THE TOPOLOGY OF CONTACT RIEMANNIAN MANIFOLDS

BY

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1. Introduction

New development in the study of contact manifolds was first given by W. M. Boothby and H. C. Wang [5], and J. W. Gray [8]. J. W. Gray defined an almost contact manifold by the condition that the structural group of the tangent bundle is reducible to $U(n) \times 1$. Later S. Sasaki [15] characterized an almost contact manifold by the existence of three tensor fields satisfying some relations, and introduced the Riemannian metric which has the natural property with respect to the almost contact structure. By these four tensor fields the study of contact manifolds came to the stage where tensor calculus is a powerful and prominent method. Two special contact Riemannian manifolds are K-contact Riemannian manifolds and Sasakian manifolds. A Sasakian manifold can be considered as an odd-dimensional analogue of a Kählerian manifold. The second Betti numbers of compact Kählerian manifolds were studied by M. Berger [1], R. L. Bishop and S. I. Goldberg [2] and others. It is natural to do research on these problems in compact Sasakian manifolds, and in fact, they were studied by S. I. Goldberg [6], [7], E. M. Moskal [13], [14], S. Tachibana [16], S. Tachibana and Y. Ogawa [17] and S. Tanno [20], etc.

In [20] we have used $(m - 1)$-homothetic deformations to get results on the first Betti numbers. We call these deformations D-homothetic deformations, where D denotes the distribution defined by a contact form η. To get results on the second Betti numbers and harmonic forms, we also utilize a D-homothetic deformation

$$g \to {}^*g = \alpha g + (\alpha^2 - \alpha)\eta \otimes \eta \tag{1.1}$$

of the associated Riemannian metric g for a positive constant α. If (ϕ, ξ, η, g) is a Sasakian structure with a contact form η, then $({}^*\phi = \phi, {}^*\xi = \alpha^{-1}\xi, {}^*\eta = \alpha\eta, {}^*g)$ is also a Sasakian structure. By studying the relations of harmonic forms with respect to g and ${}^*g$, we get

THEOREM. *A compact m-dimensional (K-contact Riemannian or) Sasakian manifold M with sectional curvature* $> -3/(m-2)$ *has the first Betti number* $b_1(M) = 0$. *If* $m = 3$, *we also have* $b_2(M) = 0$.

A harmonic 2-form w of the hybrid type (pure type, resp.) is defined similarly to the Kählerian case.

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The following theorem is originally due to S. Tachibana and Y. Ogawa [17], and E. M. Moskal [13], [14].

THEOREM. *If $m \geq 5$, a compact Sasakian manifold M with sectional curvature $> -3/(m - 2)$ has no harmonic 2-form of the pure type. And if the sectional curvature is >0, then there is no harmonic 2-form of the hybrid type. Especially then, we have $b_2(M) = 0$.*

We denote by $K(X, Y)$ the sectional curvature for the 2-plane determined by X and Y. As is well known in Kählerian manifolds holomorphic pinchings were studied by several authors. In Sasakian manifolds, we define certain pinching for ϕ-holomorphic sectional curvatures. Let

$$H = \sup \{K(X, \phi X); \quad X \in D_x, \quad x \in M\},$$

$$L = \inf \{K(X, \phi X); \quad X \in D_x, \quad x \in M\},$$

and assume that they exist and $H \neq -3$. Then the quantity

$$\mu = (L + 3)/(H + 3)$$

is an invariant of the D-homothety class of M. μ gives the degree of deviation from constancy of ϕ-holomorphic sectional curvature. When $H + 3 > 0$, we say that M is μ-holomorphically pinched.

THEOREM. *If a compact Sasakian manifold M is μ-holomorphically pinched with $\mu > \frac{1}{2}$, then $b_2(M) = 0$.*

THEOREM. *If a Sasakian manifold M is μ-holomorphically pinched with $\mu > \frac{2}{3}$, then the metric g is D-homothetically deformable to $^*g$ so that M is of strictly positive curvature with respect to $^*g$.*

THEOREM. *If a Sasakian manifold M is μ-holomorphically pinched with $\mu \geq \frac{4}{5}$, then the metric g is D-homothetically deformable to $^*g$ so that M is of Riemannian pinching $\geq \frac{1}{4}$. Thus, furthermore, if M is complete and simply connected, then M is homeomorphic with a sphere.*

From these theorems we can derive applications. For example, we have

THEOREM. *If a compact, simply connected Sasakian manifold M is μ-holomorphically pinched with $\mu > \frac{1}{2}$, and if the scalar curvature is constant, then M is globally D-homothetic with the unit sphere.*

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2. D-homothetic deformations

Let M be an m-dimensional contact Riemannian manifold, $m = 2n + 1$, with the structure tensors ϕ, ξ, η, and g which satisfy

$$\phi\xi = 0, \qquad \eta(\xi) = 1, \tag{2.1}$$

$$\phi\phi X = -X + \eta(X)\xi, \tag{2.2}$$

(2.3) $$g(X, \xi) = \eta(X),$$

(2.4) $$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\cdot\eta(Y),$$

(2.5) $$d\eta(X, Y) = 2g(X, \phi Y) = 2\phi'(X, Y)$$

for any vector fields X and Y on M ([15]). When ξ is a Killing vector field with respect to g, M is said to be a K-contact Riemannian manifold, and we have

(2.6) $$(\nabla_X \eta)(Y) = \phi'(X, Y),$$

(2.7) $$R_1(X, \xi) = (m-1)\eta(X),$$

(2.8) $$g(R(X, \xi)Y, \xi) = g(X, Y) - \eta(X)\cdot\eta(Y),$$

where ∇ is the covariant differentiation with respect to g, R_1 and R are the Ricci curvature tensor and Riemannian curvature tensor respectively [9], [18].

