THE DENSEST IRREGULAR PACKING OF THE MORDELL CUBIC NORM-DISTANCE

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1. Introduction

Let $\mathcal{S}$ be a star-domain, symmetric about 0. A set of points $\mathcal{P}$ is said to provide a packing for $\mathcal{S}$ if the domains $\{\mathcal{S} + \mathcal{P}\}$, where $P \in \mathcal{P}$, have the property that no domain $\mathcal{S} + P_0$ contains the center of another in its interior. We also say that $\mathcal{P}$ is an $\mathcal{S}$-admissible point set. A packing $\mathcal{P}$ is said to be regular if $\mathcal{P}$ is an $\mathcal{S}$-admissible lattice; it is said to be semi-regular if it is the union of a lattice $\mathcal{L}$ and a translate of $\mathcal{L}$; it is said to be irregular if it is not necessarily a lattice or a union of lattices.

The domain of action method developed by M. Rahman has been employed by Sister M. R. Von Wolff to determine that the densest irregular packing of the star-domain $\mathcal{S}_1 : | xy | \leqq 1$ has the density of an $\mathcal{S}_1$-critical lattice.

It is the purpose of this paper to exhibit further the strength of the domain of action method in the determination of the best possible irregular packing of non-convex regions. The method is applied to the star-domain $\mathcal{S}_2 : | y(3x^2 - y^2) | \leqq 1$ which is equivalent to the region

$$\mathcal{S}_2 : | x^3 - x^2y - 2xy^2 - y^3 | \leqq 1$$

for which L. J. Mordell [3] has determined the critical lattices. R. P. Bambah [1] gave another proof of this result by determining the critical determinant and the two critical lattices of the region $\mathcal{S}_2$.

Consider the square $| x | < t, | y | < t$ Let $A(t)$ denote the number of points of a set $\mathcal{P}$ in the square; then the density of $\mathcal{P}$, denoted $\mathfrak{D}(\mathcal{P})$, is defined as $\limsup_{t\to\infty} A(t)/4t^2$.

From the definition it follows that for any two-dimensional lattice $\mathcal{L}$ the density $\mathfrak{D}(\mathcal{L})$ is the reciprocal of its mesh.

A norm-distance, [2, p. 103], is a real-valued function $n(X) = n(OX)$, defined on the plane, such that $n(X)$ is

(1) nonnegative; i.e., $n(X) \geqq 0$;
(2) continuous;
(3) homogeneous; i.e., $n(tX) = | t | n(X)$, where t is any real number.

A convex distance function or Minkowski distance, m, is a norm-distance with the additional properties:

(1) $m(PQ) = 0$ implies $P = Q$.
(2) $m(PQ) \leqq m(PR) + m(RQ)$.

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Let $\mathcal{P}$ be a point set in the plane and let m be a Minkowski distance. The domain of action [4, p. 16] $D(P) = D(P, m, \mathcal{P})$ of a point P, relative to m and $\mathcal{P}$, is the set of all points X in the plane for which

$$m(PX) \leqq m(QX) \qquad \text{where } Q \in \mathcal{P},\ Q \neq P,$$

when this set is the closure of the set of all points in the plane which are closer to P than any other point of $\mathcal{P}$.

An exception to this definition occurs when there is a straight line segment in the boundary of the convex body which determines m and when P and Q lie on a line parallel to that line segment. In this case the intersection of $D(P)$ and $D(Q)$ contains interior points and the definition must be adjusted to apportion points in the common region to $D(P)$ and to $D(Q)$ equally in some consistent manner [6, p. 500]. In the following application we avoid this exceptional case by choice of the convex body defining m.

Let $| D(P) |$ denote the area of $D(P)$. If M is the greatest lower bound of $\{ | D(P) | \}$ for $P \in \mathcal{P}$, then it follows that the density $\mathfrak{D}(P)$ of the point set $\mathcal{P}$ is less than or equal to $1/M$.

Subsequent discussion will pertain to the star-domain $\mathcal{S}_2$. Henceforth we shall refer to $\mathcal{S}$ and mean always $\mathcal{S} = \mathcal{S}_2 : | y(3x^2 - y^2) | \leqq 1$.

2. The domain of action of $\mathcal{S}$

The norm-distance n determined by $\mathcal{S}$ is

$$n(P) = | y(3x^2 - y^2) |^{1/3} \qquad \text{where } P = (x, y).$$

Let the Minkowski distance m be defined by the hexagon inscribed in $\mathcal{S}$ described by the lines

$$\sqrt{3}\,| x | + | y | = 2, \qquad | y | \leqq \sqrt{3}\,| x |,$$
$$| y | = 1, \qquad | y | \geqq \sqrt{3}\,| x |.$$

Then

$$m(P, Q) = \tfrac{1}{2}(\sqrt{3}\,| x | + | y |), \qquad | y | \leq \sqrt{3}\,| x |$$
$$= | y |, \qquad | y | \geqq \sqrt{3}\,| x |.$$

Thus, if $\mathcal{P}$ is $\mathcal{S}$-admissible,

$$m(PQ) \geqq n(PQ) \geqq 1$$

for any two distinct points P and Q of $\mathcal{P}$.

Let 0 be an arbitrary point of $\mathcal{P}$ and be taken as origin. Then $D(0) = \bigcap_P D(0, m, P), P \in \mathcal{P}, P \neq 0$.

Figure 1 illustrates $\mathcal{S}$ and $D(0)$ determined by six points of an $\mathcal{S}$-admissible point set.

We will discuss in detail the domain of action of 0 with respect to a point $P_1 = (x_1, y_1)$ with the property $0 < y_1 < \sqrt{3}\, x_1$ and $\sqrt{3}\, y_1 > x_1 > 0$.

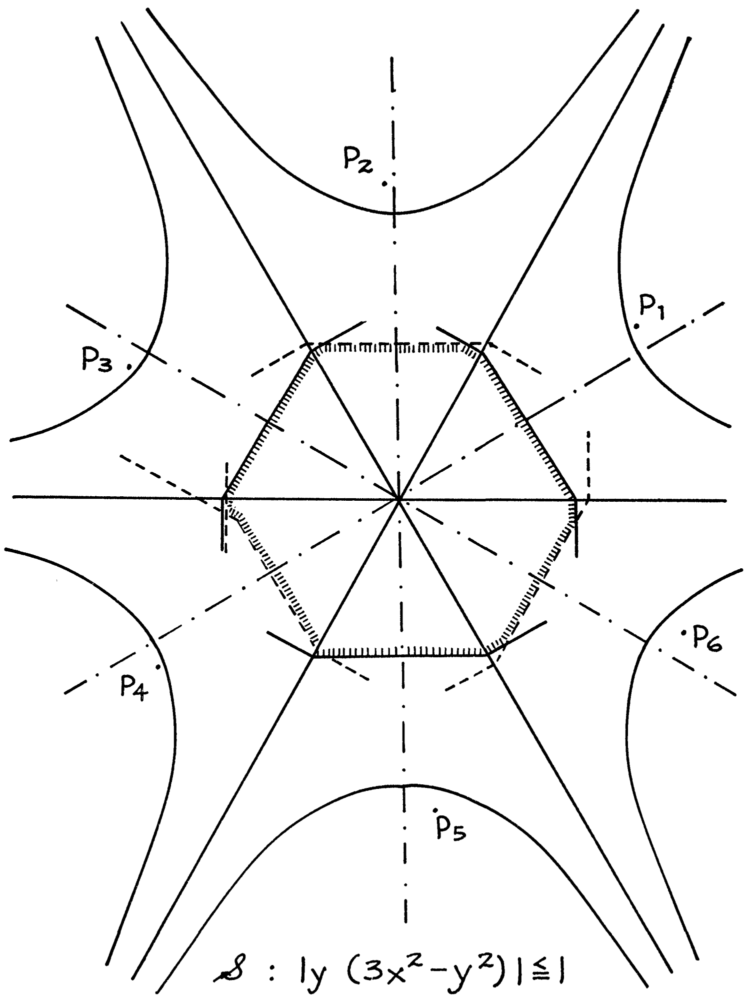


FIGURE 1

For P in any other sextant the definitions are similar.

