TERMINAL p-GROUPS

BY
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1. Introduction

Let G be a finite p-group. We shall say that G is *terminal* if it cannot be extended without upsetting the structure of its lower central series. More precisely, let G have lower central series

$$G = \gamma_1(G) > \gamma_2(G) > \cdots > \gamma_n(G) \neq \gamma_{n+1}(G) = 1.$$

Then G is terminal if, given a p-group U, $U/\gamma_{n+1}(U) \cong G$ implies that $\gamma_{n+1}(U) = 1$ (i.e. $U \cong G$.)

Blackburn [1, p. 68] has observed that the p-Sylow subgroup of the Symmetric group on p^2 symbols, for p odd, is a maximal p-group of maximal class, that is, it is a terminal p-group. The investigation presented here began as an attempt to derive Blackburn's result by homological methods. That attempt was successful, and it became clear that the method of attack could yield more general results. In particular, I was able to show that wreath products of a certain kind are always terminal. (See Section 5.)

The best known examples of p-groups of the type considered in Section 5 are Sylow subgroups of Symmetric groups (for odd primes.) Indeed, the latter groups served as a model class for my main result, Theorem 2. This turned out to be a fortunate stupidity on my part since it has turned out that Sylow subgroups of Symmetric groups can be shown to be terminal much more easily. (See Section 7.) In any event, I have left this application in Section 6 to make clearer the ancestry of my main result.

As one would expect, one can derive many of the results presented here more easily by direct group theoretical methods. (I am indebted to N. Blackburn for showing me how to do so.) The case of Sylow-Symmetric groups, in particular, can be treated almost trivially by means of a simple result of P. Hall. Also, Blackburn can derive the other application, Theorem 3, in Section 6 by a method close to his first approach. In addition, it is likely that Theorem 2 can be proved directly. Despite this, I feel that the homological approach has some merit, and it is one of the purposes of this paper to explore that approach. In particular, I hope to make clear the homological significance of the lower central series.

The homological analysis of the lower central series proceeds as follows. Given an arbitrary p-group G, it may be constructed by a simple procedure: Choose an abelian p-group K_1, and, in $H^2(K_1, Q/Z)$, choose a subgroup V_1. Make an appropriate extension to "kill off" the subgroup V_1:

$$1 \to \hat{V}_1 = A_1 \to K_2 \to K_1 \to 1.$$

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(The choice of the extension is severely limited. See Section 4.) In $H^2(K_2, Q/Z)$ choose a subgroup V_2 whose intersection with

$$\operatorname{Im}\{H^2(K_1) \to H^2(K_2)\}$$

is trivial. Again, make an appropriate extension

$$1 \to \hat{V}_2 = A_2 \to K_3 \to K_2 \to 1.$$

Continue in this way c times, where c is the class of G. If the choices above have been made properly, we have

$$G/\gamma_2 = K_1, \qquad G/\gamma_3 = K_2, \cdots, \qquad G/\gamma_c = K_{c-1}, \qquad G = K_c.$$

If one takes as coefficient group Z/pZ instead of Q/Z, the construction above may be copied to produce the series

$$(1) \qquad G = G_1 > G_2 = [G, G]G_1^p > G_3 = [G, G_2]G_2^p > \cdots.$$

Moreover, in this case the extensions at each stage are uniquely determined so that it is necessary only to specify the sub*spaces* V_i. (Again, see Section 4.)

This machinery is developed more precisely in Sections 3 and 4. It is more convenient to deal with the dual situation in *homology*. Section 2 is devoted to an outline of facts concerning the spectral sequence of a group extension. I apologize to readers mainly interested in the application to group theory for the introduction of this machinery in its cumbersome entirety. I have tried to state clearly just what is needed from this theory without being too explicit about the theory itself.

The rest of the paper is concerned with showing that a certain kind of a p-group is terminal. The homological framework introduced above is a natural one with which to investigate such questions. Namely, the cokernel of the inflation homomorphism

$$H^2(G/\gamma_c, Q/Z) \to H^2(G, Q/Z) \qquad (c = \text{the class of } G)$$

provides an upper bound for the kernel of a central extension not upsetting the lower central series of G. Hence to show a group is terminal it suffices to show this inflation is an epimorphism. It is more convenient to proceed dually and show that the coinflation $H_2(G, Z) \to H_2(G/\gamma_c, Z)$ is a monomorphism. Finally, by replacing Z by Z/pZ we may also investigate the property connected with the series (1) analogous to being terminal.

We consider only the case of odd p; indeed, the corresponding result for $p = 2$ is false. (The simplest possible example is the dihedral group of order 8 which may be extended to the quaternion group or dihedral group of order 16 without upsetting the lower central structure.)

I should like to thank M. E. Mahowald and N. Blackburn for helpful suggestions concerning crucial points in my arguments.

2. The Lyndon-Hochschild-Serre spectral sequence

We shall need to use the spectral sequences associated with a group extension

$$1 \to A \to G \to K \to 1.$$

We outline some basic facts for the convenience of the reader (see [2], [7] and [9] for details). We treat the case of homology; cohomology is dual. (Notation: If M is a G-module, we write M_G for $H_0(G, M) = M/[M, G]$.)