A Sasakian manifold is characterized by

$$[X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] + \{Y\cdot\eta(X) - X\cdot\eta(Y)\}\,\xi = 0$$

and we have

(2.9) $$\nabla_k \phi_j^i = g_{jk}\xi^i - \eta_j \delta_k^i,$$

(2.10) $$\eta_i R^i_{jkl} = g_{jk}\eta_l - g_{jl}\eta_k .$$

A Sasakian manifold is necessarily a K-contact Riemannian manifold. By the equation $\eta = 0$ we define the $(m-1)$-dimensional distribution D on M, and we define an $(m-1)$-homothetic deformation, that is, a D-homothetic deformation $g \to {}^*g$, by

(2.11) $${}^*g_{jk} = \alpha g_{jk} + \beta\eta_j\eta_k$$

for constant α and β satisfying $\alpha > 0$ and $\alpha + \beta > 0$. The inverse matrix $({}^*g^{ij})$ of $({}^*g_{jk})$ is given by

(2.12) $${}^*g^{ij} = \alpha^{-1}g^{ij} - \alpha^{-1}\beta(\alpha+\beta)^{-1}\xi^i\xi^j.$$

Denoting by W^i_{jk} the difference ${}^*\Gamma^i_{jk} - \Gamma^i_{jk}$ of Christoffel symbols, we have in a contact manifold

(2.13) $$W^i_{jk} = -\alpha^{-1}\beta(\phi^i_j\eta_k + \eta_j\phi^i_k) + 2^{-1}\beta(\alpha+\beta)^{-1}\xi^i(\nabla_j\eta_k + \nabla_k\eta_j)$$

which follows from (4.6) of [19] putting $\theta = 0$, $\nabla_j \eta_i \xi^j = 0$, etc. From now on we assume that M is a K-contact Riemannian manifold, then we have

(2.13)′ $$W^i_{jk} = -\alpha^{-1}\beta(\phi^i_j\eta_k + \eta_j\phi^i_k).$$

Putting this into

$${}^*R^i_{jkl} = R^i_{jkl} + \nabla_l W^i_{jk} - \nabla_k W^i_{jl} + W^i_{rl}W^r_{jk} - W^i_{rk}W^r_{jl},$$

we have

$$(2.14)\quad \begin{aligned} {}^*R^i_{jkl} = R^i_{jkl} &+ \alpha^{-1}\beta(2\phi^i_j\phi_{kl} + \phi^i_k\phi_{jl} - \phi^i_l\phi_{jk}) \\ &+ \alpha^{-1}\beta(\nabla_k\phi^i_j\eta_l + \nabla_k\phi^i_l\eta_j - \nabla_l\phi^i_j\eta_k - \nabla_l\phi^i_k\eta_j) \\ &+ \alpha^{-2}\beta^2(\delta^i_l\eta_j\eta_k - \delta^i_k\eta_j\eta_l). \end{aligned}$$

Contracting with respect to i and l, we have

$$(2.15)\quad {}^*R_{jk} = R_{jk} - 2\alpha^{-1}\beta g_{jk} + \alpha^{-2}\beta(2m\alpha + m\beta - \beta)\eta_j\eta_k,$$

where we have used $\nabla_r\phi^r_j = -(m-1)\eta_j$, etc. Contracting the last equation with (2.12), we get

$$(2.16)\quad {}^*S = \alpha^{-1}S - \alpha^{-2}\beta(m-1),$$

where S is the scalar curvature, and we have used (2.7).

LEMMA 2.1. *For a contact Riemannian manifold M with structure tensors ϕ, ξ, η, g, we put ${}^*\phi = \phi$, ${}^*\xi = \alpha^{-1}\xi$, ${}^*\eta = \alpha\eta$, and ${}^*g = \alpha g + (\alpha^2 - \alpha)\eta \otimes \eta$ for a positive constant α. Then $({}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$ is a contact metric structure too. If (ϕ, ξ, η, g) is a K-contact Riemannian structure (Sasakian structure, resp.), then $({}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$ is also a K-contact Riemannian structure (Sasakian structure resp.).*

Proof. If ξ is a Killing vector field with respect to g, then ${}^*\xi$ is also a Killing vector field with respect to ${}^*g$, since ξ leaves η invariant. For a Sasakian structure it is clear from the definition.

3. Harmonic 1-forms on a compact K-contact Riemannian manifold

We study the relations of harmonic 1-forms with respect to g and ${}^*g$ in this section. We note here that any contact Riemannian manifold is orientable.

First we have the following lemmas.

LEMMA 3.1 (S. Tanno [20]). *For a harmonic 1-form w on a compact K-contact Riemannian manifold, we have*

$$(3.1)\quad \xi^i w_i = 0.$$

LEMMA 3.2. *A harmonic 1-form w with respect to g on a compact K-contact Riemannian manifold is also harmonic with respect to ${}^*g$.*

Proof. Since $dw = 0$ and $\delta w = -g^{ij}\nabla_j w_i = 0$, we prove ${}^*\delta w = 0$. By definition of ${}^*\delta$ we have

$$(3.2)\quad \begin{aligned} {}^*\delta w &= -{}^*g^{ij}\,{}^*\nabla_j w_i \\ &= -{}^*g^{ij}(\nabla_j w_i - W^r_{ij} w_r). \end{aligned}$$

Then by (2.12) and (2.13)′, and Lemma 3.1, we have ${}^*\delta w = 0$, since for example

$$\xi^i\xi^j(\nabla_j w_i) = \nabla_j(\xi^i\xi^j w_i) - \nabla_j \xi^i\xi^j w_i - \xi^i\nabla_j \xi^j w_i .$$

LEMMA 3.3 (Cf. K. Yano and S. Bochner [22]). *In a compact orientable Riemannian manifold, there exists no harmonic* 1*-form* w *which satisfies*

$$\int_M R_{jk}\, w^j w^k \, d\sigma \geq 0$$

unless $\nabla w = 0$ *and then* $R_{jk}\, w^j w^k = 0$, *where* $d\sigma$ *is the volume element with respect to* g.

Now we have

THEOREM 3.4. *On a compact K-contact Riemannian manifold M there exists no harmonic* 1*-form* w *which satisfies*

$$R_{jk}\, w^j w^k + 2g_{jk}\, w^j w^k \geq 0 \tag{3.3}$$

for every point of M and which has at least one point where inequality holds. Especially, if $R_1 + 2g$ *is positive definite, then the first Betti number of M,* $b_1(M)$, *is equal to* 0.