The equations of the lines that determine $D(P, m, P_1)$ are obtained from Table 1. The lines are illustrated in Figure 2.

The notation that will be used for lines determining $D(0)$ is L_{ij} for line j in the i-th sextant, $i = 1, 2, \cdots, 6; j = 1, 2, \cdots, 8$.

From Figure 2 we see that the region bounded by the lines L_{15}, L_{12}, L_{13}, L_{17}, and L_{16} is contained in $D(0, m, P_1)$.

Formulas for vertices of $D(0)$ are illustrated in detail for $D(0, m, P_1)$ See Figure 3. Vertices of $D(P, m, P_i)$ for P_i in other sextants are found by reflecting lines from P_1 in the x or y axes or by rotation through integral multiples of $\pi/3$.

TABLE I

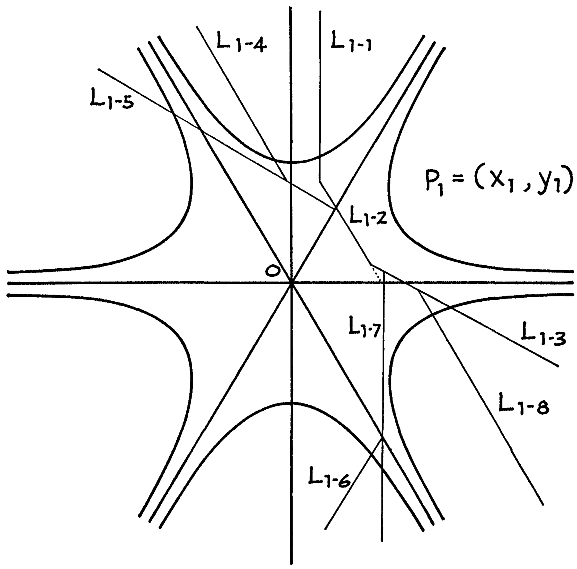
S_i	$m(X) \leqq m(P_i)$	x	y	$x-x_1$	$y-y_1$	Conclusions
S_1	$\sqrt{3}x + y \leqq \sqrt{3}\mid x - x_1 \mid + \mid y - y_1 \mid$	+	+	+	+	Does not occur
		+	+	+	−	Does not occur
		+	+	−	+	$2\sqrt{3}x \leqq \sqrt{3}x_1 - y_1$
		+	+	−	−	$2(\sqrt{3}x + y) \leqq \sqrt{3}x_1 + y_1$
	$\sqrt{3}x + y \leqq 2 \mid y - y_1 \mid$	+	+		+	Does not occur
		+	+		−	$x + \sqrt{3}y \leqq 2y_1/\sqrt{3}$
S_2	$2x \leqq \mid y - y_1 \mid + \sqrt{3}\mid x - x_1 \mid$		+	+	+	Does not occur
			+	+	−	Does not occur
			+	−	+	$y + \sqrt{3}x \leqq \sqrt{3}x_1 - y_1$
			+	−	−	$3y + \sqrt{3}x \leqq (\sqrt{3}x_1 + y_1)$
	$y \leqq \mid y - y_1 \mid$			+	+	Does not occur
				+	−	Does not occur
S_3	$\sqrt{3}\mid x \mid + \mid y \mid \leqq \sqrt{3}\mid x - x_1 \mid + \mid y - y_1 \mid$	−	+	−	+	Trivial
		−	+	−	−	Trivial
S_4	$-\sqrt{3}x - y \leqq \sqrt{3}(x_1 - x) + (y_1 - y)$	−	−	−	−	Trivial
S_5	$2\mid y \mid \leqq \mid y - y_1 \mid + \sqrt{3}\mid x - x_1 \mid$		−	−	−	$\sqrt{3}x - y \leqq \sqrt{3}x_1 + y_1$
			−	+	−	Does not occur
	$\mid y \mid \leqq \mid y - y_1 \mid$		−		−	Trivial
S_6	$\sqrt{3}x - y \leqq y_1 - y + \sqrt{3}\mid x - x_1 \mid$	+	−	+	−	Does not occur
		+	−	−	−	$2\sqrt{3}x \leqq \sqrt{3}x_1 + y_1$
	$\sqrt{3}x - y \leqq 2(y_1 - y)$	+	−		−	$\sqrt{3}x + y \leqq 2y_1$

Define

$$t(P_i) = (\sqrt{3}\mid y_i \mid - \mid x_i \mid)/2, \quad i = 1, 3, 4, 6.$$
$$= \mid x_i \mid, \quad i = 2, 5.$$

If we ignore the influence of points in the neighboring sextants, we see in Figure 4 that $D(0)$ in each sextant consists of a large triangle OAC formed by the asymptotes and the line L_{i-2} and of a small triangle K the area of which is given in terms of $t(P)$. Then $D(0)$ is seen to depend on the two functions $m(P_i) = m_i$ and $t(P_i) = t_i$ of the points which determine $D(0)$.

Notation. Consider 0 as origin and divide the plane into sextants by lines $y = 0$, $\sqrt{3}x = y$, and $\sqrt{3}x = -y$. Denote the sextant between the positive



$L_{11}: x = \dfrac{\sqrt{3}x_1 - y_1}{2\sqrt{3}}$ $\qquad L_{12}: \sqrt{3}x + y = \dfrac{\sqrt{3}x_1 + y_1}{2}$

$L_{13}: x + \sqrt{3}y = \dfrac{2y_1}{\sqrt{3}}$ $\qquad L_{14}: \sqrt{3}x + y = \sqrt{3}x_1 - y_1$

$L_{15}: x + \sqrt{3}y = \dfrac{\sqrt{3}x_1 + y_1}{\sqrt{3}}$ $\qquad L_{16}: \sqrt{3}x - y = \sqrt{3}x_1 + y_1$

$L_{17}: x = \dfrac{\sqrt{3}x_1 + y_1}{2\sqrt{3}}$ $\qquad L_{18}: \sqrt{3}x + y = 2y_1$

FIGURE 2

x-axis and the line $\sqrt{3}x = y$ by S_1. Proceeding in counterclockwise direction number the succeeding sextants S_j, $j = 2, 3, \cdots, 6$. The part of S_i between the first asymptote and the sextant bisector is denoted S_{ia}. The part of S_i between the sextant bisector and the second asymptote is denoted S_{ib}. Points on the sextant bisector may be considered as belonging to either S_{ia} or S_{ib}.

A point $P \in S_i$ is denoted P_i.

Frequently in reference to a point P_i we shall speak of the points P_j, $j = 1 \pm 2$. This is always understood to mean j is congruent to 1 ± 2 modulo 6.

When $m(P_i) > m(P_{i+1})$, then lines $L_{i-1,5}$ and $L_{i+1,7}$ affect the domain of action in Sextant i. When this occurs we shall refer to each of these lines as the cutoff from Sextant $i \pm 1$.