Let M be a right G-module. Choose a G-projective resolution X of Z and a K-projective resolution Y of Z. Then the double complex $(M \otimes_A X) \otimes_K Y$ has total homology $H_*(G, M)$, and its "first" spectral sequence with E_2-term $H_*(K, H_*(A, M))$ converges to that total homology. The spectral sequence —from E_2 on—is independent of the resolutions used and is a functor on triples (G, A, M). Thus, given $\phi : G \to G'$ with $\phi(A) \leq A'$ and $f : M \to M'$, ϕ-semilinear, there are induced maps of the corresponding spectral sequences which on the E_2-level and in the limit are the usual homomorphisms. In particular, the edge homomorphisms

$$H_q(A, M)_K \to H_q(G, M) \qquad \text{(co-restriction)}$$

and

$$H_p(G, M) \to H_p(K, M_A) \qquad \text{(co-inflation)}$$

are those induced by the group homomorphisms $A \to G$ and $G \to K$. (Actually the first homomorphism is a *factor* of the co-restriction

$$H_q(A, M) \to H_q(G, M).)$$

What the spectral sequence yields is a filtration of $H_n(G, M)$ for each n for which the factors are subgroups of factor groups of the terms $H_p(K, H_q(A, M))$ with $p + q = n$. The particular subfactor groups are determined by computing kernels and images of differentials in the spectral sequence. In particular, we have the following filtrations for $n = 1$ and 2.

$$n = 1: \quad H_1(G, M) = F_1 H_1 \geq F_0 H_1$$

where

$$F_1 H_1/F_0 H_1 = \operatorname{Im}\{H_1(G, M) \to H_1(K, M_A)\} = H_1(K, M_A),$$

and

$$F_0 H_1 = \operatorname{Im}\{H_1(A, M) \to H_1(G, M)\} = \operatorname{coker}\{d_{2,0}^2 : H_2(K, M_A) \to H_1(A, M)_K\}.$$

$$n = 2: \quad H_2(G, M) = F_2 H_2 \geq F_1 H_2 \geq F_0 H_2$$

where

$$F_2 H_2/F_1 H_2 = \operatorname{Im}\{H_2(G, M) \to H_2(K, M_A)\} = \operatorname{Ker}\{d_{2,0}^2 : H_2(K, M_A) \to H_1(A, M)_K\}$$

$$F_1 H_2/F_0 H_2 = \operatorname{Coker}\{d_{3,0}^2 : H_3(K, M_A) \to H_1(K, H_1(A, M))\}$$

and

$$F_0 H_2 = \mathrm{Im}\{H_1(A, M) \to H_1(G, M)\}$$
$$= \mathrm{Coker}\{d^3_{3,0} : E^3_{3,0} \to E^3_{0,2}\}$$

where

$$E^3_{3,0} = \mathrm{Ker}\{d^2_{3,0} : H_3(K, M_A) \to H_1(K, H_1(A, M))\}$$

and

$$E^3_{0,2} = \mathrm{Coker}\{d^2_{2,1} : H_2(K, H_1(A, M)) \to H_2(A, M)_K\}.$$

Hence the filtration in these dimensions is determined once the differentials $d^2_{2,0}$, $d^2_{3,0}$, $d^2_{2,1}$ and $d^3_{3,0}$ are known. (Often part of this information is summarized in the so-called exact fundamental sequence $H_2(G, M) \to H_2(K, M_A) \to H_1(A, M)_K \to H_1(G, M) \to H_1(k, M_A) \to 0$.)

Suppose A acts trivially on M, i.e., $M_A = M$. Then the homomorphisms $d^2_{p,q}$ can be described quite explicitly. (See Charlap and Vasquez [3] for the dual case.) We shall need to know $d^2_{2,0}$ in detail.

Let $\varepsilon \in H^2(K, A/A')$ be the characteristic class of the induced extension

$$1 \to A/A' \to G/A' \to K \to 1.$$

We have $H_1(A, M) = M \otimes A/A'$. Let $\phi : M \to \mathrm{Hom}\,(A/A', H_1(A, M))$ be defined by $\phi(b)(h) = b \otimes h$. Then there is defined a homomorphism

$$H_2(K, M) \to \mathrm{Hom}\,(H^2(K, A/A'), H_0(K, H_1(A, M))$$

[2, Chap X1, Sec. 7] which yields a pairing

$$\cap : H_2(K, M) \otimes H^2(K, A/A') \to H_0(K, H_1(A, M))$$

called the cap product (induced by ϕ.) Then, dual to Hochschild-Serre [6], we have

$$d^2_{2,0}(X) = -X \cap \varepsilon, \qquad X \in H_2(K, M). \tag{2}$$

We shall be particularly interested in the case in which the sequence

$$1 \to A \to G \to K \to 1$$

splits. Suppose also, as above, that A acts trivially on M. Then K appears as a subgroup of G complementing A. Co-restricting from this subgroup splits the homomorphism $H_n(G, M) \to H_n(K, M)$ so that, in particular, it is an epimorphism. It follows that the homomorphisms $d^2_{2,0}$ and $d^3_{3,0}$ must be trivial. Thus, we have the additional information

$$\begin{aligned} F_0 H_1 &= H_1(A, M)_K \leq H_1(G, M), \\ F_2 H_2/F_1 H_2 &= H_2(K, M), \\ F_1 H_2/F_0 H_2 &= H_1(K, H_1(A, M)), \\ F_0 H_2 &= \mathrm{Coker}\{d^2_{2,1} : H_2(K, H_1(A, M)) \to H_2(A, M)_K\}. \end{aligned} \tag{3}$$

3. Some characteristic series

Let $k = Z/qZ$ where q is a natural number, possibly zero. If G is a group, let G act trivially on k and write $H_n(G, k) = H_n(G)$. Define the series of normal subgroups

$$G_1 = G, \cdots, \quad G_{n+1} = \operatorname{Ker}\{G_n \to H_1(G_n)_G\},$$

that is,

$$\begin{aligned} G_n &= \gamma_n(G) & \text{if} \quad k = Z, \\ &= [G, G_{n-1}]G_{n-1}^q & \text{if} \quad k = Z/qZ. \end{aligned}$$

Suppose next that $1 \to A \to U \to G \to 1$ is a central extension. We shall be particularly interested in the case in which $A \to H_1(A)$ is an isomorphism. (If $k = Z$, this is automatic; otherwise it means that the exponent of A divides q.) We shall assume below that this is so and identify the two groups.