Proof. Assume that there is a harmonic 1-form w for which (3.3) holds, then we have

$$\int_M [R_1(w, w) + 2g(w, w)]\, d\sigma > 0. \tag{3.4}$$

As M is compact $g(w, w)$ is bounded and we have some positive number ε such that

$$\int_M [R_1(w, w) + (2 - \varepsilon)g(w, w)]\, d\sigma > 0.$$

On the other hand by Lemma 3.1, (2.12) and (2.15) we have

$$\begin{aligned} {}^*R_{jk}\, {}^*w^j\, {}^*w^k &= \alpha^{-2}\, {}^*R_{jk}\, w^j w^k \\ &= \alpha^{-2}\, (R_{jk}\, w^j w^k - 2\alpha^{-1}\beta g_{jk}\, w^j w^k). \end{aligned} \tag{3.5}$$

Since $\alpha^{-1}\beta = \alpha - 1 \to -1$ as $\alpha \to 0$, we can choose α so that $-2\alpha^{-1}\beta > 2 - \varepsilon$, that is $2\alpha < \varepsilon$. Then we have

$$\int {}^*R_1(w, w)\, d^*\sigma = \alpha^{(m+1)/2} \int {}^*R_1(w, w)\, d\sigma > 0,$$

where we have used the relation $d^*\sigma = \alpha^{(m+1)/2}\, d\sigma$ which is seen from (10.1) in [19]. The last inequality contradicts the fact that w is a harmonic 1-form with respect to g and hence ${}^*g$.

THEOREM 3.5. *Let M be a compact K-contact Riemannian manifold with sectional curvature $>-3/(m-2)$. Then we have $b_1(M) = 0$. If $m = 3$, we also have $b_2(M) = 0$.*

Proof. We take an orthonormal basis $(\xi, e_1, \cdots, e_{2n})$ for which R_1 is diagonal. If sectional curvature $>-K$ for some positive constant K, we have

$$R_{ii} > 1 - (m-2)K \qquad \text{for } i = 1, \cdots, 2n.$$

Then we have $R_1(w, w) + 2g(w, w) > (3 - (m-2)K)g(w, w)$ for any harmonic 1-form w, completing the proof.

4. Harmonic 1-forms on compact Sasakian manifolds

In a contact Riemannian manifold, an orthonormal frame

$$(\xi, e_\lambda, e_{\lambda^*} = \phi e_\lambda), \qquad \lambda = 1, \cdots, n,$$

is called a ϕ-basis. By $K(X, Y)$ we mean the sectional curvature for the 2-plane determined by X and Y.

In a Sasakian manifold we have (for example [16])

$$R_{ir}\phi_j^r = -R_{jr}\phi_i^r. \tag{4.1}$$

This relation implies that for an eigen-vector X of R_1, ϕX is also an eigen-vector. Thus we have a ϕ-basis for which only $R_{00} = R_1(\xi, \xi) = m - 1$, $R_{\lambda\lambda}$, $R_{\lambda^*\lambda^*}$ may be non-vanishing components of R_1. Now we put $K_{0\lambda} = K(\xi, e_\lambda)$, $K_{0\lambda^*} = K(\xi, e_{\lambda^*})$, $K_{\lambda\mu} = K(e_\lambda, e_\mu) = R_{\lambda\mu\mu\lambda}$, $\lambda \neq \mu$, $K_{\lambda\lambda} = K_{\lambda^*\lambda^*} = 0$, etc.; then we have

$$K_{0\lambda} = K_{0\lambda^*} = 1, \qquad \lambda = 1, \cdots, n,$$

$$R_{\lambda\lambda} = 1 + \textstyle\sum_\mu (K_{\mu\lambda} + K_{\mu^*\lambda}), \tag{4.2}$$

$$R_{\lambda^*\lambda^*} = 1 + \textstyle\sum_\mu (K_{\mu\lambda^*} + K_{\mu^*\lambda^*}). \tag{4.3}$$

On the other hand we know that (see [17])

$$K_{\lambda^*\mu^*} = K_{\lambda\mu}, \qquad K_{\lambda\mu^*} = K_{\lambda^*\mu}. \tag{4.4}$$

Therefore if we assume that $K_{\lambda\mu} + K_{\lambda\mu^*} > -(2 - \delta_{\lambda\mu})K$ for some positive constant K, then we get

$$R_1(w, w) + 2g(w, w) \geq (3 - (2n-1)K)g(w, w) \tag{4.5}$$

for any harmonic 1-form w. By Theorem 3.4 we have

THEOREM 4.1. *Let M be a compact Sasakian manifold. If the sectional curvatures satisfy*

$$K_{\lambda\mu} + K_{\lambda\mu^*} > -3(2 - \delta_{\lambda\mu})/(m-2)$$

$$(\textit{or especially} \quad \textstyle\sum_\mu (K_{\lambda\mu} + K_{\lambda\mu^*}) > -3),$$

then $b_1(M) = 0$.

Remark. As a corollary of this theorem we have the result corresponding to Theorem 3.5 in a Sasakian manifold.

5. Harmonic 2-forms on compact Sasakian manifolds

To begin with we state the following

PROPOSITION 5.1. *If a compact Sasakian manifold M is of dimension 3 and if the sectional curvature is > -3, then we have $b_2(M) = 0$.*

This is contained in Theorem 3.5.

From now on in this section we assume that $m \geq 5$.

LEMMA 5.2 (S. Tachibana [16]). *For a harmonic p-form w, $p \leq n$, on a compact Sasakian manifold we have*

$$\xi^i w_{ii_2\cdots i_p} = 0, \tag{5.1}$$

$$\phi^{ij} w_{iji_3\cdots i_p} = 0. \tag{5.2}$$

LEMMA 5.3. *If w is a harmonic p-form with respect to g on a compact Sasakian manifold, then it is also harmonic with respect to ${}^*g$.*

Proof. First we assume that w is a harmonic p-form with $p \leq n$. Since $dw = 0$ and $\delta w = 0$, we show ${}^*\delta w = 0$. By definition we have

$${}^*g^{ij}\,{}^*\nabla_j w_{ii_2\cdots i_p} = {}^*g^{ij}(\nabla_j w_{ii_2\cdots i_p} - W^r_{ij} w_{ri_2\cdots i_p} - \sum W^r_{i_a j} w_{ii_2\cdots r\cdots i_p}).$$

By (2.12), (2.13)′ and Lemma 5.2, we have ${}^*\delta w = 0$. The difference between proofs of Lemma 3.2 and Lemma 5.3 is the use of the relation (5.2). If the degree of w is greater than n, we take its adjoint with respect to g and ${}^*g$. It is verified that one differs from the other by a constant factor which is a rational power of α.