For simplicity and when there is no possibility of confusion $D(0)$ will be referred to as D and $m(0P)$ will be replaced by $m(P)$, $m(P_i)$ by m_i, and $t(P_i)$ by t_i.

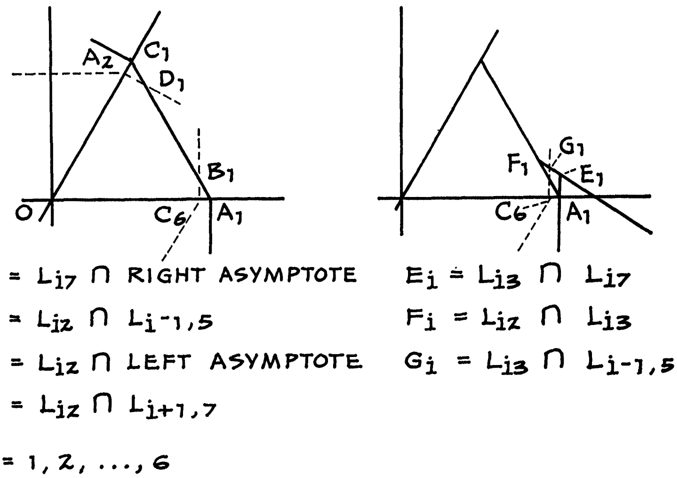


FIGURE 3
Vertices of $D(0, m, P_1)$

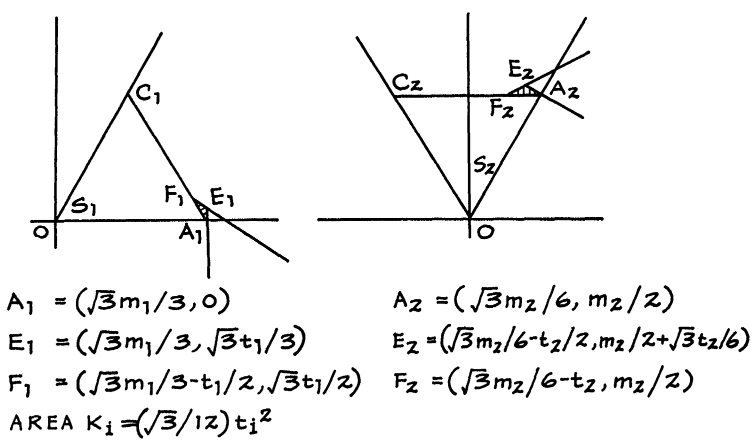


FIGURE 4
Formulas for triangles $K_i = A_iE_iF_i$

$\mathfrak{L}^{**}$ denotes the critical lattice generated by two generators with minimal m-distance A_1, A_2 in S_{1a} and S_{2a}.

$D^{**}(0) = D^{**}$ denotes the domain of action of $\{A_i\}$, $i = 1, 2, \cdots, 6$.

$\mathfrak{L}^{*}$ denotes the critical lattice generated by two independent points in $\{B_i\}$ where each B_i is the reflection in the bisector of S_i of the point A_i defined above. $D^{*}(0) = D^{*}$ represents the domain of action of $\{B_i\}$, $i = 1, 2, \cdots, 6$.

$m^{*} = m(A) = m(B)$, where A is any minimal generator of $\mathfrak{L}^{**}$, B is any minimal generator of $\mathfrak{L}^{*}$. $t^{*} = t(A) = t(B)$ for the same points A and B.

$\Delta M_i = m(P_i) - m(B) = m(P_i) - m^{*}$. When the meaning is clear, ΔM_i will be abbreviated Δ_i. $\Delta t_i = t(P_i) - t^{*}$.

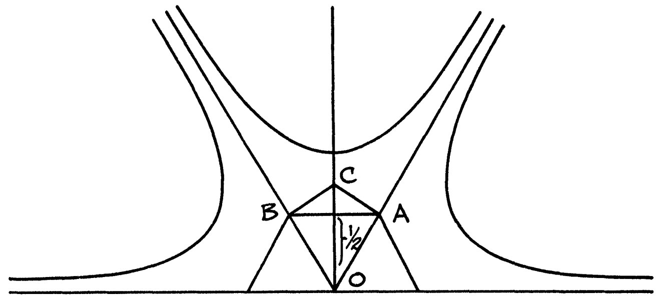


FIGURE 5

Let $D_0(0) = \{X \mid m(X) \leqq \frac{1}{2}\}$. $D(0)$ always contains $D_0(0)$. $|D_0(0)| =$ area of $D_0(0) = \sqrt{3}/2$.

If $\{P_i\}$, $i = 1, 2, \cdots 6$, determines $D(0)$, let $m(\boldsymbol{P}) = \min\{m(P_i)\}$. Then the basic hexagon of $\{P_i\}$ is formed by lines $L_{i-2} : 2m(X) = m(\boldsymbol{P}), X = (x, y)$.

Let $\mathcal{S}_i = \{\mathcal{S} + P_i\}$ be a translate of $\mathcal{S}$ with center P_i. Then H_{ij} indicates the boundary curve of $\mathcal{S}_i$ in the j-th sextant (P_i considered as origin).

3. Some properties of the domain of action of $\mathcal{S}$

$\mathcal{P}$ is an $\mathcal{S}$-admissible point set. Our concern is with points 0 in $\mathcal{P}$ such that $D(0)$ is small. If, for some $0 \in \mathcal{P}$, $|D(0)| < \Delta(\mathcal{S})$, then at the outset we may restrict our considerations to points 0 with the following properties.

(i) There is a point of $\mathcal{P}$ in each S_i.

For, if there is no point of $\mathcal{P}$ in some sextant, say S_2, then triangle ABC of area $\sqrt{3}/36$ is in $D(0)$. See Figure 5. Then

$$\begin{aligned} |D(0)| > |D_0(0)| + \text{area } ABC &= \sqrt{3}/2 + \sqrt{3}/36 \\ &= .9141 \cdots > \Delta(\mathcal{S}) \\ &= .87656773 \cdots. \end{aligned}$$

(ii) If $D(0)$ is small, then $m(P_i) < 1.04$, $i = 1, 2, \cdots, 6$.

By "small" we mean $D(0)$ such that $|D(0)| \leqq |D^*(0)| = \Delta(\mathcal{S})$. The restriction on the domain of $\{P_i\}$ follows from the fact that P_i determines at least a trapezoid of area

$$T_i = \sqrt{3}/12\,(2X_i - 1)(5 - 6X_i),$$

where $2X_i = m(P_i)$, in addition to the area of $D_0(0)$.

Thus, if $m(P_i) = 1.04$, for even one P_i,

$$|D(0)| > |D_0(0)| + T_i = \sqrt{3}/2 + .0108 > \Delta(\mathcal{S}).$$

Figure 6 illustrates the domain of $\{P_i\}$ for small $D(0)$. Each P_i lies in the closed region defined by arc $R_{ia}\hat{P}_i R_{ib}$ on H_{0i} and chord $R_{ia} R_{ib}$, where $m(R_{ia}) = m(R_{ib}) = 1.04$. $\hat{P}_i$ is that point in S_i with $m(P_i) = 1$.