Suppose $n > 0$, and consider the commutative diagram of group extensions

$$(4) \qquad \begin{array}{ccccccccc} 1 \to & A & \to & U & \to & G & \to 1 \\ & \downarrow & & \downarrow = & & \downarrow & \\ 1 \to & AU_n & \to & U & \to & G/G_n & \to 1. \end{array}$$

(4) induces a corresponding diagram of fundamental exact sequences.

$$(5) \qquad \begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \operatorname{Ker}\phi_n & \to & \operatorname{Ker}\iota & & & & \\ & & \downarrow & & \downarrow & & & & \\ H_2(U) & \to & H_2(G) & \xrightarrow{\psi} & A & \longrightarrow & H_1(U) & \to & H_1(G) & \to 0 \\ \downarrow = & & \downarrow \phi_n & & \downarrow \iota & & \downarrow = & & \downarrow & \\ H_2(U) & \to & H_2(G/G_n) & \to & H_1(AU_n)_U & \to & H_1(U) & \to & H_1(G/G_n) & \to 0. \end{array}$$

Examination of the diagram shows that $\psi(\operatorname{Ker}\phi_n) = \operatorname{Ker}\iota$. However. $H_1(AU_n)_U = AU_n/U_{n+1}$ and ι is the homomorphism induced by inclusion. Hence, we have

$$(6) \qquad \psi(\operatorname{Ker}\phi_n) = A \cap U_{n+1},$$

where $\phi_n : H_2(G) \to H_2(G/G_n)$ is the co-inflation homomorphism.

Let $\phi : U \to G$ be an epimorphism. We shall call ϕ an n-covering of G if $\operatorname{Ker}\phi \leq U_{n+1} \cap Z(U)_q$. (Clearly, the definition depends on q.)

PROPOSITION 1. *Let $\phi : U \to G$ be an epimorphism whose kernel is central and of exponent dividing q. ϕ is an n-covering of G if and only if*

$$\operatorname{Ker}\phi_n + \operatorname{Ker}\psi = H_2(G).$$

(The notation is the same as above.)

Proof. According to (6), we have $A \cap U_{n+1} = A$ if and only if $\operatorname{Ker} \phi_n + \operatorname{Ker} \psi = H_2(G)$.

(Say two homomorphisms are *disjoint* if they satisfy this condition.)

If ϕ and ϕ' are two n-coverings of G, we shall say they are equivalent if there is a commutative diagram

$$\begin{array}{ccc} U & & \\ & \searrow^{\phi} & \\ F \downarrow & & G \\ & \nearrow_{\phi'} & \\ U' & & \end{array}$$

with F an isomorphism. Let ϕ be an n-covering of G, and denote as above by $\psi : H_2(G) \to \operatorname{Ker} \phi$ the co-transgression which is certainly an epimorphism. Put

$$A = H_2(G)/\operatorname{Ker} \psi \cong \operatorname{Ker} \phi.$$

It is clear that equivalent coverings yield the same factor group A. In fact, any covering yielding this factor group A may be replaced by an equivalent covering with kernel A and for which the co-transgression is the canonical projection of $H_2(G)$ onto A.

Thus, writing $\operatorname{Cov}(G) = \bigcup_{n>0} \operatorname{Cov}_n(G)$ for the set of equivalence classes of coverings of G, we have defined a function π from $\operatorname{Cov}(G)$ to the set of factor groups of $H_2(G)$. In the next section, we shall see that the function π provides almost a complete classification of coverings of G in the most interesting cases.

In case G is a p-group there are some simplifications. Namely, suppose $G_n >_{\neq} G_{n+1} = (1)$. Then, an epimorphism $\phi : U \to G$ is an n-covering if and only if $U_{n+1} = \operatorname{Ker} \phi$ and $U_{n+2} = (1)$. In particular, G has no n-coverings, that is, it is terminal in the sense of Section 1, if and only if $H_2(G)$ has no factor groups disjoint from ϕ_n, that is, if and only if $\operatorname{Ker} \phi_n \leq pH_2(G)$.

4. The group extension

If G is any group and A a k-module on which G acts trivially, we may define the homomorphism

$$\alpha : H^2(G, A) \to \operatorname{Hom}_k(H_2(G), A)$$

by $\alpha(\varepsilon)(\gamma) = -\gamma \cap \varepsilon$. Hence, if ε is the characteristic class of the central extension

$$1 \to A \to U \to G \to 1,$$

then $\alpha(\varepsilon) = \psi : H_2(G) \to A$ is the co-transgression. (See Section 2.) Moreover, if $q = 0$ or q is prime, then the Universal Coefficient Theorem provides an exact sequence

$$0 \to \operatorname{Ext}_k(H_1(G), A) \xrightarrow{\beta} H^2(G, A) \xrightarrow{\alpha} \operatorname{Hom}_k(H_2(G), A) \to 0.$$

(The homomorphism β is quite easy to describe. Namely, given an abelian extension of $H_1(G)$ by A, pull it back to an extension of G by means of the homomorphism $G \to H_1(G)$.) We shall suppose in what follows that the Universal Coefficient Theorem applies. Then, given any factor group A of $H_2(G)$, it arises from some covering of G; that is, the function π is surjective. Moreover, it is quite easy to see that the set of equivalence classes of coverings which yield this factor group A is in one-to-one correspondence with the set $\mathrm{Ext}_k\,(H_1(G), A)$. Thus, if $H_1(G)$ is k-free (as it always is if $k = Z/pZ$, p a prime,) then there is a one-to-one correspondence between equivalence classes of coverings of G and factor groups of $H_2(G, k)$. If in addition G is a p-group with $G_n >_{\neq} G_{n+1} = (1)$, then there is a one-to-one correspondence between equivalence classes (over G) of p-groups U such that $U/U_{n+1} = G$, $U_{n+2} = (1)$ and factor groups A of $H_2(G)$. This provides a small simplification of the extension problem for p-groups in that a classification of U's up to isomorphism requires only the knowledge of how the automorphisms of G permute the factor groups of $H_2(G)$.

5. A general class of terminal p-groups

We shall consider a special kind of wreath product. As we shall see in Section 6, the special hypotheses listed below are just those which arise when considering the Sylow subgroups of the Symmetric groups.

