}$ is then P-reflective in view of Isbell's proof of Freyd's theorem, given on p. 1276 of [5] and sketched here. Let $X \in \mathcal{C}$ be arbitrary. Let $\{f_\alpha : X \to A_\alpha\}$ be a representative set of morphisms for which $A_\alpha \in \mathcal{A}$ and $f_\alpha \in P$. Let $f : X \to \prod \{A_\alpha\}$ be determined by $p_\alpha f = f_\alpha$ for all α. Factor $f = f_1 f_0$. Then $f_0 \in P$ is a reflection map reflecting X into $\mathcal{A}$. Finally $\mathcal{A}$ is replete since every equivalence is in I and $\mathcal{A}$ is closed under the formation of I-subobjects.

Conversely assume that $\mathcal{A}$ is P-reflective. It is well known that $\mathcal{A}$ is then closed under the formation of products. As for I-subobjects, assume $A \in \mathcal{A}$ and that $f : X \to \mathcal{A}$ is in I. Let $e : X \to X^*$ reflect X into $\mathcal{A}$. Then there exists $g : X^* \to A$ such that $f = ge$. Since $f \in I$ it follows that $e \in I$ by (2) of proposition 1.1. But $e \in P$ as $\mathcal{A}$ is P-reflective. Thus e is an equivalence and $X \in \mathcal{A}$ as $\mathcal{A}$ is replete.

COROLLARY. *Let $\mathcal{C}$ be a category with products and let (I, P) be a right bicategory structure on $\mathcal{C}$ such that $\mathcal{C}$ is P-co-well-powered. Let $\mathcal{A}$ be a full subcategory which is closed under the formation of products. Let $\mathcal{B}$ be the full subcategory of all I-subobjects of members of $\mathcal{A}$. Then $\mathcal{B}$ is the smallest P-reflective subcategory containing $\mathcal{A}$.*

Proof. Since I is composition closed, $\mathcal{B}$ is closed under the formation of I-subobjects. Moreover, $\mathcal{B}$ is also closed under the formation of products in view of (8) of Proposition 1.1.

PROPOSITION 1.3. *Let $\mathcal{C}$ be a category with pushouts and coproducts. Let P be a class of epimorphisms of $\mathcal{C}$ for which $\mathcal{C}$ is P-co-well-powered. Then there is a unique I for which (I, P) is a right bicategory structure on $\mathcal{C}$ if the following conditions are satisfied:*

(1) *P contains all equivalences.*

(2) *P is composition closed.*

(3) *$e_i \in P$ for all i implies $\sum \{e_i\} \in P$.*

(4) *If e, f, g and h form a pushout diagram with $fe = hg$ and if $e \in P$ then $h \in P$.*

Proof. The necessity of (1)–(4) follows from the definition and from Proposition 1.1. The uniqueness of I follows from (4) of proposition 1.1. As for the existence of I, assume that (1)–(4) are satisfied. Define

$$I = \{g \mid g = fe \quad \text{and} \quad e \in P \quad \text{imply} \quad e \quad \text{is an equivalence}\}.$$

We first claim that I is composition closed. To this end assume that $g \in I$, $h \in I$ and that $hg = fe$ with $e \in P$. It suffices to show that e is an equivalence. Construct morphisms m and n so that e, g, m and n form a pushout diagram with $me = ng$. Since $fe = hg$ there exists a unique morphism r for which $rm = f$ and $rn = h$. But $n \in P$ by (4). Thus n is an equivalence as $h \in I$. This implies $g = n^{-1}me$ and so e is an equivalence as $g \in I$.

It is readily shown that all equivalences are in I since $P \subseteq E$. Thus (I, P) satisfies B_0, B_1 and B_3. It remains to show that B_2 is also satisfied. To this end let $f\,X\colon \to Y$ be a given morphism. Let $\{e_i\}$ be a representative subset of the class of all morphisms in P, with domain X, which are right factors of f. (In other words, each e_i is such a right factor which means there exists a morphism g_i for which $g_i e_i = f$. Moreover if $e \in P$ is any other right factor of f then e is equivalent to at least one e_i in the sense that there is an equivalence h for which $e = he_i$.) The existence of such a representative set, $\{e_i\}$, follows from the hypothesis that $\mathcal{C}$ is P-co-well-powered.