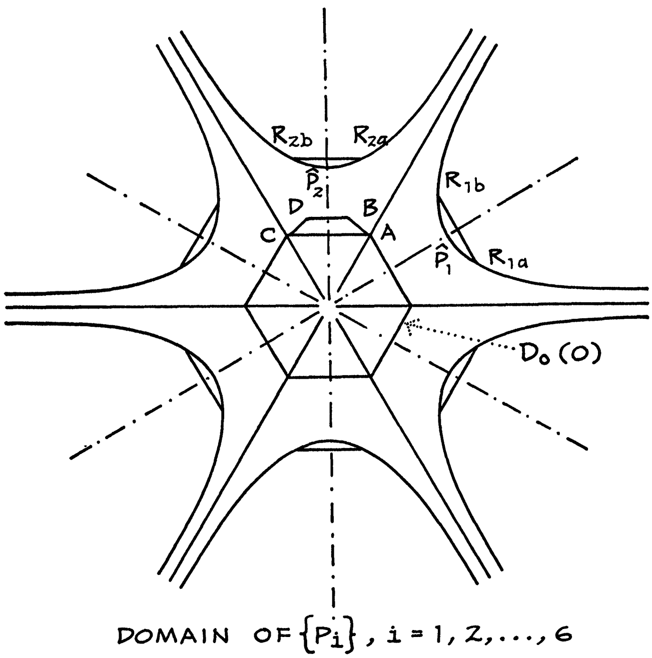


FIGURE 6

(iii) There cannot be two points in the same sextant which have a significant influence on a small $D(0)$.

This follows from (ii) and the admissibility requirement on the P_i.

(iv) Six and only six points P_i influence a small $D(0)$.

This follows from (i) and (iii).

(v) The following lemma is due to M. Rahman [4, p. 37]. If $\mathcal{P}$ is $\mathcal{S}$-admissible and $P_i \in \mathcal{P}$, $i = 1, 2, \cdots, 6$, then $m(p_i) \geqq m^*$ for at least four values of i.

Define $\mathcal{R}_i$ to be the regions described by chords $R_{ia} R_{ib}$ and arcs $R_{ia} \hat{P}_i R_{ib}$ on H_{0i}, (Figure 6), where $m(R_{ia}) = m(R_{ib}) = r_i > 1$, and $r_i = r_j$, $1 \leq i < j \leq 6$.

Consider $\mathcal{R}_i$ and $\mathcal{R}_j$ for any $i = 1, 2, \cdots, 6; j \equiv i \pm 2 \pmod 6$. The proof of (v) depends on the fact that for $P_i \in \mathcal{R}_i$, $P_j \in \mathcal{R}_j$, $n(P_iP_j)$ is maximum when $P_i = A_i$, (or B_i) while $P_j = A_j$ (or B_j).

Yet for $P_i = A_i$, $P_j = A_j$ (or $P_i = B_i$, $P_j = B_j$) $n(P_iP_j) = 1$. So for P_i in an $\mathcal{S}$-admissible point set $\mathcal{P}$, P_i and P_j cannot both have $m(P_k) < m^*$, $k = i, j$.

4. Statement of the problem

$\mathcal{S} : | y(3x^2 - y^2) | \leqq 1$. $\{\mathcal{P}\}$ is the set of all $\mathcal{S}$-admissible point sets. Find the l.u.b. of $\mathcal{D}(\mathcal{P})$, $\mathcal{P} \in \{\mathcal{P}\}$. This will be determined by proving that for P

in any $\mathcal{S}$-admissible point set, $|D(P)| \geq \Delta(\mathcal{S})$, which is equivalent to the assertion

$$\mathfrak{D}(\mathcal{P}) \leqq 1/\Delta(\mathcal{S}).$$

THEOREM 1. *Let* 0 *be an element of an* $\mathcal{S}$*-admissible point set* $\mathcal{P}$. *Then*

$$|D(0)| \geqq \Delta(\mathcal{S}).$$

THEOREM 2. *Let* 0 *be an element of an* $\mathcal{S}$*-admissible point set* $\mathcal{P}$. *Then* $|D(0)| = \Delta(\mathcal{S})$ *if and only if* 0 *together with the points of* $\mathcal{P}$ *contributing to* $D(0)$ *are points of a critical lattice of* $\mathcal{S}$.

5. Outline of the proof of Theorem 1

Consider the possible situations regarding the m-distance of the six points P_i which determine $D(0)$.

There are three cases

(1) $m_i \geqq m^*$ for all six values of i.
(2) $m_i < m^*$ for only one value of i.
(3) $m_i < m^*$ for two values of i.

We prove that in each of these cases $|D(0)| \geqq \Delta(\mathcal{S})$. Since (1), (2) and (3) are the only possible situations, Lemmas 2, 3, and 4 will prove Theorem 1.

LEMMA 2. *If* $m(P_i) \geqq m^*$ *for all six values of* i, *then* $|D(0)| \geqq \Delta(\mathcal{S})$.

LEMMA 3. *If* $m(P_i) < m^*$ *for only one value of* i, *then* $|D(0)| \geqq \Delta(\mathcal{S})$.

LEMMA 4. *If* $m(P_i) < m^*$ *for two values of* i, *then* $|D(0)| \geqq \Delta(\mathcal{S})$.

6. A method for proving a function positive in a given domain

In the proof of the result of Theorem 1 it will be necessary to demonstrate that a function is positive in a neighborhood of critical lattice points. Lemma 1 establishes the method that is central to the proof.

LEMMA 1. *Given a twice differentiable function* $f(x)$ *on the interval* $[a, b]$ *of the real line with the properties*

(1) $f(a) \geqq 0$
(2) $f'(a) > 0$
(3) $|f'(x)| \leqq M$ *for some* M *and all* $x \in [a, b]$
(4) $\mu \leqq f''(x)$ *for some* μ *and all* $x \in [a, b]$,

(a) *then there exists a point* a' *in* $[a, b]$ *such that* $f(x) > 0$ *for* $x(a, a']$ *and further,*

(b) *if there exist* $n + 1$ *equidistant points* $x_0, x_1, \cdots, x_n$, *satisfying*

(i) $a \leqq a' = x_0 < x_1 < \cdots < x_n = b$
(ii) $f(x_i) > 0, i = 0, 1, \cdots, n$
(iii) $f(x_i) > (M(b - a')/2n = C, i = 0, 1, \cdots, n,$

then $f(x) > 0$ *for* $x \in [a', b]$.

Proof of (a). *Case* 1. $\mu \geqq 0$. $\mu \geqq 0$. Then $a' = b$. For $f''(x) \geqq 0$ implies that $f'(x)$ is non-decreasing and, since $f'(a) > 0$, it follows that $f'(x) > 0$ in $[a, b]$.

Case 2. $\mu < 0$. Choose $\varepsilon > 0$ such that $f'(a) > \varepsilon \cdot |\mu| = -\varepsilon \cdot \mu$ Consider $x \varepsilon [a, a']$ where $a' = a + \varepsilon$. Then by the mean value theorem

$$f'(x) - f'(a) = f''(x^*)(x - a) \geqq \mu \cdot \varepsilon \quad \text{for all } x \in [a, a'] \text{ and some } x^* \in (a, a').$$

Then $f'(x) \geqq f'(a) + \mu \cdot \varepsilon > 0$ for all $x \in [a, a']$. Thus $f'(x)$ is positive in $[a, a']$ and so $f(x) > 0$ in $(a, a']$.