Let K be a finite p-group (with p odd) and K_1 a proper subgroup of K. Let q be a power of p and let

$$A = \mathrm{Hom}_{K_1}\,(Z(K), Z/qZ) \cong Z(K) \otimes_{K_1} Z/qZ$$

be the indicated induced K-module. We shall be interested in the semi-direct product $G = K \cdot A$.

Theorem 2. *With K and A as above, and p odd, assume the following:* (A) *The transfer $\mathfrak{V} : K/K' \to K_1/K_1'$ is trivial,* (B) $(K_1/K_1')^q = 1$, *and* (C) *the last nontrivial term of the lower central series of $G = K \cdot A$ is contained in A^K, (the subgroup of A of elements fixed by K).*

Then the semi-direct product $K \cdot A$ is a terminal p-group.

Proof. According to Section 3, it is sufficient to show that

$$H_2(G, Z) \to H_2(G/\gamma_c(G), Z)$$

is a monomorphism where γ_c is the last nontrivial term of the lower central series of G. Hypothesis (C) is designed precisely to make the above homomorphism a factor of $H_2(G) \to H_2(G/A^K)$. Hence it suffices to prove that the latter homomorphism is a monomorphism, and this we proceed to do.

Let B be the K-module A/A^K. Then $\bar{G} = G/A^K$ is the semi-direct product $K \cdot B$. Each of the products $K \cdot A$ and $K \cdot B$ yields a spectral sequence and there are induced homomorphisms of corresponding constituents of these spectral sequences which we must analyze in detail.

Referring to Section 2, we see that in order to show that $H_2(G) \to H_2(\bar{G})$ is a monomorphism, it suffices to show that the three homomorphisms of corresponding factors of the filtrations are monomorphisms. Since we are dealing with split extensions, by (3), we have the following simplifications:

(a) $F_2 H_2(G)/F_1 H_2(G) = H_2(K) \to F_2 H_2(\bar{G})/F_1 H_2(\bar{G}) = H_2(K)$ is the identity, certainly a monomorphism.

(b) $F_1 H_2(G)/F_0 H_2(G) \to F_1 H_2(\bar{G})/F_0 H_2(\bar{G})$ is the induced homomorphism

$$H_1(K, A) \to H_1(K, B). \tag{7}$$

(Here we have used the facts $H_1(A, Z) = A, H_1(B, Z) = B$.)

(c) $F_0 H_2(G) = \operatorname{Coker} d^2_{2,1} \to F_0 H_2(\bar{G}) = \operatorname{Coker} \bar{d}^2_{2,1}$ is induced by the obvious homomorphism

$$H_2(A)_K \to H_2(B)_K . \tag{8}$$

In order to prove the theorem, we shall show first (I) that both differentials $d^2_{2,1}$ and $\bar{d}^2_{2,1}$ are trivial and next (II), that the homomorphisms (7) and (8) are monomorphisms.

Before beinning the proof of (I), we discuss some preliminaries. First we outline some facts about the homology of abelian groups.

Let $A = (Z/qZ)^l$. We have natural isomorphisms

$$H_1(A, Z) \cong A, \qquad H_2(A, Z) \cong \Lambda^2 A$$

where the exterior product may be formed either over Z or Z/qZ.

Suppose next $q > 0$. Then $H_1(A, Z/qZ)$ is still naturally isomorphic to A, and the coefficient sequence

$$0 \to Z \xrightarrow{q} Z \to Z/qZ \to 0$$

yields a split exact sequence

$$0 \to \Lambda^2 A \to H_2(A, Z/qZ) \to A \to 0.$$

The splitting is not natural. However, in case that K is a group acting on A *by permuting the factors,* we have an isomorphism

$$H_2(A, Z/qZ) \cong \Lambda^2 A \oplus A$$

of K-modules.

Secondly, it will be useful to investigate the homomorphism $A^K \to A$. Suppose, as above, that A is induced from the trivial K_1-module Z/qZ but allow $q = 0$. Then by Shapiro's Lemma [2, Chap. X, 7.4], we have a natural isomorphism $H_n(K, A) \cong H_n(K_1, Z/qZ)$. Moreover, identifying $Z/qZ = A^K$, we know that the homomorphism induced by $A^K \to A$ when composed with the Shapiro isomorphism yields the (co-) transfer $H_n(K, Z/qZ) \to H_n(K_1, Z/qZ)$.

(See [2, Chap. XII, Exercises 7–10].) In particular, $H_1(K, A^K) \to H_1(K, A)$ is simply the group theoretic transfer and by hypothesis (A) is trivial.

(I) First we show
$$d^2_{2,1} : H_2(K, A) \to H_2(A)_K$$
is trivial.

Let F be the free group on the *set* of generators K/K_1. Let K act on F by permuting the generators. Then, the homomorphism $\rho : F \to A$ defined by

$$\rho(\bar{x}) = x \otimes 1 \in Z(K) \otimes_{K_1} Z/qZ,$$

where $x \in K$ represents $\bar{x} \in K/K_1$, is a K-epimorphism. Hence, there is induced a homomorphism $K \cdot F \to K \cdot A$ of semi-direct products and, also, homomorphisms of corresponding spectral sequences. Consider, in particular, the case in which the coefficient group is Z/qZ. We have the commutative diagram

$$\begin{array}{ccc} H_2(K, H_1(F, Z/qZ)) & \to & H_2(F, Z/qZ)_K = 0 \\ \downarrow & & \downarrow \\ H_2(K, H_1(A, Z/qZ)) & \to & H_2(A, Z/qZ)_K . \end{array}$$

Since $H_1(F, Z/qZ) \to H_1(A, Z/qZ)$ is an isomorphism, it follows that

$$d^2_{2,1} : H_2(K, H_1(A, Z/qZ)) \to H_2(A, Z/qZ)_K$$
is trivial.