LEMMA 5.4 (K. Yano and S. Bochner [22]). *In a compact Riemannian manifold, there exists no harmonic p-form w which satisfies*

$$F_p(w) \geq 0,$$

unless $\nabla w = 0$ and then $F_p(w) = 0$, where

$$F_p(w) = R_{ij} w^{ii_2\cdots i_p} w^j{}_{i_2\cdots i_p} + 2^{-1}(p-1) R_{ijkl} w^{iji_3\cdots i_p} w^{kl}{}_{i_3\cdots i_p}. \tag{5.3}$$

By (2.12) and (2.14), we have

$${}^*R_{ijkl} = \alpha R_{ijkl} + \beta(2\phi_{ij}\phi_{kl} + \phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk}) + [*], \tag{5.4}$$

where [*] is the term which contains η. Thus by (2.12), (2.15), (5.4) and Lemma 5.2, we have for a harmonic form w

$$\begin{aligned} {}^*F_p(w) &= (R_{ij} - 2\alpha^{-1}\beta g_{ij})\alpha^{-(p+1)} w^{ii_2\cdots i_p} w^j{}_{i_2\cdots i_p} \\ &\quad + 2^{-1}(p-1)[\alpha R_{ijkl} + \beta(2\phi_{ij}\phi_{kl} + \phi_{ik}\phi_{jl} \\ &\quad - \phi_{il}\phi_{jk})]\alpha^{-(p+2)} w^{iji_3\cdots i_p} w^{kl}{}_{i_3\cdots i_p} \\ &= \alpha^{-(p+1)} F_p(w) - \alpha^{-(p+2)}\beta[2w^{i_1\cdots i_p} w_{i_1\cdots i_p} \\ &\quad - (p-1)\phi^i_k \phi^j_l w^{kli_3\cdots i_p} w_{iji_3\cdots i_p}]. \end{aligned} \tag{5.5}$$

Let w be a harmonic 2-form with respect to g; we decompose w as follows [13], [17]:

$$w = \overset{1}{w} + \overset{2}{w}, \tag{5.6}$$

where

$$\overset{1}{w}(X, Y) = 2^{-1}[w(X, Y) + w(\phi X, \phi Y)],$$
$$\overset{2}{w}(X, Y) = 2^{-1}[w(X, Y) - w(\phi X, \phi Y)].$$

Then $\overset{1}{w}$ and $\overset{2}{w}$ are harmonic and

$$\overset{1}{w}(\phi X, \phi Y) = \overset{1}{w}(X, Y), \tag{5.7}$$

$$\overset{2}{w}(\phi X, \phi Y) = -\overset{2}{w}(X, Y). \tag{5.8}$$

A harmonic form w is said to be of the hybrid type if it satisfies (5.7) and of the pure type if it satisfies (5.8) respectively.

First we consider a harmonic 2-form w of the hybrid type.

LEMMA 5.5 (S. Tachibana and Y. Ogawa [17]). *Let M be a Sasakian manifold and w be a harmonic 2-form of the hybrid type with respect to g. Then there is a ϕ-basis for which only $w_{\lambda\lambda^*}$ may be non-zero components of w and*

$$F_2(w) = \textstyle\sum_{\lambda\neq\mu} (K_{\lambda\mu} + K_{\lambda\mu^*})(w_{\lambda\lambda^*} - w_{\mu\mu^*})^2 - 2 \sum (w_{\lambda\lambda^*})^2. \tag{5.9}$$

Since the condition (5.2) is written as $\sum w_{\lambda\lambda^*} = 0$, we have

$$\textstyle\sum_{\lambda\neq\mu} (w_{\lambda\lambda^*} - w_{\mu\mu^*})^2 = 2n \sum(w_{\lambda\lambda^*})^2. \tag{5.10}$$

We denote by (w, w) the (local) inner product:

$$(w, w) = (1/p!)w^{i_1\cdots i_p}w_{i_1\cdots i_p} .$$

LEMMA 5.6. *Assume that $K_{\lambda\mu} + K_{\lambda\mu^*} > 2 K$, $\lambda \neq \mu$, for some positive constant K; then we have*

$$^*F_2(w) \geq 2\alpha^{-3}(2n K - \alpha)(w, w). \tag{5.11}$$

Proof. We take a ϕ-basis stated in Lemma 5.5. Then by (5.5), (5.7) and (5.10) we have

$$\begin{aligned} ^*F_2(w) &= \alpha^{-3}[\textstyle\sum (K_{\lambda\mu} + K_{\lambda\mu^*})(w_{\lambda\lambda^*} - w_{\mu\mu^*})^2 - 2 \sum (w_{\lambda\lambda^*})^2] \\ &\quad - \alpha^{-4}\beta[4 \textstyle\sum (w_{\lambda\lambda^*})^2 - 2 \sum (w_{\lambda\lambda^*})^2] \\ &\geq 2\alpha^{-3}K \textstyle\sum (w_{\lambda\lambda^*} - w_{\mu\mu^*})^2 - 2(\alpha^{-4}\beta + \alpha^{-3}) \sum (w_{\lambda\lambda^*})^2 \\ &\geq [4nK\alpha^{-3} - 2\alpha^{-3}(\alpha^{-1}\beta + 1)] \textstyle\sum (w_{\lambda\lambda^*})^2. \end{aligned}$$

Thus we have (5.11).

THEOREM 5.7. *A compact Sasakian manifold M, $m \geq 5$, which satisfies*

$$K_{\lambda\mu} + K_{\lambda\mu^*} > 2K, \qquad \lambda \neq \mu$$

for some positive constant K has no harmonic 2-form of the hybrid type. Especially if the sectional curvature is strictly positive, then M has no harmonic 2-form of the hybrid type.

Proof. If we take a positive constant α so that $\alpha < 2n\, K$, then we have $^*F_2(w) \geq 0$ for any harmonic 2-form of the hybrid type by Lemma 5.6. Therefore we have $^*F_2(w) = 0$ and by (5.11) we get $w = 0$.

Next we consider a harmonic 2-form of the pure type.