Proof of (b). For any point $x \in [a', b]$ it follows from the mean value theorem that

$$| f(x) - f(x_j) | = | f'(x^*) | | x - x_j | < (M(b - a'))/2n = C$$

where x_j is the partition point nearest to x and $x^*(x, x_j)$ or $x^* \in (x_j, x)$ as the case may be. Then

$$-C < f(x) - f(x_i) < C.$$

So $f(x) = f(x) - f(x_i) > -C + C = 0$ for all x in $[a', b]$.

The aim of the remaining lemmas is to compare $D(0)$ for 0 in a critical lattice with a $D(0)$ determined by points in a neighborhood of a critical lattice. Proposition 4 shows that the battle is waged in two alternate sextants at a time and the outcome depends largely on property (v) of the domain of action stated above.

7. Proof of Lemma 2

LEMMA 2. *If* $m(P_i) \geqq m^*, i = 1, 2, \cdots, 6$, *then* $|D(0)| \geqq \Delta(\mathcal{S})$.

Proof. Compare $D(0)$ with $D^{**}(0)$. If $D(0)$ contains $D^{**}(0)$, then the lemma is proved, since in this case

$$| D(0) | \geqq | D^{**}(0) |.$$

Even if in each sextant S_i we have

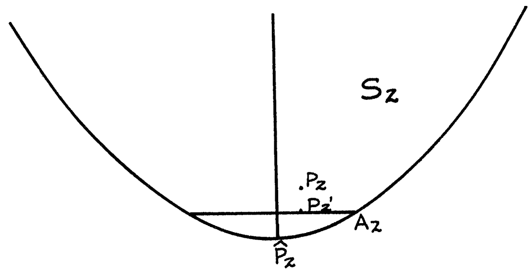
$$(1) \qquad | D(0) \cap S_i | \geqq | D^{**}(0) \cap S_i |,$$

then the result follows, since

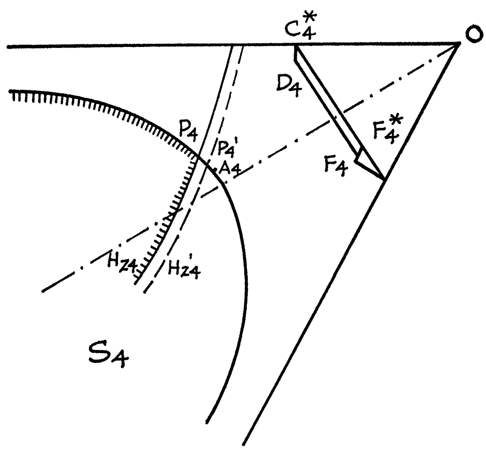
$$\textstyle\sum_{i=1}^{6} | D \cap S_i | \geqq \sum_{i=1}^{6} | D^{**} \cap S_i | = \Delta(\mathcal{S}).$$

Suppose condition (1) fails to be satisfied for at least one i. Say $P_i \in S_{ia}$. Then $| D \cap S_i | < | D^{**} \cap S_i |$. By hypothesis $m_i \geqq m^*$. Then t_i must be less than t^*.

It is sufficient to prove that whenever t_i is less than t^* it follows that the minimum gain in S_{i+2} is greater than the maximum loss in S_i. By applying this inequality to three alternate sextants it follows that the total loss due to $t_i < t^*$ for one or more values of i is compensated by the total gain in three



RELATIVE POSITIONS OF Pz AND Pz'.



SHADED AREA IN S4 INDICATES ADMISSIBLE REGION FOR P4 SUCH THAT m (P4) IS SMALL

FIGURE 7

alternate sextants. Applying this result to two sets of alternate sextants completes the proof of Lemma 2.

Prove that the minimum gain in S_{i+2} is greater than the maximum loss in S_i. There is no loss of generality in assuming that P_i is in S_{ia}. By a symmetric argument the result for P_i in S_{ib} follows.

$P_i \in S_{ia}$. Define P_i' to be the point Q in S_{ia} such that $t(Q) = t(P_i)$ and $m(Q) = m^*$.

Define $\Delta K_i =$ area $(K_i^* - K_i)$ where triangle K_i is determined by P_i' and triangle K_i^* is determined by A_i.

K_i represents the maximum area loss in S_i. Since $m(P_i) \geqq m^*$, there is no area loss in the sextants adjacent to S_i. Hence ΔK_i is greater than or equal to the loss in S_i and $S_{i\pm 1}$ due to $t_i < t^*$.

Determine the minimum area gain in S_j. There is no loss of generality in letting $i = 2, j = 4$. See Figure 7.

For $Q_4 \in S_4$ such that $m(Q_4)$ is small we have $m(Q_4) \geqq m(P_4)$ where

$$P_4 = H_{04} \cap H_{24}$$

and H_{24} has center P_2.

Further, if $P_4' = H_{04} \cap H_{24}'$, where H_{24}' has center P_2',

$$m(P_4) > m(P_4') > m^*.$$

Let

$$\Delta M_4 = m(Q_4) - m^* \quad \text{and} \quad \Delta M_4' = m(P_4') - m^*.$$

Then $\Delta M_4 > \Delta M_4' > 0$.

The gain $| D \cap S_4 | - | D^{**} \cap S_4 |$ is at least equal to the area of trapezoid $C^*F^*FD_4$. Denote this area T_4.

T_4 is an increasing function of $m(P_4)$ so $T_4 > T_4' > 0$, where T_4 denotes the area determined by arbitrary admissible Q_4 in S_4 and T_4' denotes the area determined by P_4'. Then T_j' is less than or equal to the area gain in S_j.

We now prove that T_j' is greater than ΔK_i.

Suppose $i = 2$. Let $g(x_2, y_2, x_4, y_4) = T_4' - \Delta K_2$.

For $P_2 \in S_{2a}$, $t(P_2) < t^*$ only when $x_2 \in [0, x_2^*) = [0, .063717 \cdots)$.

The aim is to show that $g(x_2, y_2, x_4, y_4) > 0$ in $[0, x_2^*)$.

Consider P_2' and P_4' as defined above. $P_2' = (x_2, y_2)$. $P_4' = (x_4, y_4)$.

The points 0, P_2', and P_4' are related by the conditions

$$(2) \quad \begin{aligned} &(a) \quad m(P_2') = m^* \\ &(b) \quad n(OP_4') = 1 \\ &(c) \quad n(P_2'P_4') = 1 \end{aligned}$$

Then $g(x_2, y_2, x_4, y_4)$ may be considered as a function of one independent variable, say x_2. Then $g(x_2, y_2, x_4, y_4) = f(x_2)$.

We know that $f(x_2) = 0$ at x_2^*. $A_2 = (x_2^*, y_2^*)$. If we can prove that f is a decreasing function of x_2 in $[0, x_2^*]$, then it will follow that f is positive in $[0, x_2^*)$.

$$df/dx_2 = df/dz_1 = \textstyle\sum_{i=1}^4 A_i Z_i,$$

where $A_i = \partial f/\partial z_i$ and $Z_i = dz_i/dz_1$ and $z_1 = x_2$; $z_2 = y_2$; $z_3 = z_4$; $z_4 = y_4$.

$$f(x_2) = T_4' - \Delta K_2 = \tfrac{1}{2}\Delta M_4' \cdot W - (\sqrt{3}/12)(x_2^{*2} - x_2^2)$$

where $W = \frac{2}{3}\sqrt{3}m^* - 2x_2^* = 1.034 \cdots$; and

$A_1 = \frac{1}{6}\sqrt{3}x_2$; $A_2 = 0$; $A_3 = -\frac{1}{4}\sqrt{3}W$; $A_4 = -\frac{1}{4}W$.