Consider next the spectral sequence for coefficient group Z. The homomorphism $Z \to Z/qZ$ allows us to compare it with the spectral sequence above so that we have the commutative diagram

$$\begin{array}{ccc} H_2(K, H_1(A, Z)) & \xrightarrow{\quad 3 \quad} & H_2(A, Z)_K \\ 1 \big\downarrow & & \big\downarrow 2 \\ H_2(K, H_1(A, Z/qZ)) & \xrightarrow{\quad 4 \quad} & H_2(A, Z/qZ)_K . \end{array}$$

Arrow 4 is trivial, arrow 1 is the identity, and since $H_2(A, Z) = \Lambda^2 A$ is a K-direct summand of $H_2(A, Z/qZ)$, arrow 2 is a monomorphism. Hence arrow 3 is trivial as required.

We next show $\bar{d}^2_{2,1} : H_2(K, B) \to H_2(B)_K$ is trivial. Let

$$\mathcal{A} = \mathrm{Hom}_{K_1}(Z(K), Z) \cong Z(K) \otimes_{K_1} Z$$

be the K-module induced from the trivial K_1-module Z. Consider $\mathcal{A}^K \cong Z$ as above and put $\mathcal{B} = \mathcal{A}/\mathcal{A}^K$.

Notice that, as above, $\mathcal{A}^K$ consists of the diagonal elements of $\mathcal{A} = Z^{(K:K_1)}$ and is a Z direct summand of $\mathcal{A}$, but not a K-direct summand. In particular $\mathcal{B}$ is free abelian.

The following diagram summarizes the relation of these groups to A and B:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Z & \xrightarrow{q} & Z & \rightarrow & Z/qZ & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{A} & \xrightarrow{q} & \mathcal{A} & \rightarrow & A & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{B} & \xrightarrow{q} & \mathcal{B} & \rightarrow & B & \rightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Consider also the commutative diagram

$$\begin{array}{ccc}
K\cdot\mathcal{A} & \rightarrow & K\cdot A \\
\downarrow & & \downarrow \\
K\cdot\mathcal{B} & \rightarrow & K\cdot B
\end{array}$$

of semi-direct products and the induced homomorphisms of corresponding spectral sequences. We may construct from this situation the following diagram.

$$\begin{array}{ccc}
H_2(K, A) & \xrightarrow{1} & H_2(A)_K \\
\downarrow{\scriptstyle 4} & & \downarrow \\
H_2(K, B) & \xrightarrow{2} & H_2(B)_K \\
\uparrow{\scriptstyle 5} & & \uparrow \\
H_2(K, \mathcal{B}) & \xleftarrow{3} & H_2(\mathcal{B})_K .
\end{array}$$

I claim that arrow 3 is trivial and that the images of arrow 4 and 5 span $H_2(K, B)$. Since we showed above that arrow 1 is trivial, it will follow that arrow 2 is trivial as required.

To prove that the two images span $H_2(K, B)$, consider the following diagram which also arises from the situation described above.

$$\begin{array}{ccccccc}
 & & H_2(K, A) & \xrightarrow{6} & H_1(K, \mathcal{A}) & \xrightarrow{7} & H_1(K, \mathcal{A}) \\
 & & \downarrow{\scriptstyle 4} & & \downarrow{\scriptstyle 3} & & \\
H_2(K, \mathcal{B}) & \xrightarrow{5} & H_2(K, B) & \longrightarrow & H_1(K, \mathcal{B}) & & \\
 & & & & \downarrow{\scriptstyle 2} & & \\
 & & & & H_0(K, Z) & & \\
 & & & & \downarrow{\scriptstyle 1} & & \\
 & & & & H_0(K, \mathcal{A}). & &
\end{array} \tag{10}$$

Arrow 7 is multiplication by q. Since $H_1(K, \mathfrak{A}) \cong H_1(K_1, Z) = K_1/K_1'$, hypothesis (B) tells us that arrow 7 is trivial; hence 6 is an epimorphism.

Again, arrow 1 is the (co-) transfer $H_0(K, Z) = Z \to H_0(K_1, Z) = Z$ which is multiplication by the index $(K:K_1)$, certainly a monomorphism. By exactness arrow 2 is trivial and arrow 3 is an epimorphism. Given that arrows 3 and 6 are epimorphisms, it follows immediately from the diagram that $H_2(K, B)$ is the sum of the images of arrows 4 and 5.

Finally, we show that $d_{2,1}^2 : H_2(K, \mathfrak{B}) \to H_2(\mathfrak{B})_K$. Since K is finite and $\mathfrak{B}$ is finitely generated, $H_2(K, \mathfrak{B})$ is finite. Hence it suffices to show that $H_2(\mathfrak{B})_K$ is a free abelian group.[1] For this purpose we investigate the induced homomorphism

$$H_2(\mathfrak{A}) = \Lambda^2\mathfrak{A} \to H_2(\mathfrak{B}) = \Lambda^2\mathfrak{B}.$$

It is certainly an epimorphism. To determine its kernel, proceed as follows. Let N generate the subgroup $\mathfrak{A}^K \cong Z$ of $\mathfrak{A}$. Define the K-homomorphism $\mathfrak{A} \to \Lambda^2\mathfrak{A}$ by $a \to a \wedge N$. Its kernel is clearly $\mathfrak{A}^K$ so that there is induced a monomorphism $\mathfrak{B} \to \Lambda^2\mathfrak{A}$. A closer examination shows that

$$0 \to \mathfrak{B} \to \Lambda^2\mathfrak{A} \to \Lambda^2\mathfrak{B} \to 0$$

is an exact sequence of K-modules. Thus we get the exact sequence

$$H_0(K, \mathfrak{B}) \xrightarrow{\ 1\ } H_0(K, \Lambda^2\mathfrak{A}) \xrightarrow{\ 2\ } H_0(K, \Lambda^2\mathfrak{B}) \to 0. \tag{11}$$

I claim now that $H_0(K, \mathfrak{B})$ is finite and $H_0(K, \Lambda^2\mathfrak{A})$ is free. It follows that arrow 1 is trivial so that 2 is an isomorphism and $H_2(\mathfrak{B})_K = H_0(K, \Lambda^2\mathfrak{B})$ is free.