LEMMA 5.8 (E. M. Moskal [13]). *For a ϕ-basis for which R_{00}, $R_{\lambda\lambda}$ and $R_{\lambda^*\lambda^*}$ may be only non-vanishing components of R_1, a harmonic 2-form w of the pure type satisfies*

$$F_2(w) = \textstyle\sum_{\lambda,\mu} (\sum_\nu (K_{\lambda\nu} + K_{\lambda\nu^*}))[(w_{\lambda\mu})^2 + (w_{\lambda\mu^*})^2 + (w_{\lambda^*\mu})^2 + (w_{\lambda^*\mu^*})^2]. \tag{5.12}$$

LEMMA 5.9. *Assume that*

$$K_{\lambda\mu} + K_{\lambda\mu^*} > -(2 - \delta_{\lambda\mu})K'$$

for some positive constant K'; then a harmonic 2-form w of the pure type satisfies

$$^*F_2(w) \geq 2\alpha^{-3}[-\, 3\alpha^{-1}\beta - (2n - 1)K'] \;(w, w). \tag{5.13}$$

Proof. First we have

$$\textstyle\sum_\mu (K_{\lambda\mu} + K_{\lambda\mu^*}) > -(2n - 1)K'. \tag{5.14}$$

Putting (5.12) and (5.14) into (5.5) we have

$$\begin{aligned} ^*F_2(w) &= \alpha^{-3}F_2(w) - \alpha^{-4}\beta[4(w, w) + 2(w, w)] \\ &\geq -2(2n - 1)\alpha^{-3}K'(w, w) - 6\alpha^{-4}\beta(w, w). \end{aligned}$$

THEOREM 5.10. *A compact Sasakian manifold M, $m \geq 5$, which satisfies*

$$K_{\lambda\mu} + K_{\lambda\mu^*} > -3(2 - \delta_{\lambda\mu})/(m - 2)$$

$$(\textit{especially } \textstyle\sum_\mu (K_{\lambda\mu} + K_{\lambda\mu^*}) > -3)$$

has no harmonic 2-form of the pure type. In particular if the sectional curvature $> -3/(m - 2)$, then there is no harmonic 2-form of the pure type.

Proof. As M is compact there is a positive number ε such that

$$K_{\lambda\mu} + K_{\lambda\mu^*} > -(3 - \varepsilon)(2 - \delta_{\lambda\mu})/(m - 2).$$

So we can take a positive number α so that $(-3\alpha^{-1}\beta - 3 + \varepsilon) > 0$, and by (5.13) we have $^*F_2(w) \geq 0$ for any harmonic 2-form of the pure type. And we have $w = 0$.

It is well known that a complete Riemannian manifold of Riemannian pinching > 0 is compact. Therefore by Proposition 5.1, Theorem 5.7 and Theorem 5.10, we have

THEOREM 5.11 (E. M. Moskal [14]). *A complete Sasakian manifold M with Riemannian pinching* >0 *has the second Betti number* $b_2(M) = 0$.

Remark. Under the additional condition that ξ is regular, Theorem 5.11 was proved by S. I. Goldberg [6]. And in Kählerian case the similar fact was obtained by R. L. Bishop and S. I. Goldberg [2]. In Sasakian manifolds without regularity condition, the first result was based on the condition that the sectional curvature is greater than $1/(m-1)$ [17], [13].

6. μ-holomorphic pinchings of Sasakian manifolds

In a Kählerian manifold N with a complex structure J and Hermitian metric G, if there are two positive constants λ and H' such that for any $X \in N_p$, $p \in N$, the J-holomorphic sectional curvature $K(X, JX)$ satisfies

$$\lambda H' \leq K(X, JX) \leq H', \tag{6.1}$$

then N is said to be λ-holomorphically pinched. After normalization of the Kählerian metric $G \to \bar{G} = H'G$, we have $\lambda \leq \bar{K}(X, JX) \leq 1$.

In a Sasakian manifold, analogously to the Kählerian case, we want to define certain pinching for ϕ-holomorphic sectional curvatures. One of the standard properties of a K-contact Riemannian manifold is that the sectional curvature for each 2-plane which contains ξ is equal to 1. So by a usual homothetic deformation of an associated Riemannian metric of a contact structure η, the resulting Riemannian metric is not associated with η or a constant multiple of it. This is why we consider a D-homothety (1.1) as a normalization.

LEMMA 6.1. *For a ϕ-basis in a Sasakian manifold we have*

$$^*K_{\lambda\mu} = \alpha^{-1}K_{\lambda\mu}, \tag{6.2}$$

$$^*K_{\lambda\mu^*} = \alpha^{-1}[K_{\lambda\mu^*} + 3(1-\alpha)\delta_{\lambda\mu}], \tag{6.3}$$

$$(\textit{especially} \quad ^*K_{\lambda\lambda^*} + 3 = \alpha^{-1}(K_{\lambda\lambda^*} + 3)).$$

Proof. First we note that for a ϕ-basis $(\xi, e_\lambda, e_{\lambda^*})$ the related $^*\phi$-basis is given by

$$(^*\xi = \alpha^{-1}\xi, \qquad ^*e_\lambda = \alpha^{-1/2}e_\lambda, \qquad ^*e_{\lambda^*} = \alpha^{-1/2}e_{\lambda^*}).$$

Let X and Y be orthonormal vectors with respect to g in D_x. Then by (2.11) and (2.14), we have

$$\begin{aligned} ^*K(X, Y) &= {}^*g(^*R(X, Y)X, Y)/^*g(X, X)^*g(Y, Y) \\ &= \alpha^{-1}[K(X, Y) - 3\alpha^{-1}\beta(\phi'(X, Y))^2]. \end{aligned}$$

On the other hand, since $\phi'(e_\lambda, e_\mu) = 0$ and $\phi'(e_\lambda, e_{\mu^*}) = -\delta_{\lambda\mu}$, we have (6.2) and (6.3).

Now assume that H and L defined by

$$H = \sup \{K(X, \phi X); \quad X \in D_x, \quad x \in M\},$$
$$L = \inf \{K(X, \phi X); \quad X \in D_x, \quad x \in M\}$$

exist and that $H + 3 > 0$; then μ defined by

$$\mu = (L + 3)/(H + 3) \tag{6.4}$$

is an invariance of the D-homothety class of M, as is seen by Lemma 6.1. And in this case we say that M is μ-holomorphically pinched.