From conditions (3) define

$$g_1 = m(P_2') - m^*$$

$$g_2 = n(OP_4') - 1$$

$$g_3 = n(P_2'P_4') - 1.$$

Then equations

$$\frac{dg_i}{dz_1} = \sum_{j=1}^{4} \frac{\partial g_i}{\partial z_j} \cdot \frac{dz_j}{dz_1} = \sum_{j=1}^{4} \frac{\partial g_i}{\partial d_j} \qquad Z_j = 0, i = 1, 2, 3 \tag{3}$$

give a system which we can solve for the Z_j since the determinant of the coefficients is not zero.

Let $X_4 = \partial g_2/\partial x_4$; $Y_4 = \partial g_2/\partial y_4$; $X_{24} = \partial g_3/\partial x_2$; $Y_{24} = \partial g_3/\partial y_2$. Then from (3) we have

$$Z_1 = 1$$

$$Z_2 = 0$$

$$X_4 Z_3 + Y_4 Z_4 = 0$$

$$X_{24} Z_1 + Y_{24} Z_2 - X_{24} Z_3 - Y_{24} Z_4 = 0$$

from which we obtain

$$Z_3 = X_{24} Y_4/D \quad \text{and} \quad Z_4 = -X_{24} X_4/D$$

where $D = X_{24} Y_4 - Y_{24} X_4$.

$$f'(x_2) = A_1 + A_3 Z_3 + A_4 Z_4 = \tfrac{1}{6}\sqrt{3}x_2 - \tfrac{1}{4}\sqrt{3}W \cdot Z_3 - \tfrac{1}{4}W \cdot Z_4 .$$

To prove that $f'(x_2)$ is negative in $[0, x_2^*]$; i.e., that

$$-A_3 Z_3 > A_1 + A_4 Z_4$$

or

$$A_3 \cdot (Y_4/X_4) \cdot Z_4 > A_4 Z_4 + A_1 . \tag{4}$$

It is not difficult to show that Z_4 and X_4 are both negative in $[0, x_2^*]$. Thus (4) is equivalent to

$$A_3 Y_4 > A_4 X_4 + A_1(X_4/Z_4) \tag{5}$$

Computer estimates verify that the left side of (5) is greater than $\sqrt{3}/4 \cdot W \cdot (1.8) = .8059$ while the right side of (5) is less than $\frac{1}{4} \cdot W \cdot (2.42) + .0476 = .6892$.

Then (4) is satisfied and thus $f'(x_2) < 0$ and therefore f is a decreasing function in $[0, x_2^*]$. $f(x_2^*) = 0$ and f decreasing imply that f is positive in $[0, x_2^*)$.

Then $T_4' - \Delta K_2 > 0$ which proves the claim that the area gain in S_j is greater than the area loss in S_i . So

$$t_i < t^* \Rightarrow T_{i-2}' > \Delta K_i . \tag{6}$$

Consider three alternate sextants S_i , S_{i+2} , S_{i-2} . If $t_j < t^*$ in two or three sextants, the corresponding inequalities hold:

$$\begin{aligned} T_{i+2}' &> \Delta K_i \\ T_i' &> \Delta K_{i-2} \\ T_{i-2}' &> \Delta K_{i+2} . \end{aligned} \tag{7}$$

Adding the inequalities (7) it follows that

(8) $$\sum_l T_1' > \sum_l \Delta K_1, \qquad l = i, i+2, i-2.$$

Applying (8) to both sets of alternate sextants the proof of Lemma 2 is complete.

8. Further properties of $\mathcal{S}$

In the proofs of Lemmas 3 and 4 we shall need some inequalities, properties of $\mathcal{S}$ which we now state and prove as propositions.

(a) PROPOSITION 1. *Suppose* $P_i \in S_{ia}$ *and* $m(P_i) \leqq m^*$. *Then* $2\Delta M_{i-2} \geqq t_2^{*2} - t_i^2$ *for* $0 \leqq t(P_i) \leqq t^*$.

Suppose $P_i \in S_{ib}$ *and* $m(P_i) \leqq m^*$. *Then* $2\Delta M_{i+2} \geqq t^{*2} - t_i^2$ *for* $0 \leqq t(P_i) \leqq t^*$.

This proposition provides a relation between $\Delta m(P_j)$ and $\Delta t(P_i)$, $j \equiv i + 2 \pmod 6$.

Proof. Assume that $P_i \in S_{ia}$. By a symmetric argument the result for $P_i \in S_{ib}$ will follow.

Let $i = 2$. By hypothesis $m(P_2) \leqq m^*$.

Let P_2' be that point in S_{2a} such that $t(P_2') = t(P_2)$ and $m(P_2') = m^*$.

Let $P_6 = H_{06} \cap H_{25}$ and $P_6' = H_{06} \cap H_{25}'$ where H_{25} and H_{25}' have centers P_2 and P_2' respectively. See Figure 8.

Clearly H_{25} intersects H_{06} below H_{25}'. Then $m(P_6) \geqq m(P_6')$.

Any point in S_{6a} has m-distance greater than $m(P_6)$ hence greater than or equal to $m(P_6')$ which is minimum when $P_6' = A_6$, a minimal generator of $\mathcal{L}^{**}$. Then $\Delta M_6 \geqq \Delta M_6' \geqq 0$ while $t(P_2') = t(P_2)$.

Therefore, if we prove the result for P_2' and P_6', it will be true for P_2 and arbitrary points $P \in S_6$.

Let $P_2' = (x_2, y_2)$ and $P_6' = (x_6, y_6)$.

Then we assume that P_2' and P_6' are related by the conditions

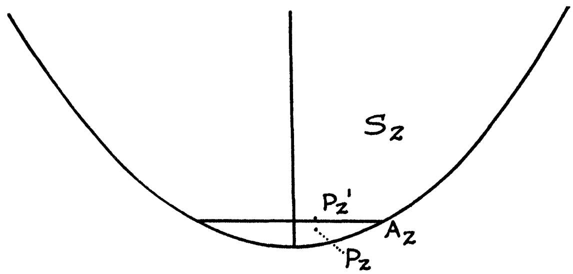
(1)
(i) $m(P_2') = m^*$
(ii) $n(OP_6') = 1$
(iii) $n(P_2' P_6') = 1$.

Let $g(x_2, y_2, x_6, y_6) = 2\Delta M_6 - t^{*2} + t_2^2$. The function g may be considered to be a function of a single variable, say x_2. Then $g(x_2, y_2, x_6, y_6) = f(x_2)$. $P_2 \in S_{2a}$ and $m(P_2') \leqq m^*$ imply $x_2 \in [0, x_2^*]$ where $A_2 = (x_2^*, y_2^*)$.

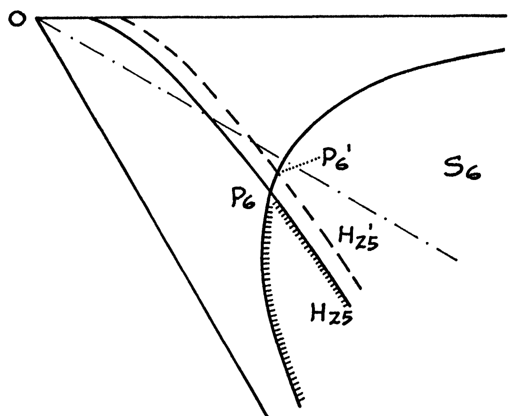
Prove that $f(x_2) > 0$ in $[0, x_2^*)$. We know that $f(x_2^*) = 0$. If $f'(x_2) < 0$ in $[0, x_2^*)$, then it will follow that f decreases to 0 and is, therefore, positive in $[0, x_2^*)$.

$$df/dx_2 = df/dz_1 = \sum_{i=1}^4 A_i Z_i$$

where $A_i = \partial f/\partial z_i$ and $Z_i = dz_i/dz_1$ and $z_1 = x_2$; $z_2 = y_2$; $z_3 = x_6$; $z_4 = y_6$. $A_1 = 2x_2$; $A_2 = 0$; $A_3 = \sqrt{3}$; $A_4 = -1$.