To prove the former contention, consider the exact sequence

$$H_0(K, Z) \xrightarrow{\ 3\ } H_0(K, \mathfrak{A}) \xrightarrow{\ 4\ } H_0(K, \mathfrak{B}) \to 0.$$

As we saw above, arrow 3 is the endomorphism of Z produced by multiplication by $(K:K_1)$ so that $H_0(K, \mathfrak{B})$ is $Z/(K:K_1)Z$ and is finite. In order to prove that $H_0(K, \Lambda^2\mathfrak{A})$ is free, we use the following lemma.

LEMMA. *Let K be finite, and let M be a Z-free K-module. Then M_K is Z-free if and only if $H^{-1}(K, M) = (0)$.*

Proof. $H^{-1}(K, M) = M_N/[K, M]$ where M_N is the kernel of the norm homomomorphism. If $M_N = [K, M]$, then $H_0(K, M) = M/M_N = \text{Im } N$ which is free since it is a subgroup of M. The converse is clear since $H^{-1}(K, M)$ is in any event a torsion group and it is a subgroup of $H_0(K, M)$.

To derive the desired result from the lemma, consider the homomorphism

$$\phi : \Lambda^2\mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$$

[1] I am indebted to unpublished work of Charlap and Vasquez for this argument. In a paper which will appear in the future, Charlap and Vasquez prove a much more general fact about the vanishing of d^2. Since I need only the special case I have included a separate proof here of that case.

defined by $x \wedge y \to x \otimes y - y \otimes x$ where x, y range over a basis of $\mathcal{A}$. The composite $\psi \cdot \phi$ of ϕ with the defining epimorphism

$$\psi : \mathcal{A} \otimes \mathcal{A} \to \Lambda^2 \mathcal{A}$$

is multiplication by 2. Since K is of odd order, it follows that $H^{-1}(H, \Lambda^2\mathcal{A})$ is a direct summand of $H^{-1}(K, \mathcal{A} \otimes \mathcal{A})$ since both groups are finite of odd order.

However, since $\mathcal{A} \otimes \mathcal{A}$ is clearly the Z-module of a permutation representation of K, it is a direct sum of K-modules, each of which is induced from the trivial module Z for some subgroup of K. It follows by Shapiro's Lemma that $H^{-1}(K, \mathcal{A} \otimes \mathcal{A})$ is a direct sum of groups of the form $H^{-1}(H, Z)$ with H ranging over some family of subgroups of K. Thus $H^{-1}(K, \mathcal{A} \otimes \mathcal{A})$ is trivial, and the desired result follows. (One may show quite easily, but with more writing, that $\Lambda^2\mathcal{A}$ is a direct sum of induced modules of the desired kind.)

(II) (7) and (8) are monomorphisms.

First we consider

$$H_1(K, A) \to H_1(K, B). \tag{7}$$

This homomorphism can be fitted into the sequence

$$H_1(K, A^K) \xrightarrow{1} H_1(K, A) \xrightarrow{2} H_1(K, B).$$

As above, arrow 1 is the transfer $K/K' \to K_1/K_1'$ and is trivial by hypothesis (A).

To show that

$$H_2(A)_K \to H_2(B)_K \tag{8}$$

is a monomorphism, we use the groups $\mathcal{A}$ and $\mathcal{B}$ as above. We have the exact commutative diagram of K-modules.

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{B} & \to & \Lambda^2\mathcal{A} & \to & \Lambda^2\mathcal{B} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B & \to & \Lambda^2 A & \to & \Lambda^2 B & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{12}$$

whose bottom row is constructed as above. (12) induces the exact diagram

$$\begin{array}{ccccccc} \mathcal{B}_K & \xrightarrow{1} & (\Lambda^2\mathcal{A})_K & \xrightarrow{2} & (\Lambda^2\mathcal{B})_K & \longrightarrow & 0 \\ \Big\downarrow 2 & & \Big\downarrow & & \Big\downarrow & & \\ B_K & \xrightarrow{3} & (\Lambda^2 A)_K & \xrightarrow{4} & (\Lambda^2 B)_K & \longrightarrow & 0. \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

Since we saw in (I) that arrow 1 is trivial, and since arrow 2 is an epimorphism, arrow 3 must also be trivial, and arrow 4 is a monomorphism as required.

This completes the proof of Theorem 2.

We may extend the notion of being terminal. Say that a p-group G is p-terminal if it cannot be extended without uspetting the "p-central series"

$$G_1 = G > G_2 = [G, G_1]G_1^p > \cdots > G_n = [G, G_{n-1}]G_{n-1}^p > \cdots$$

More generally, if q' is a power of p, define the notion of a q'-terminal p-group in the obvious way.

If we examine the proof of Theorem 2, it is clear what modifications must be made in order to conclude that G is q'-terminal. We must at each state attempt to replace the coefficient group of Z by $Z/q'Z$. First, if $q' \geq q$, then $H_1(B, Z) \to H_1(B, Z/qZ)$ is an isomorphism and similarly for A. It follows from (I) that

$$d_{2,1}^2 : H_2(K, H_1(B, Z/q'Z)) \to H_2(B, Z/q'Z)_K$$

is trivial. (Similarly for A.)

If $q' \geqq q$, the remark above shows easily that

$$H_1(K, H_1(A, Z/q'Z)) \to H_1(K, H_1(B, Z/q'Z))$$

is a monomorphism.

To investigate

$$H_2(A, Z/q'Z)_K \to H_2(B, Z/q'Z)_K$$

examine the diagram

$$\begin{array}{ccccccccc} 0 & \to & H_2(A) & \to & H_2(A, Z/q'Z) & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H_2(B) & \to & H_2(B, Z/q'Z) & \to & B & \to & 0 \end{array}$$

which induces

$$\begin{array}{ccccccc} & & & & (A^K)_K & & \\ & & & & \downarrow 4 & & \\ H_2(A)_K & \to & H_2(A, Z/q'Z) & \to & A_K & \to & 0 \\ 1\downarrow & & 2\downarrow & & 3\downarrow & & \\ H_2(B)_K & \to & H_2(B, Z/q'Z) & \to & B & \to & 0. \end{array}$$

Arrow 3 will be a monomorphism if and only if arrow 4 is trivial. However, the latter homomorphism is the (co-) transfer $H_0(K, Z/qZ) \to H_0(K_1, Z/qZ)$ which is simply multiplication on Z/qZ by $(K:K_1)$. Hence if $(K:K_1) \geq q$, the proof goes through and we get the following theorem.