Let $e = \sum \{e_i\}$. Thus there exist coprojections c_i and $\bar{c}_i$ for which

$ec_i = \bar{c}_i e_i$. By (3), $e \in P$. Let g be the morphism determined by the equations $g\bar{c}_i = g_i$ and let d be determined by $dc_i = 1_X$.

Let s and t be morphisms such that e, d, s and t form a pushout diagram with $td = se$. Since $fd = ge$ (as $fdc_i = gec_i$ for all i) there exists a morphism r such that $f = rt$ and $g = rs$.

We claim that $f = rt$ is the desired factorization of f. First note that $t \in P$ by (4). To show that $r \in I$, assume $r = uv$ with $v \in P$. Then $f = uvt$ and so $vt \in P$ is a right factor of f. Hence without loss of generality we may as well assume $vt = e_i$ and hence $u = g_i$ for some i. Then $s\bar{c}_i vt = s\bar{c}_i e_i = t$. Since $t \in E$ we conclude that $s\bar{c}_i v = 1$, a suitable identity morphism. Thus v is an equivalence as $v \in E$. We have proven that $r \in I$.

Finally we claim that such factorizations are unique to within an equivalence. Assume $f = f_1 f_0 = gh$ with $f_1, g \in I$ and $f_0, h \in P$. Obtain s and t so that f_0, h, s and t form a pushout diagram with $sf_0 = th$. Since $f_1 f_0 = gh$ there exists r such that $rs = f_1$ and $rt = g$. By (4) we see that $s, t \in P$ and by definition of I we see that s and t are equivalences. It follows that $e = t^{-1}s$ is the desired equivalence for which $ef_0 = h$ and $ge = f_1$.

2. Conditions equivalent to reflectivity

In this section we obtain a sort of two-stage converse to the Freyd-Isbell theorem. That is given a well-behaved category $\mathcal{C}$ we prove that $\mathcal{A} \subseteq \mathcal{C}$ is reflective if there exists $\mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ such that $\mathcal{A}$ is reflective in $\mathcal{B}$ and $\mathcal{B}$ is reflective in $\mathcal{C}$ by virtue of the Freyd-Isbell theorem. This result is stated in the form of a set of conditions on $\mathcal{A}$ which are necessary and sufficient for reflectivity. For convenience we have only considered the case in which $\mathcal{A}$ is replete. (The extension to the general case is trivial.) Our approach depends on the following definition which is due to Maranda [13].

Definition. Let $\mathcal{A} \subseteq \mathcal{C}$ be given. A morphism $f : X \to Y$ is *injective* with respect to $\mathcal{A}$ if for all morphisms $g : X \to A$ with $A \in \mathcal{A}$ there exists at least one $h : Y \to A$ for which $hf = g$.

The class of all morphisms of $\mathcal{C}$ which are injective with respect to $\mathcal{A}$ is denoted by $\Psi_{\mathcal{C}}(\mathcal{A})$ or simply $\Psi(\mathcal{A})$ if there is no danger of confusion.

Dually $f : X \to Y$ is *projective* with respect to $\mathcal{A}$ if for all $g : A \to Y$ with $A \in \mathcal{A}$ there exists $h : A \to X$ such that $fh = g$. The class of all morphisms of $\mathcal{C}$ which are projective with respect to $\mathcal{A}$ is denoted by $\Omega_{\mathcal{C}}(\mathcal{A})$ or simply $\Omega(\mathcal{A})$.

Lemma 2.1. *Let (I_0, P_0) be a well-founded bicategory structure on the complete category $\mathcal{C}$. Let $\mathcal{A}$ be a full subcategory closed under the formation of products. Let $\mathcal{B}$ be the full subcategory of all I_0-subobjects of members of $\mathcal{A}$. Let $P = \Psi_{\mathcal{B}}(\mathcal{A}) \cap E_{\mathcal{B}}$. It then follows that*:

(1) $P \subseteq I_0$.

(2) *If $\mathcal{B}$ is P-co-well-powered there exists a unique I such that (I, P) is a right bicategory structure on $\mathcal{B}$.*

Proof. (1) *Let* $f: X \to Y$ be a member of P. Then $X \in \mathcal{B}$, as $f \in \Psi_{\mathcal{B}}(\mathcal{A})$, and so there exists $g: X \to A$ with $g \in I_0$ and $A \in \mathcal{A}$. Since f is injective with respect to $\mathcal{A}$ there exists h for which $hf = g$. Thus $f \in I_0$ by (2) of Proposition 1.1.