PROPOSITION 6.2. *If a Sasakian manifold M is μ-holomorphically pinched, we can find a Riemannian metric $^*g$ (D-homothetic to g) so that $^*H = 1$ and $^*L = 4\mu - 3$ with respect to $(\phi, {}^*\xi, {}^*\eta, {}^*g)$.*

Proof. It is enough to put $\alpha = (H + 3)/4$ in (1.1).

A meaning of the quantity μ is as follows: Let x be an arbitrary point of M and take a sufficiently small regular neighborhood U of x with respect to ξ. Then we have a local fibering

$$\pi : U \to V = U/\xi.$$

Since U is Sasakian, V is Kählerian and we denote the structure tensors on V by J, G. They satisfy

$$\phi u^* = (Ju)^*, \qquad g = \pi^*G + \eta \otimes \eta,$$

where u^* on U is the horizontal lift of u on V with respect to η.

PROPOSITION 6.3. *If a Sasakian manifold M is μ-holomorphically pinched, then V is μ-holomorphically pinched with respect to the Kählerian structure (J, G).*

Proof. The sectional curvatures on U and V are related by

$$K(u^*, v^*) = K(u, v)\cdot\pi - 3[\phi'(u^*, v^*)]^2 \tag{6.5}$$

for any $u, v \in V_{\pi x}$. (Cf. (5.8) in [20], or [11], etc.) Then Proposition 6.3 follows from (6.5) easily.

LEMMA 6.4. *In a Sasakian manifold M we have*

$$K(X + B\xi, Y) = B^2 + (1 - B^2)K(X, Y)$$

for any orthonormal pair $(X + B\xi, Y)$ in M_x, $x \in M$, such that $X, Y \in D_x$.

Proof. This follows from (2.8) and (2.10).

Note that any 2-plane is determined by a pair of this form.

LEMMA 6.5 (E. M. Moskal [13]). *Let M be a Sasakian manifold; then* *for any* X, $Y \in D_x$, *we have*

$$(6.6)\quad \begin{aligned} g(R(X, Y)X, Y) = (\tfrac{1}{32})[&3D(X + \phi Y) + 3D(X - \phi Y) \\ &- D(X + Y) - D(X - Y) - 4D(X) - 4D(Y) - 24P(X, Y)], \end{aligned}$$

where $D(X) = g(R(X, \phi X)X, \phi X)$ *and*

$$P(X, Y) = [g(X, Y)]^2 - g(X, X)g(Y, Y) + [\phi'(X, Y)]^2.$$

Especially if X *and* Y *are orthonormal, denoting* $H(X) = K(X, \phi X)$ *and* $g(X, \phi Y) = \cos\theta$, *we have*

$$(6.7)\quad \begin{aligned} K(X, Y) = (\tfrac{1}{8})[&3(1 + \cos\theta)^2 H(X + \phi Y) \\ &+ 3(1 - \cos\theta)^2 H(X - \phi Y) - H(X + Y) - H(X - Y) \\ &- H(X) - H(Y) + 6\sin^2\theta]. \end{aligned}$$

LEMMA 6.6. *For an orthonormal pair* X, $Y \in D_x$ *in a Sasakian manifold we have*

$$(6.8)\quad \begin{aligned} K(X, Y) + \sin^2\theta K(X, \phi Y) = (\tfrac{1}{4})[&(1 + \cos\theta)^2 H(X + \phi Y) \\ &+ (1 - \cos\theta)^2 H(X - \phi Y) + H(X + Y) + H(X - Y) - H(X) \\ &- H(Y) + 6\sin^2\theta]. \end{aligned}$$

Proof. Replacing Y by ϕY in (6.6) and adding the resulting equation to (6.7), we get (6.8).

Remark. (6.7) and (6.8) are also obtainable from the corresponding identities in [3] by virtue of (6.5).

7. The first Betti numbers and μ-holomorphic pinchings

We assume that a Sasakian manifold M is μ-holomorphically pinched. Then by (6.8) for a ${}^*\phi$-basis in Proposition 6.2 we have

$$(7.1)\qquad 4\mu - 2 \le {}^*K_{\lambda\mu} + {}^*K_{\lambda\mu^*} \le 4 - 2\mu, \qquad \lambda \ne \mu.$$

Thus we have

$$(7.2)\quad \begin{aligned} \textstyle\sum_\mu ({}^*K_{\lambda\mu} + {}^*K_{\lambda\mu^*}) &= \textstyle\sum_{\lambda\ne\mu} ({}^*K_{\lambda\mu} + {}^*K_{\lambda\mu^*}) + {}^*K_{\lambda\lambda^*} \\ &\ge 4n\,\mu - 2n - 1. \end{aligned}$$

Therefore by Theorem 4.1, we have

THEOREM 7.1. *Let* M *be a compact Sasakian manifold which is* μ*-holomorphically pinched with* $\mu > (m - 3)/2(m - 1)$; *then* $b_1(M) = 0$.

8. The second Betti numbers and μ-holomorphic pinchings

By (7.1) and Theorem 5.7, or 5.10, we have

PROPOSITION 8.1. *If a compact Sasakian manifold M is μ-holomorphically pinched with $\mu > \frac{1}{2}$, then there is no harmonic 2-form of the hybrid type.*

PROPOSITION 8.2. *If a compact Sasakian manifold M is μ-holomorphically pinched with $\mu > (m-3)/2(m-1)$, then there is no harmonic 2-form of the pure type.*

By these we have

THEOREM 8.3. *If a compact Sasakian manifold M, $m \geq 5$, is μ-holomorphically pinched with $\mu > \frac{1}{2}$, then $b_2(M) = 0$.*

COROLLARY 8.4. *If a compact Sasakian manifold M, $m \geq 5$, is μ-holomorphically pinched with $\mu > \frac{1}{2}$ and if the scalar curvature S is constant, then it is an η-Einstein Sasakian manifold.*

Proof. This follows from the above Theorem and Corollary 5.7 in [20].