THEOREM 2′. *Let $q' \geq q$ be powers of the odd prime p. Let K be a p-group, K_1 a subgroup, A the K-module $Z(K) \otimes_{K_1} Z/qZ$. Suppose*

(A) *the transfer, $K/K' \to K_1/K_1'$, is trivial,*

(B) $(K_1/K_1')^q = 1$,

(C) *the last nontrivial term of the q'-central series is contained in* A^K, *and*
(D) $(K:K_1) \geq q$.
Then the semi-direct product $K \cdot A$ *is* q'*-terminal.*

6. Some applications

It is natural at this point to ask whether the peculiar hypotheses of Theorem 2 do actually occur in interesting cases.

First, we remark that condition (C) will be true provided the intersection of the conjugates of K_1 in K is trivial. In this case, A^K is the center of $G = K \cdot A$ and certainly contains the last nontrivial term of the lower central series of G. (Also, it contains the last nontriival term of the lower q'-central series for q' any power of p.)

The simplest example in which the hypotheses of Theorem 2 apply is that in which K_1 is trivial itself.

THEOREM 3. *Let K be a nontrivial finite p-group, p an odd prime. Let A be the group algebra of K over Z/qZ where q is a power of p. The semi-direct product $K \cdot A$ is a terminal p-group. (Also, it is q'-terminal for each power of p, $q' \geq q$.)*

Remarks. 1. That the hypothesis $q' \geq q$ is necessary in theorems of this kind is clear from the example $P[Z/p^2Z(P)]$ (where P is cyclic of order p). This group is *not* p-terminal.

2. As was remarked in Section 1, the dihedral group of order 8 falsifies Theorem 3 for $p = 2$.

The simplest way to construct more terminal p-groups is to form direct products.

PROPOSITION 4. *A direct product of non-abelian terminal p-groups is a terminal p-group. (Similarly for q-terminal p-groups, q a power of p.) More generally, if the factors have unequal class, only those with the largest class need be terminal.*

Proof. Let G_1 and G_2 be p-groups of class c_1 and c_2 respectively. Suppose $c_1 \geq c_2$ for convenience. If $c_1 \geq c_2$, the last nontrivial term of the lower central series of $G = G_1 \times G_2$ is $\gamma_{c_1}(G_1) \times (1)$, otherwise if $c_1 = c_2 = c$, it is $\gamma_c(G_1) \times \gamma_c(G_2)$. Let $\bar{G}_1 = G_1/\gamma_{c_1}(G_1)$ and $\bar{G}_2 = G_2$ or $G_2/\gamma_c(G_2)$ as required.

We show that

$$\operatorname{Ker}\{H_2(G_1 \times G_2, Z) \to H_2(\bar{G}_1 \times \bar{G}_2, Z)\} \leq pH_2(G_1 \times G_2, Z).$$

By the Künneth Theorem for direct products [2, Chap. VI, Sec. 3], we have

$$H_2(G_1 \times G_2, Z) \cong H_2(G_1) \oplus H_1(G_1) \otimes H_1(G_2) \oplus H_2(G_2)$$

and similarly for $\bar{G}_1 \times \bar{G}_2$. By hypothesis,

$$\operatorname{Ker}\{H_2(G_i) \to H_2(\bar{G}_i)\} \leq pH_2(G_i), \qquad i = 1, 2.$$

Also, $H_1(G_i) \to H_1(\bar{G}_i)$ are isomorphisms for $i = 1, 2$. The contention above follows easily.

(The proof for the case of q-terminal groups is exactly the same if q is prime. If q is not prime, the argument becomes a bit more involved since the Künneth formula is harder to apply. Alternately, it is possible to argue by means of the Lyndon spectral sequence in all cases.)

We come now to the application which motivated Theorem 2.

THEOREM 5. *Let p be an odd prime. The p-Sylow subgroups of the Symmetric groups S_n—for $n \geq p^2$—are terminal p-groups.*

Proof. The p-Sylow subgroup of the Symmetric group S_n is the direct product of subgroups S_r, each of which is itself the p-Sylow subgroup of the Symmetric group S_{p^r} for some $p^r \leq n$. Also, if $n \geq p^2$, then there is at least one factor S_r with $r \geq 2$. See [5, Chap. 5, Sec. 5.9.] We shall show that the groups S_r—for $r \geq 2$—are terminal p-groups. Since the class of each such S_r is strictly greater than 1, it will follow by Proposition 4 that the Sylow subgroup of S_n is terminal even if it contains cyclic factors isomorphic to S_1.

To prove that S_r is terminal for $r \geq 2$, we exhibit this group as a semi-direct product of the type dealt with in Theorem 2. For this purpose we utilize the construction of these groups as iterated wreath products starting with a cyclic group P of order p. (See [5, Chap. 5, Sec. 5.9].)

For convenience in notation consider S_{r+1} with $r \geq 1$. It may be realized as the semi-direct product $S_r \cdot A$ where A is a vector space of dimension p^r over Z/pZ and S_r—being a transitive permutation group of degree p^r—acts on A by permuting a fixed choice of basis elements. It is clear, then, that A is the S_r-module induced from the trivial S_r^*-module Z/qZ, where S_r^* is the subgroup of S_r fixing, say, the first symbol. We must show that S_r^* satisfies hypotheses (A), (B), and (C) of Theorem 2 (with $q = p$). Since the intersection of the conjugates of S_r^* is the subgroup fixing all symbols, it is trivial and hypothesis (C) follows immediately. We shall establish (A) and (B) by induction on r.