As for (2), observe that $\mathcal{B}$ is a reflective subcategory of $\mathcal{C}$ by the corollary to Theorem 1.2. It follows that $\mathcal{B}$ is complete and hence Proposition 1.3 can be applied to $\mathcal{B}$. A straightforward verification shows that P satisfies (1)–(4) of Proposition 1.3.

THEOREM 2.2. *Let $\mathcal{C}$ be a complete category and let (I_0, P_0) be a well-founded bicategory structure on $\mathcal{C}$. Let $\mathcal{A}$ be a replete full subcategory of $\mathcal{C}$.*

$\mathcal{A}$ is then a reflective subcategory of $\mathcal{C}$ iff the following three conditions are satisfied:

(1) *$\mathcal{A}$ is closed under the formation of products.*

(2) *$\mathcal{B}$ is P-co-well-powered (where $\mathcal{B}$ and P are defined as in the above Lemma).*

(3) *$\mathcal{A}$ is closed under the formation of $(I_0 \cap I)$-subobjects (where I is the class of morphisms for which (I, P) is a right bicategory structure on B; the existence of I is guaranteed by the above conditions and the above Lemma).*

Proof. Assume that $\mathcal{A}$ is reflective. Then it is well known that (1) must hold. We claim that $\mathcal{A}$ is P-reflective in $\mathcal{B}$. To prove this choose $B \in \mathcal{B}$ and let $e : B \to A$ be the reflection map. It is clear that $e \in \Psi_{\mathcal{B}}(\mathcal{A})$, hence it remains to prove that $e \in E_{\mathcal{B}}$. Assume that $fe = ge$ where $f, g : A \to B'$ and where $B' \in \mathcal{B}$. By definition of $\mathcal{B}$ there exists $h : B' \to A'$ with $A' \in \mathcal{A}$ and $h \in I_0$. Thus $hfe = hge$. But e is a reflection map and $A' \in \mathcal{A}$ hence $hf = hg$. This implies $f = g$ as $h \in I_0 \subseteq M$. Thus $e \in E_{\mathcal{B}}$ and so $e \in P$ showing that $\mathcal{A}$ is P-reflective in $\mathcal{B}$.

It remains to prove (2) for then $\mathcal{A}$ is closed under the formation of I-subobjects (by theorem 1.2) and (3) follows. Choose $B \in \mathcal{B}$ and let $f : B \to B'$ be in P. Let $e : B \to A$ be the reflection map which reflects B into $\mathcal{A}$. It suffices to show that B' is an I_0-subobject of A for then every P-quotient of B is an I_0-subobject of A and A has a representative set of I_0-subobjects. Since $f \in \Psi(\mathcal{A})$ there is an $h : B' \to A$ with $hf = e$. Thus $h \in E_{\mathcal{B}}$ as $e \in E_{\mathcal{B}}$. Moreover it can readily be shown that $h \in \Psi(\mathcal{A})$ using the facts that $e \in \Psi(A)$ and $f \in E_{\mathcal{B}}$. Thus $h \in P \subseteq I_0$ and so B' is an I_0-subobject of A.

Conversely assume (1), (2) and (3) hold. Then by (1) and the corollary to the Freyd-Isbell theorem, $\mathcal{B}$ is reflective in $\mathcal{C}$. It thus suffices to prove that $\mathcal{A}$ is reflective in $\mathcal{B}$. By the Freyd-Isbell theorem it suffices to show that $\mathcal{A}$ is closed under the formation of I-subobjects. To prove this let $f : X \to A$ be given with $f \in I$ and $A \in \mathcal{A}$. Then $X \in \mathcal{B}$ (as I is a class of $\mathcal{B}$ morphisms) and so there exists $g : X \to A'$ with $g \in I_0$ and $A' \in \mathcal{A}$. Consider $h : X \to A \times A'$ for which $p_1 h = f$ and $p_2 h = g$. Then by (2) of Proposition 1.1, we see that $h \in I$, as $p_1 h \in I$, and $h \in I_0$ as $p_2 h \in I_0$. Hence $h \in I \cap I_0$ and $X \in \mathcal{A}$ by (1) and (3).