RELATIVE POSITIONS OF P_z AND P_z'.



SHADED AREA INDICATES ADMISSIBLE REGION OF S_6.

FIGURE 8

From conditions (1) define

$$g_1 = m(P_2') - m^*$$
$$g_2 = n(OP_6') - 1$$
$$g_3 = n(P_2' P_6') - 1.$$

Then equations

$$\frac{dg_i}{dz_1} = \sum_{j=1}^{4} \frac{\partial g_i}{dz_j} \frac{dz_j}{dz_1} = \sum_{j=1}^{4} \frac{\partial g_i}{\partial z_j} \qquad z_j = 0, i = 1, 2, 3 \tag{2}$$

give a system which we can solve for Z_i, since the determinant of the coefficients is not zero. Let

$$X_6 = \partial g_2/\partial x_6\,; \qquad Y_6 = \partial g_2/\partial y_6\,; \qquad X_{26} = \partial g_3/\partial x_2\,; \qquad Y_{26} = \partial g_3/\partial y_2\,.$$

Then from (2) we have

$$Z_1 = 1; \qquad Z_2 = 0;$$

$$X_{26} Z_1 + Y_{26} Z_2 - X_{26} Z_3 - Y_{26} Z_4 = 0$$

$$X_4 Z_3 + Y_4 Z_4 = 0$$

from which we obtain

$$Z_4 = \frac{X_{26}}{X_{26} \dfrac{Y_6}{X_6} - Y_{26}}$$

$$Z_3 = -(Y_6/X_6) Z_4 .$$

Now

$$f'(x_2) = A_1 + A_3 Z_3 + A_4 Z_4 = 2x_2 - \sqrt{3}(Y_6/X_6) Z_4 - Z_4 .$$

Prove that $f'(x_2) < 0$ in $[0, x_2^*]$, i.e., that

(3) $$2x_2 - \sqrt{3}(Y_6/X_6) Z_4 < Z_4 .$$

Since it can easily be shown that in $[0, x_2^*]$, Z_4 is positive, it is possible to determine a lower bound for Z_4 in this domain.

$$Z_4 = \frac{-X_{26}}{X_{26} \dfrac{Y_6}{X_6} - Y_{26}} \geqq \frac{|X_{26}|}{|X_{26}| \dfrac{Y_6}{|X_6|} + |Y_{26}|}$$

$$> \frac{7.25}{(7.94)\dfrac{(1.18)}{(2.8)} + 5.83} = 79009232.$$

Since $Z_4 > 0$, (3) is equivalent to

(4) $$2x_2/Z_4 + \sqrt{3}(Y_6/|X_6|) < 1$$

then

$$2x_2/Z_4 + \sqrt{3}(Y_6/|X_6|) < .128/.79 + (1.733)(1.174/2.8$$
$$= .89194202.$$

Then (4) is satisfied and thus $f'(x_2) < 0$. So f is a decreasing function in $[0, x_2^*]$. $f(x_2^*) = 0$ and f decreasing imply that f is positive in $[0, x_2^*)$.

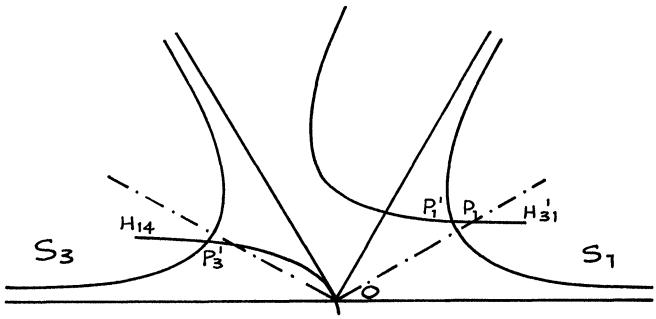
(b) PROPOSITION 2. *If* $m(P_i) < m^*$ *then* $6\Delta M_j > 5 \mid \Delta M_i \mid, j = i \pm 2$.

Proof. (a) Let $i = 1$. $P_1 \in S_{1b}$. By symmetry, what is true of $P_1 \in S_{1b}$ will be true of $P_i \in S_{ib}$.

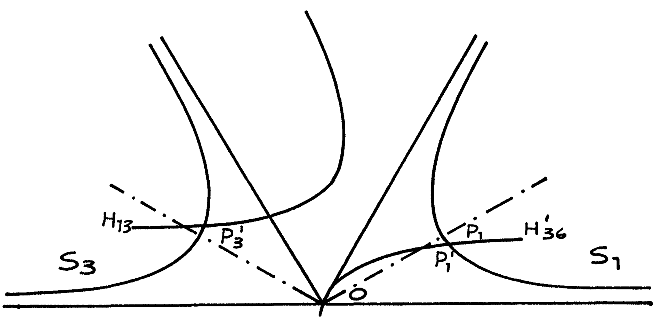
By hypothesis $m(P_1) < m^*$. Hence $\Delta M_1 < 0$. Recall that

$$\Delta M_1 = m(P_1) - m^*.$$

From property (v) we know that $m(P_3) \geqq m^*$.



a. $P_1 \in S_{1b}$. RELATIVE POSITIONS OF P_1, P_1', AND P_3'.



b. $P_1 \in S_{1a}$. RELATIVE POSITIONS OF P_1, P_1', AND P_3'.

FIGURE 9

Prove that $6\Delta M_3 > 5 \mid \Delta M_1 \mid$.

See Figure 9a. $P_3' = H_{03} \cap H_{14}$; $P_1' = H_{01} \cap H_{31}'$ where H_{31}' has center P_3'. P_3' is the point in the admissible region of S_3 with minimum m-distance from 0. So

(1) $$\Delta M_3 \geqq \Delta M_3'.$$

$m(P_1') < m(P_1)$ since $m(P_1)$ is an increasing function of x_1 on H_{31}'. Then

(2) $$\mid \Delta M_1' \mid > \mid \Delta M_1 \mid$$

Then it suffices to prove that

(3) $$6\Delta M_3' > 5 \mid \Delta M_1' \mid.$$

Let $g(x_1, y_1, x_3, y_3) = 6\Delta M_3' - 5 \mid \Delta M_1' \mid = 6\Delta M_3' + 5\Delta M_1'$. Prove that g is positive for $P_1' \in S_{1b}$, $m(P_1') < m^*$.

The points 0, P_1', and P_3' are related by the conditions

$$(4)\quad \begin{aligned} &\text{(i)} \quad n(OP_1') = 1 \\ &\text{(ii)} \quad n(P_3' P_1') = 1 \\ &\text{(iii)} \quad n(OP_3') = 1. \end{aligned}$$

Then $g(x_1, y_1, x_3, y_3)$ may be considered as a function of one independent variable, say x_1. Let $g(x_1, y_1, x_3, y_3) = f(x_1)$.