If $m = 3$, there is only one ϕ-holomorphic plane at each point. Therefore by Proposition 5.1 we have

THEOREM 8.5. *If a compact Sasakian manifold M, $m = 3$, is μ-holomorphically pinched with $\mu > 0$, then $b_2(M) = 0$.*

9. The lower bound of $K(X, Y)$ in terms of H and L

By (6.7) for $X, Y \in D_x$, we have

$$K(X, Y) \geq (\tfrac{1}{8})[6L - 4H + 6 + 6(L-1)\cos^2\theta]. \tag{9.1}$$

If we calculate many cases, then finally we see that in the sequel we need only the case $1 \leq L \leq H$. And we have

PROPOSITION 9.1. *Assume that* $1 \leq L \leq H$. *Then*

$$K(X, Y) \geq (3 + 3L - 2H)/4 \tag{9.2}$$

holds for any X, Y, in D_x.

The upper bound will be given by Lemma 12.3.

10. Deformability to a space of positive curvature

THEOREM 10.1. *If a Sasakian manifold M, $m > 5$, is μ-holomorphically pinched with $\mu > \frac{2}{3}$, then the metric g is D-homothetically deformable to the metric ${}^*g$ so that M is of Riemannian ${}^*\delta$-pinching with ${}^*\delta > 0$ with respect to ${}^*g$.*

Proof. It is enough to show that the lower bound is positive. By Proposition 6.2 we can assume that

$$4\mu - 3 = L \leq H(X) \leq H = 1, \qquad X \in D_x.$$

Take α so that $\alpha \leq \frac{2}{3}$ and define $g \to {}^*g$ by (1.1). Then

(10.1) $$^*H = (4 - 3\alpha)/\alpha,$$

(10.2) $$^*L = (4\mu - 3\alpha)/\alpha \geq 1.$$

Therefore we can apply Proposition 9.1 and

$$^*K(X, Y) \geq (3\mu - 2)/\alpha > 0$$

for any X, Y in D_x ; then Lemma 6.4 completes the proof.

THEOREM 10.2. *If a Sasakian manifold M, $m = 3$, is μ-holomorphically pinched with $\mu > 0$, then g is D-homothetically deformable to $^*g$ so that M is of positive curvature with respect to $^*g$.*

11. Applications of preceding theorems

The following shows that completeness is an invariant of the D-homothety class.

LEMMA 11.1. *If a contact Riemannian manifold M is complete with respect to g, then it is also complete with respect to $^*g$.*

Proof. Let $l = x(t)$, $0 \leq t \leq 1$, be any C^∞-curve *in* M. We decompose the tangent vector $\partial x(t)/\partial t$ as

$$\partial x(t)/\partial t = v(t) + u(t)\xi,$$

where $v(t) \in D_{x(t)}$ and $u(t)$ is a real number for each t. As the length $^*| l |$ of l with respect to $^*g$ is given by

$$^*| l | = \int_0^1 [\alpha g(v(t), v(t)) + \alpha^2(u(t))^2]^{1/2}\, dt,$$

when $\alpha \geq 1$, we have

$$\sqrt{\alpha}\, | l | \leq {}^*| l | \leq \alpha\, | l |,$$

and when $\alpha < 1$, we have

$$\alpha\, | l | \leq {}^*| l | \leq \sqrt{\alpha}\, | l |, \qquad \text{Q.E.D.}$$

The next two results are due to E. M. Moskal:

PROPOSITION 11.2 (E. M. Moskal [13, 14]). *Let M be a compact, simply connected, m-dimensional Einstein-Sasakian manifold. If M is of strictly positive curvature, then M is isometric with the unit sphere S^m.*

The assumption of strict positiveness of the curvature may be replaced by $K_{\lambda\mu} + K_{\lambda\mu^*} > 0$ for $\lambda \neq \mu$.

PROPOSITION 11.3 (E. M. Moskal [13, 14]). *If a complete, simply connected Sasakian manifold M is of Riemannian pinching >0 and has the constant scalar curvature S, then M is D-homothetic with the unit sphere S^m. M is isometric with S^m if $S = m(m - 1)$.*

Remark 11.4. If we remove the assumption of simple connectedness in Proposition 11.3, then the conclusion is that M is D-homothetically deformable to a space of constant curvature.

COROLLARY 11.5. *A 5-dimensional complete Sasakian manifold of Riemannian pinching* >0 *is a homology sphere over the field of real numbers.*

THEOREM 11.6. *Let M be a 5-dimensional, simply connected, complete Sasakian manifold of Riemannian pinching* >0. *If the torsion part of the integral second homology group vanishes, then M is diffeomorphic with a sphere.*

Proof. By Corollary 11.5 each free part of the integral homology group $H_r(M; Z)$ or cohomology group $H^r(M; Z)$ vanishes for $r = 1, \cdots, 4$. As M is simply connected $H_1(M; Z) = H_3(M; Z) = 0$ by the duality for the torsion part. Thus $H_r(M; Z) = 0$ for $r = 1, \cdots, 4$ and also the homotopy groups $\pi_r(M) = 0$ for $r = 1, \cdots, 4$ by the Hurewicz isomorphism. M is a homotopy sphere by the Whithead homotopy type theorem. Finally by the generalized Poincaré conjecture for dim $M = 5$ (for example [12]), M is diffeomorphic with a sphere.

Remark. If the contact structure is regular, the condition on the torsion part is unnecessary. (See Theorem 3 in [7].)

THEOREM 11.7. *Any homogeneous (with respect to g) Sasakian manifold of positive curvature is of constant curvature with respect to the D-homothetically deformed metric.*

Proof. Since M is homogeneous, M is complete and the scalar curvature is constant. Hence Theorem 11.7 follows from Proposition 11.3 and Remark 11.4.

As for Corollary 11.5 and Theorem 11.6, by Theorem 7.1 and Theorem 8.3, we have

COROLLARY 11.8. *A 5-dimensional compact Sasakian manifold of μ-holomorphic pinching* $>\frac{1}{2}$ *is a homology sphere over the field of real numbers.*

COROLLARY 11.9. *Let M be a 5-dimensional, simply connected, compact Sasakian manifold of μ-holomorphic pinching* $>\frac{1}{2}$. *If the torsion part of the integral homology group vanishes, then M is diffeomorphic with a sphere.*

THEOREM 11.10. *Assume that a compact Sasakian manifold M, $m \geq 5$, is μ-holomorphically pinched with $\mu > \frac{1}{2}$ and the scalar curvature S is constant. Then*

(i) *M is an η-Einstein manifold.*
(ii) *By a D-homothety of g, M is of constant curvature with respect to $^*g$.*
(iii) *In particular, if M is simply connected, it is D-homothetic with S^m.*

Proof. By Proposition 6.2 we assume that $4\mu - 3 \leq H(X) \leq 1, X \in D_x$. We put $\alpha = (S + m - 1)/(m^2 - 1)$. Then we see that α is positive and that $^*S = m(m-1)$ by (2.16), and hence M is Einstein with respect to $^*g$.