Remark. The notion of a reflection can be thought of as a generalization of Birkhoff's notion of a "completion process," given in [1]. It is of interest that Birkhoff showed that each completion process determines a closely related notion of "closed subsystem". Among other things closed subsystems of complete systems are then complete.

If we translate [1] into category theory, the completions become examples of reflections and the closed subsystems correspond to $(I_0 \cap I)$-subobjects.

3. Generation of reflective subcategories

Given $\mathfrak{A} \subseteq \mathfrak{C}$, under what conditions does there exist a smallest replete, reflective subcategory $\mathfrak{A}^* \subseteq \mathfrak{C}$ for which $\mathfrak{A} \subseteq \mathfrak{A}^*$? The answer to this question is not known. Even if $\mathfrak{C}$ is the category of uniform spaces and uniform maps, it is an open question as to whether every $\mathfrak{A} \subseteq \mathfrak{C}$ admits such an $\mathfrak{A}^*$; see [8, p. 33]. In this section we shall obtain conditions on $\mathfrak{A}$ and $\mathfrak{C}$ under which such an $\mathfrak{A}^*$ can be constructed.

THEOREM 3.1. *Let $\mathfrak{C}$ be complete and let (I_0, P_0) be a well-founded bicategory structure on $\mathfrak{C}$. Let $\mathfrak{A} \subseteq \mathfrak{C}$ be replete and closed under the formation of products. Let $\mathfrak{B} \subseteq \mathfrak{C}$ be the subcategory consisting of I_0-subobjects of members of $\mathfrak{A}$. Let $P = \Psi_{\mathfrak{B}}(\mathfrak{A}) \cap E_{\mathfrak{B}}$.*

Assume that $\mathfrak{B}$ is P-co-well-powered. Let I be such that (I, P) is a right bicategory structure on $\mathfrak{B}$ (see Lemma 2.1). Then $\mathfrak{A}^*$, the full subcategory of all I-subobjects of members of $\mathfrak{A}$, is the smallest replete reflective subcategory for which $\mathfrak{A} \subseteq \mathfrak{A}^*$.

Proof. $\mathfrak{A} \subseteq \mathfrak{A}^*$ is clear and $\mathfrak{A}^*$ is reflective in $\mathfrak{B}$ by the corollary to Theorem 1.2. Since $\mathfrak{B}$ is reflective in $\mathfrak{C}$, by the same corollary, it follows that $\mathfrak{A}^*$ is reflective in $\mathfrak{C}$.

Next, let us assume $\mathfrak{D} \subseteq \mathfrak{C}$ is replete and reflective with $\mathfrak{A} \subseteq \mathfrak{D}$. To prove $\mathfrak{A}^* \subseteq \mathfrak{D}$ let $X \in \mathfrak{A}^*$ be given. Then there exists $g : X \to A$ with $g \in I$ and $A \in \mathfrak{A}$. Let $h : X \to D$ reflect X into $\mathfrak{D}$. It suffices to prove that h is an equivalence.

Since $A \in \mathfrak{A} \subseteq \mathfrak{D}$ there exists $d : D \to A$ such that $dh = g$. Let $e : D \to B$ reflect D into $\mathfrak{B}$. Then $e \in P_0$ by the corollary to Theorem 1.2. Moreover, since $A \in \mathfrak{A} \subseteq \mathfrak{B}$ there exists $f : B \to A$ such that $fe = d$. Hence $feh = g$. Thus $eh \in I$ by (2) of Proposition 1.1.

We claim that $h \in E_{\mathfrak{D} \cup \mathfrak{B}}$. Assume $rh = sh$ where $r, s : D \to Y$ and $Y \in \mathfrak{D} \cup \mathfrak{B}$. If $Y \in \mathfrak{D}$ then $r = s$ as h is a reflection map. If $Y \in \mathfrak{B}$ there exists $t : Y \to A'$ with $t \in I_0$ and $A' \in \mathfrak{A}$. Thus $trh = tsh$. But $A' \in \mathfrak{A} \subseteq \mathfrak{D}$ and h is a reflection map so $tr = ts$. Hence $r = s$ as $t \in I_0 \subseteq M_{\mathfrak{C}}$.