For this purpose, we use the associativity of the wreath product. Namely, S_r may be constructed as the semi-direct product $P \cdot (S_{r-1})^p$. Here the p-fold direct product $(S_{r-1})^p$ is the subgroup of S_r stabilizing a partition of the p^r symbols into p blocks, each with p^{r-1} symbols, the first block containing the first symbol. Also, P is the subgroup of S_{p^r} permuting these blocks cyclicly. (P acts on S_{r-1}^p in the obvious way.) (See [5].) Clearly, S_r^* is the subgroup—$S_{r-1}^{*p} \times (S_{r-1})^{p-1}$ of $(S_{r-1})^p$. Since S_{r-1}/S'_{r-1} is elementary abelian, and since $S_0^* = (1)$, hypothesis (B) follows by induction.

Finally, we prove hypothesis (A), the triviality of the transfer

$$H_1(S_r) \to H_1(S_r^*).$$

Since transfer is transitive it suffices to prove that the intermediate transfer

$$H_1(S_{r-1}^p) \to H_1(S_{r\,1}^* \times S_{r-1}^{p-1})$$

is trivial. However, transfer behaves simply with respect to direct products. Namely, let $G_1 \geq H_1$, $G_2 \geq H_2$ be a pair of groups with subgroups. Put $m_i = (G_i:H_i)$. We have

$$H_1(G_1 \times G_2) = H_1(G_1) \oplus H_2(G_2);$$

similarly for $H_1 \times H_2$. Also, denoting by

$$\mathfrak{V}_i : H_i(G_i) \rightarrow H_1(H_i), \qquad i = 1, 2$$

and

$$\mathfrak{V} : H_1(G_1 \times G_2) \rightarrow H_1(H_1 \times H_2)$$

the appropriate transfers, we have

$$\mathfrak{V} = m_2 \mathfrak{V}_1 + m_1 \mathfrak{V}_2 . \tag{13}$$

Let $r \geq 2$. Since $(S_{r-1}:S_{r-1}^*) \geq p$ and since $H_1(S_{r-1})$ is elementary abelian, formula (13) establishes the usual induction hypothesis, and we descend to the case $r = 1$. In that case S_1 is generated by a cycle of length p and S_1^* is trivial so that hypothesis (A) is established for $r \geq 1$. This completes the proof.

Remark. To prove these groups are q-terminal for q a power of p, one may repeat the (valid) mod q argument paralleling the argument given above. Alternately, since the factors of the lower central series are elementary abelian, it is possible to show that all the notions of terminality are the same.

7. Epilogue

As was mentioned in the introduction, the applications in Section 6 may be obtained more easily by direct group theoretical methods. In this section I shall show how to prove Theorem 5 more easily by homological methods. In particular, I shall prove still another generalization which was suggested by N. Blackburn. In what follows below, consider only p-groups.

Let $1 \rightarrow A \rightarrow U \rightarrow G \rightarrow 1$ be a central extension with $A^q = 1$. Let V be a normal subgroup of U containing A, and put $L = V/A$ As in Section 3, denote the co-transgression by $\psi : H_2(G, Z/qZ) \rightarrow A$. Then, as in formula (6), we have

$$\psi(\text{Ker } \{H_2(G, Z/qZ) \rightarrow H_2(G/L, Z/qZ)\} = ([U, V]V^q) \cap A. \tag{14}$$

In particular, let $L = \zeta(G)$ be the center of G and put $q = 0$. As in Section 3, we have

PROPOSITION 6. *Ker* $\{H_2(G) \rightarrow H_2(G/\zeta(G))\} \leq pH_2(G)$ *if and only if whenever* $U/A \cong G$ *with* A *central, then* $\zeta(U)/A \cong \zeta(G)$.

(If this condition is satisfied, say that G is *unicentral.*)

Remarks. 1. If $\zeta(G)$ is cyclic of prime order, the G is unicentral if and only if it is is terminal. In particular, the groups S_r are terminal since Theorem 7 below applies to them. (This proof is still too hard from an absolute point of view.)

2. If G is unicentral, it can never be true that $G = U/\zeta(U)$, that is, G cannot be the group of inner automorphisms of another group.

3. Examination of the proof of Theorem 4 shows that we are really showing that the groups under consideration are unicentral.

THEOREM 7 (Blackburn). *Let K be a finite group. Suppose H is unicentral and also $\zeta(H) \leq [H, H]$. Then the standard wreath product $K \cdot H^k$, $k = |K|$, is unicentral.*

Proof. Let $\bar{H} = H/\zeta(H)$. Since $\zeta(K \cdot H^k) < \zeta(H)^k$, it suffices to prove that

$$\text{Ker}\,\{H_2(K \cdot H^k) \to H_2(K \cdot \bar{H}^k)\} \leq pH_2(K \cdot H^k).$$

Since $H_k(H^k) = Z(K) \otimes H_1(H)$ is cohomologically trivial for K, the spectral sequence yields a natural decomposition.

$$H_2(KH^k) = H_2(K) \oplus H_2(H^k)_K$$

and similarly for $K\bar{H}^k$. Hence, we need only prove that

$$\text{Ker}\,\{H_2(H^k)_K \to H_2(\bar{H}^k)_K\} \to pH_2(H^k)_K.$$

Since H is unicentral and since $\zeta(H) \leq [H, H]$, the *proof* of Proposition 4 shows that H^k is unicentral. Hence, the fundamental exact sequence for

$$1 \to \zeta(H)^k \to H^k \to \bar{H}^k \to 1$$

yields the exact sequence

$$0 \to Q \to H_2(H^k) \to H_2(\bar{H}^k) \to \zeta(H)^k \to 0,$$

where $Q \leq pH_2(H^k)$. Since the right hand end is isomorphic with $Z(K) \otimes \zeta(H)$ and, thus, is a cohomologically trivial K-module, we have the exact sequence

$$Q \to H_2(H^k)_K \to H_2(\bar{H}^k)_K .$$

The desired inclusion follows.

COROLLARY 8. *Let Q be a generalized quaternion group, and let K be any finite group. Then the standard wreath product $K \cdot Q^k$ is unicentral (and terminal.)*

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