COROLLARY 11.11. *Any compact homogeneous (with respect to g) Sasakian manifold of μ-holomorphic pinching* $> \frac{1}{2}$ *is of constant curvature with respect to the D-homothetically deformed metric.*

12. Curvature as an average

For this notion we refer to R. L. Bishop and S. I. Goldberg [3]. The terminology for J in a Kählerian manifold may be used for ϕ in a Sasakian manifold. By (6.7), if ϕ-holomorphic sectional curvature $H(X)$ is constant, $H > -3$, then we have

$$K(X, Y) = (\tfrac{1}{4})[H + 3 + 3(H - 1) \cos^2 \theta] \tag{12.1}$$

for any X, Y in D_x. This implies that by a D-homothety $g \to {}^*g$ such that $^*H = 1$, we have $^*K(X, Y) = 1$, namely $^*g$ is of constant curvature.

For a ϕ-holomorphic section $(X, \phi X)$ we have $K(X, \phi X) = H$, and for an anti-ϕ-holomorphic section (X, Y) (i.e. $g(X, Y) = 0$ and $g(X, \phi Y) = 0$) we have $K(X, Y) = (H + 3)/4 = A$. Then (12.1) is

$$K(X, Y) = H - 3A \sin^2 \theta + 3 \sin^2 \theta \tag{12.2}$$

for any pair X, Y in D_x. Generalizing this, analogously to [3], we obtain

PROPOSITION 12.1. *Let X, Y be an orthonormal pair in D_x, $x \in M$. We define $H(X, Y)$ and $A(X, Y)$ by*

$$H(X, Y) = \frac{1}{\pi} \int_0^\pi H(X \cos \gamma + Y \sin \gamma)\, d\gamma, \tag{12.3}$$

$$A(X, Y) = \frac{1}{\pi} \int_0^\pi K(X \cos \gamma + Y \sin \gamma, - \phi X \sin \gamma + \phi Y \cos \gamma)\, d\gamma. \tag{12.4}$$

Then we have

$$K(X, Y) = H(X, Y) - 3A(X, Y) \sin^2 \theta + 3 \sin^2 \theta. \tag{12.5}$$

Proof. By calculation we have

$$H(X, Y) = (\tfrac{1}{4})[H(X) + H(X + Y) + H(X - Y) + H(Y)], \tag{12.6}$$

$$A(X, Y) = (\tfrac{1}{2})[K(X, \phi Y) + K(X + Y, - \phi X + \phi Y)]. \tag{12.7}$$

Substituting (12.6) and (12.7) into the right hand side of (12.5), finally we see that (12.5) holds.

By (6.7) with $\cos \theta = 0$ we have

LEMMA 12.2. *If (X, Y) defines an anti-ϕ-holomorphic section we have*

$$(3 + 3L - 2H)/4 \le K(X, Y) \le (3 + 3H - 2L)/4. \tag{12.8}$$

Next by (6.7), (12.5) and (12.6) we get

$$K(X, Y) \leq (\tfrac{1}{4})[2H + 2H\cos^2\theta + 6\sin^2\theta] - A(X, Y)\sin^2\theta.$$

On the other hand by (12.4) and (12.8), we have

$$A(X, Y) \geq (3 + 3L - 2H)/4.$$

Therefore

LEMMA 12.3. *For any orthonormal pair (X, Y) in D_x, we have*

$$K(X, Y) \leq H + (\tfrac{3}{4})(1 - L)\sin^2\theta. \tag{12.9}$$

13. Riemannian pinchings

It is known as the sphere theorem that

A simply connected complete Riemannian manifold M, $m = 2n + 1$, with δ-pinching is homeomorphic with a sphere, if $\delta \geq \frac{1}{4}$ [10].

In contact Riemannian manifolds the situation is very different from the usual cases, because we can apply D-homothetic deformations and δ-pinching may be changed to $^*\delta$-pinching in such a way that $\delta < \frac{1}{4}$ turns to $^*\delta \geq \frac{1}{4}$.

PROPOSITION 13.1. *If a Sasakian manifold M is μ-holomorphically pinched (>0), then g is D-homothetically deformable to $^*g$ so that M is $^*\delta$-pinched with*

$$^*\delta \geq (3\mu - 2)/(4 - 3\mu). \tag{13.1}$$

Proof. By Proposition 6.2 we assume that $4\mu - 3 \leq H(X) \leq 1$, $X \in D_x$. Now we put $\alpha = \mu$, and consider a deformation (1.1). Then we have $^*H = (4 - 3\mu)/\mu$ and $^*L = 1$ by (10.1) and (10.2). By Lemma 12.3 and Proposition 9.1 we have

$$^*\delta \geq (3 + 3^*L - 2^*H)/4^*H = (3\mu - 2)/(4 - 3\mu).$$

THEOREM 13.2. *If a Sasakian manifold M is μ-holomorphically pinched with $\mu \geq \frac{4}{5}$, then we have a Riemannian metric $^*g$ D-homothetic to g so that M is of Riemannian pinching $^*\delta \geq \frac{1}{4}$.*

Further if M is complete and simply connected, it is homeomorphic with a sphere.

Example 13.3. If ϕ-holomorphic sectional curvature in a Sasakian manifold satisfies

$$0 \leq H(X) \leq \tfrac{3}{4} \tag{13.2}$$

for every X in D_x, then g is deformable to $^*g$ so that M is of Riemannian pinching $^*\delta \geq \frac{1}{4}$.

Remark. Proposition 13.1 and Theorem 13.2 may be also obtained by local fibering and (6.5) from the corresponding results in [2]. The converse is also true. A little difference is that in our case equality is contained in $\mu \geq \frac{4}{5}$.

COROLLARY 13.4. *Let B be a complete Hodge manifold with holomorphic pinching $\geq \frac{1}{5}$; then it has the homotopy type of complex projective space.*

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