It is easily proven that $eh \in \Psi_{\mathfrak{B}}(\mathfrak{A})$ as e and h are reflection maps and $\mathfrak{A} \subseteq \mathfrak{D} \cap \mathfrak{B}$. Since $e \in P_0 \subseteq E_{\mathfrak{C}}$ and $h \in E_{\mathfrak{D} \cup \mathfrak{B}}$ we see that $eh \in E_{\mathfrak{B}}$ and so $eh \in P$. But $eh \in I$ has been shown above, hence eh is an equivalence. It now follows that h is an equivalence with $h^{-1} = (eh)^{-1}e$. (The fact that $h(eh)^{-1}e = 1$ follows from $h(eh)^{-1}eh = h$ and $h \in E_{\mathfrak{D} \cup \mathfrak{B}}$.)

The following characterizations of $\mathfrak{A}^*$ are used in our example concerning simply connected spaces.

PROPOSITION 3.2. (We use the notation and hypothesis of the previous theorem.) *Let $X \in \mathfrak{B}$ be given. Then the following statements are equivalent:*

(1) $X \in \mathfrak{A}^*$.

(2) *$f : X \to Y$ and $f \in P$ imply f is an equivalence.*

(3) *$f : X \to Y$ and $Y \in \mathfrak{B}$ imply $f \in I$.*

Proof. (1) $\Rightarrow$ (2). Assume $f : X \to Y$ is in P and that $X \in \mathfrak{A}^*$ which means there exists $g : X \to A$ for which $g \in I$ and $A \in \mathfrak{A}$. Since $f \in P \subseteq \Psi(\mathfrak{A})$ there exists $h : Y \to A$ with $hf = g$. This implies $f \in I$ by (2) of Proposition 1.1 and so $f \in I \cap P$ implying f is an equivalence.

(2) $\Rightarrow$ (3). Let $f : X \to Y$ be given with $Y \in \mathfrak{B}$. Factor $f = f_1 f_0$, with $f_1 \in I$ and $f_0 \in P$. Then f_0 is an equivalence by (2), so $f \in I$.

(3) $\Rightarrow$ (1). Since $X \in \mathfrak{B}$ and since $\mathfrak{A}^*$ is a P-reflective subcategory of $\mathfrak{B}$, there exists $e : X \to A$ for which $e \in P$ and $A \in \mathfrak{A}^*$. By (3) we see $e \in I$ and so e is an equivalence and $X \in \mathfrak{A}^*$.

Since our main example concerns coreflectivity, it is convenient to summarize the dual of our results:

THEOREM 3.3 (Dual of previous results). *Let $\mathfrak{C}$ be a complete category with a well-founded bicategory structure, (I_0, P_0). Let $\mathfrak{A} \subseteq \mathfrak{C}$ be closed under the formation of coproducts. Let $\mathfrak{B}$ be the full subcategory of P_0-quotients of members of $\mathfrak{A}$. Let $I = \Omega_{\mathfrak{B}}(\mathfrak{A}) \cap M_{\mathfrak{B}}$. Let $\mathfrak{A}^*$ be the full subcategory consisting of all $X \in \mathfrak{B}$ such that $f : Y \to X$ and $f \in I$ imply f is an equivalence. Then:*

(1) *$\mathfrak{A}$ is coreflective in $\mathfrak{C}$ iff $\mathfrak{B}$ is I-well-powered and $\mathfrak{A} = \mathfrak{A}^*$.*

(2) *If $\mathfrak{B}$ is I-well-powered then $\mathfrak{A}^*$ is the smallest replete coreflective subcategory which contains $\mathfrak{A}$.*

4. Simply connected spaces, an example

It is well known that certain reasonably well-behaved spaces have universal simply connected covering spaces (e.g. see [4, p. 93 ff.]). This implies that for certain categories of spaces with base points, the simply connected objects form a coreflective subcategory. In this section we shall apply Theorem 3.3 to the simply connected spaces. As a result we shall show that every pointed topological space has a generalized universal simply connected covering space. For pathological spaces, however, "simple connectedness" may have to be interpreted in the sense of Chevalley, [2, p. 44], and fiber maps with totally pathwise disconnected fibers will, when necessary, be used in place of genuine covering maps. Nonetheless, this generalized universal "covering" has the usual universal factoring properties and coincides with the standard universal covering if such a covering exists.

To be formal, we shall let $\mathfrak{C}$ be the category of all pointed topological spaces. $\mathfrak{A}$ shall be the full subcategory of all coproducts of simply connected and locally pathwise connected members of $\mathcal{C}$. (The term "simply connected" is being used in the sense of Hu [4, p. 42].)

Let (I_0, P_0) be the bicategory structure for which P_0 is the class of onto morphisms of $\mathcal{C}$ and I_0 is the class of into-homeomorphisms. Then (I_0, P_0) is well-founded.

We shall let $\mathcal{B}$ be the full subcategory of all P_0-quotients (i.e. images) of members of $\mathcal{A}$. It is easy to prove that $\mathcal{B}$ is then the full subcategory of pathwise connected spaces. As shown in the last example of [9], $\mathcal{B}$ is well-powered and hence I-well-powered where $I = \Omega_{\mathcal{B}}(\mathcal{A}) \cap M_{\mathcal{B}}$.

Thus (2) of Theorem 3.3 is applicable and there exists a smallest coreflective subcategory $\mathcal{A}^*$ for which $\mathcal{A} \subseteq \mathcal{A}^*$. We now seek topological descriptions of I and $\mathcal{A}^*$. We first note that by the dual of Lemma 2.1 there exists P such that (I, P) is a left bicategory structure on $\mathcal{B}$.

We claim that every $f \in I$ is a fiber map with totally pathwise disconnected fibers. To prove this let S be a simplex and let $g : S \to S \times [0, 1]$ be continuous. Then $g \in P$ by the dual of (3) of Proposition 3.2 as $S \times [0, 1] \in \mathcal{A}^*$. Hence any commutative diagram of the form indicated by the figure can be filled in at r with commutativity preserved (in view of (6) of Proposition 1.1, which is self-dual).

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "g"'] & {} \arrow[d, "f \in I"] \\
S \times [0,1] \arrow[r] \arrow[ur, dashed, "r"] & {}
\end{tikzcd}$$

This clearly implies that f is a fiber map (see [4, p. 63]). Moreover f has totally pathwise disconnected fibers since $f \in M_{\mathcal{B}}$.

We do not know if every fiber map with totally pathwise disconnected fibers is in I, but every covering in the sense of Hu is in I (Theorem 16.2 on p. 89 of [4]).

We note that $\mathcal{A}$ is contained in the coreflective subcategory of all locally pathwise connected spaces (this subcategory is coreflective even if the base-points are ignored, see Theorem A of [10]). It follows that every member of $\mathcal{A}^*$ is locally pathwise connected.

By Theorem 3.3, a space X is in $\mathcal{A}^*$ iff X admits no non-trivial fiberings by members of I. Thus if $X \in \mathcal{A}^*$ then X is simply connected in the sense of Chevalley (i.e. X has no non-trivial coverings since every such covering is in I. Observe that the Chevalley covering maps coincide with the Hu covering maps as X is locally pathwise connected. Unlike [2], we do *not* require that all spaces considered be T_2.).

We have shown that there exists $\mathcal{A}^*$, a class of spaces which are "simply-connected" in a sense between Chevalley and Hu. Each space X, with basepoint, admits a coreflection map $p : X^* \to X$ where $X^* \in \mathfrak{A}^*$ and where p is a fiber map with totally pathwise disconnected fibers.

Remarks. (1) Let $\mathfrak{D}$ be the largest full subcategory of $\mathfrak{B}$ for which $I \subseteq \Omega_{\mathfrak{B}}(\mathfrak{D})$ (i.e. $D \in \mathfrak{D}$ iff $I \subseteq \Omega_{\mathfrak{B}}(\{D\})$). Then $\mathfrak{D}$ is easily shown to be I-coreflective in $\mathfrak{B}$ by the dual of the Freyd-Isbell theorem. (To show $\mathfrak{D}$ is closed under P-quotients requires the use of (6) of Proposition 1.1.) Thus $\mathfrak{A}^* \subseteq \mathfrak{D}$ and so $I \subseteq \Omega_{\mathfrak{B}}(\mathfrak{A}^*)$.

It follows that the coreflection map $p : X^* \to X$ coreflecting X into $\mathfrak{A}^*$ is a "universal I-map." For if $f : Y \to X$ is in I then there exists $g : X^* \to Y$ for which $p = fg$ as $f \in I \subseteq \Omega_{\mathfrak{B}}(\mathfrak{A}^*)$ and $X^* \in \mathfrak{A}^*$. Moreover $g \in I$ by the dual of (3) of Proposition 1.1.

(2) When dealing with pathwise connected spaces it is often permissible to ignore the base point. In general the base point can be freely moved along a path in view of the following results:

LEMMA 4.1. *Let*

$$f : (Y, y_0) \to (X, x_0)$$

be in I. Assume that $h : [0, 1] \to X$ is continuous with $h(0) = x_0$ and $h(1) = x_1$. Since $f \in I$ there exists a unique $\bar{h} : [0, 1] \to Y$ such that $\bar{h}(0) = y_0$ and $f\bar{h} = h$. Let $y_1 = \bar{h}(1)$. Then

$$f : (Y, y_1) \to (X, x_1)$$

is also in I.

Proof. To show that $f : (Y, y_1) \to (X, x_1)$ is in $\Omega(\mathfrak{A})$, let

$$g : (A, a_1) \to (X, x_1)$$

be a morphism with $(A, a_1) \in \mathfrak{A}$. We may as well assume $A \cap [0, 1] = \emptyset$. Let $A \oplus [0, 1]$ be the space obtained from $A \cup [0, 1]$ by identifying a_1 with 1. Define

$$g \oplus h : (A \oplus [0, 1], 0) \to (X, x_0)$$

in the obvious way. Since $f : (Y, y_0) \to (X, x_0)$ is in I there exists m for which $fm = g \oplus h$. Clearly $m_{|[0,1]} = \bar{h}$ and so $m(a_1) = \bar{h}(1) = y$. Let $\bar{g} = m_{|A}$. Then $g = f\bar{g}$ and so $f : (Y, y_1) \to (X, x_1)$ is in $\Omega(\mathfrak{A})$. The lemma now follows easily.

COROLLARY. *Let x_0 and x_1 be in the same path component of X. Then $(X, x_0) \in \mathfrak{A}^*$ iff $(X, x_1) \in \mathfrak{A}^*$. In general the coreflection of (X, x_0) is the same as the coreflection of (X, x_1) except for a change of base point.*

Proof. Follows readily by applying the dual of (2) of Proposition 3.2.

COROLLARY. *Let X be pathwise connected and let $p : X^* \to X$ be the coreflection map which coreflects X into $\mathfrak{A}^*$. (Since X is pathwise connected we may* *suppress the base point.)* *Suppose that* $p(x_0) = p(x_1)$. *Then there exist as unique automorphism* $g : X^* \rightarrow X^*$ *for which* $g(x_0) = x_1$ *and* $pg = p$.

Proof. Let $x = p(x_0) = p(x_1)$. Then

$$p : (X^*, x_0) \rightarrow (X, x) \quad \text{and} \quad p : (X^*, x_1) \rightarrow (X, x)$$

are both coreflections of (X, x) and hence there exists a unique

$$g : (X, x_0) \rightarrow (X, x_1)$$

for which $pg = p$. There also exists $h : (X, x_1) \rightarrow (X, x_0)$ with $ph = p$ and it is easily shown that h is the inverse of g.

(3) Consider the space $X = R \times R \backslash \{(2^{-n}, 0)\}$. This is the space of Example 9 on p. 232 of [11]. Let $p : X^* \rightarrow X$ be the universal pro-covering of X, which is defined on p. 228 of [11]. It can be shown that X^* is simply connected in the sense of Hu and that $p \in I$, hence $p : X^* \rightarrow X$ is the coreflection of X into $\mathfrak{A}^*$. It can also be verified that $p^{-1}(0, 0)$ is not discrete, hence the coreflection maps are not always covering maps. I do not know how closely the coreflection is related to the universal pro-covering which Lubkin defined for semi-simple spaces (see [11]).

(4) The above techniques can be applied to other notions of simple connectedness. For example it can readily be shown that the class of all pathwise connected spaces which are simply connected in the sense of Chevalley form a coreflective subcategory. For by Theorem 3.3 the pathwise connected and Chevalley simply connected spaces generate a coreflective subcategory and by the above arguments every member of this coreflective subcategory is pathwise connected and Chevalley simply connected.

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