Prove that $f(x_1) > 0$ in $(a, b] = (.83768249 \cdots, .86602543 \cdots]$, i.e. when P_1' lies on H_{01} between B_1 and $P_1 = (\sqrt{3}/2, 1/2)$.

$$f'(x_1) = \textstyle\sum_{i=1}^4 A_i Z_i,$$

where $A_i = \partial f/\partial z_i$ and $Z_i = dz_i/dz_1$ and $z_1 = x_1$; $z_2 = y_1$; $z_3 = x_3$; $z_4 = y_3$.

If we let

$$g_1 = n(OP_1') - 1$$

$$g_2 = n(P_3' P_1') - 1$$

$$g_3 = n(OP_3') - 1,$$

then equations

$$\frac{dg_i}{dz_1} = \sum_{j=1}^{4} \frac{\partial g_i}{\partial z_j} \frac{dz_j}{dz_1} = \sum_{j=1}^{4} \frac{\partial g_i}{\partial z_j} Z_j = 0, \qquad i = 1, 2, 3,$$

give a system which we can solve for Z_i, since the determinant of the coefficients is not zero.

We use the derivative argument described above (Lemma 1). Let $n = 1$. We know that f' and f'' are rational functions of the variables z_i, $i = 1, 2, 3, 4$. Constraints (4) enable us to determine the domain of the variables whenever $x_1 \in [a, b]$.

An IBM 1620 computer was employed to find that $\mu = -33.0$ and $M = 1.7$. The computer also verified that $C = 0.3$ is a lower bound for $\{f(a'), f(b)\}$.

Then we know that $f(x_1) > 0$ and hence f is increasing in

$$[a, a'] = [.83768249 \cdots, .84068249 \cdots],$$

since ε is found to be 0.003. Since n is greater than $M \cdot (b - a')/2C = .4446/.6$, it follows that $f(x_1) > 0$ for $x_1 \in [a', b]$. Then we have proved f positive for $x_1 \in (a, b]$. Hence, if $P_i \in S_{ib}$ and $m(P_i) < m^*$, then

$$6\Delta M_{i+2} > 5 \mid \Delta M_i \mid.$$

(b) Let $i = 1$. $P_1 \in S_{1a}$. By symmetry what is true of $P_1 \in S_{1a}$ will be true of $P_i \in S_{ia}$.

See Figure 9b. $P_3 = H_{03} \cap H_{13}$. $P_1' = H_{01} \cap H_{36}'$. H_{36}' has center P_3'. Since $m(P_3) \geqq m(P_3')$, $\Delta M_3 \geqq \Delta M_3'$.

Also $m(P_1) > m(P_1')$, since $m(P_1)$ is an increasing function of x_1 on H_{36}'. Thus $\mid \Delta M_1' \mid > \mid \Delta M_1 \mid$.

Prove that $6\Delta M_3' > 5 \mid \Delta M_1' \mid$.

The points 0, P_1', and P_3' are related by conditions

$$(5) \quad \begin{array}{ll} \text{(iv)} & n(OP_1') = 1 \\ \text{(v)} & n(OP_3') = 1 \\ \text{(vi)} & n(P_3' P_1') = 1. \end{array}$$

Let $f(x_1)$ be defined as in part (a) with the difference that now

$$x_1 \in [b, d] = [.8660 \cdots, .9014 \ldots].$$

We have $f(d) = 0$. Prove that $f(x_1) > 0$ in $[b, d)$.

Let $n = 1$. We know that f' and f'' are rational functions of the four variables. Conditions (5) enable us to determine the domain of the variables whenever $x_1 \in [b, d]$. Constant values were found to be $\mu = -3.3$, $M = 1.05$. $C = 0.3$ was verified to be a lower bound for $\{f(b), f(d')\}$. In this case $f'(x_1)$ is negative and hence f is decreasing in

$$[d', d] = [.88140032 \cdots, .90140032 \cdots],$$

since ε is found to be equal to 0.02.

Since $n = 1$ is greater than $M(d' - b)/2C = .168/.6$, it follows that $f(x_1)$ is positive for x_1 in $[b, d']$.

The fact that f is positive in $[b, d']$ and f decreases to zero in $[d', d]$ give f positive in $[b, d]$ which proves the proposition in case (b).

The truth of the proposition for cases (a) and (b) implies its truth for all $i = 1, 2, \cdots, 6$.

Note that by a reflection in the bisector of S_i it follows from parts (a) and (b) above that we have

$$(6) \qquad 6\Delta M_{i-2} > 5 \mid \Delta M_i \mid \quad \text{if} \quad m(P_i) < m^*.$$

9. Proposition 3

Proposition 3 establishes the fact that when $m_i < m^*$ for one value of i, the area of $D(0)$ in three alternate sextants is always greater than

$$\tfrac{1}{2} \mid D^*(0) \mid = \tfrac{1}{2}\Delta(\mathcal{S}).$$

Notation. Suppose $m_i \neq m^*$. Denote the change of area in the i-th sector from the area of $D^* \cap S_i$ by

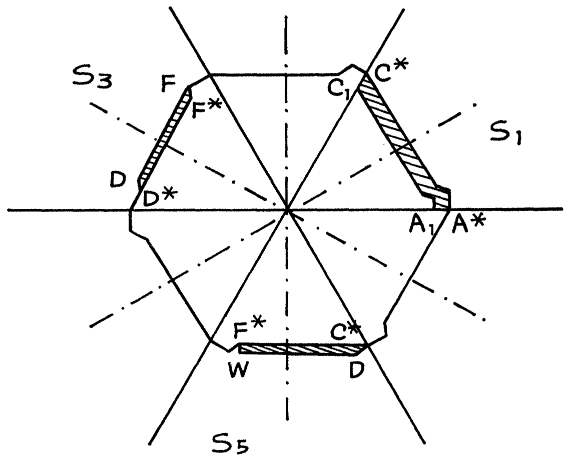
$$I_i^+, \quad \text{if } P_i \text{ determines an area gain}$$

$$I_i^-, \quad \text{if } P_i \text{ determines an area loss.}$$

Also let $\Delta M = \min\{\Delta M_{i+2}, \Delta M_{i-2}\}$.

PROPOSITION 3. *If $m_i < m^*$, then $I_i^- < \sum_j I_j^+$, $j \equiv i \pm 2 \pmod 6$.*

Proof. We must first determine the maximum area loss due to $m_i < m^*$.



SHADED REGION IN S_1 INDICATES MAXIMUM AREA LOSS DUE TO $m_1 < m^*$: I_1^+.
SHADED REGIONS IN S_3 AND S_5 INDICATE MINIMUM AREA GAINS DUE TO $m_1 < m^*$: I_3^+ AND I_5^+.

FIGURE 10

Let $i = 1$. Assume $P_1 \in S_{1b}$. I_1^- denotes area loss in Sextant 1. Figure 10 illustrates I_1^-.

In general I_i^- in S_i is determined by the following components: T_i = area of trapezoid $A_i A_i^* C_i^* C_i$; and $(K_i^* - K_i)$.

In all cases

$$(1) \qquad I_i^- \leqq \text{area } T_i + \text{area } (K_i^* - K_i).$$

Determine the minimum area gain in S_{i+2}. Figure 10 illustrates the case for $i = 1$. We see that

$$(2) \qquad I_{i+2}^+ + I_{i-2}^+ \geqq \text{area } (D^*F^*FD)_{i+2} + \text{area } (C^*F^*WD)_{i-2}.$$

The minimal case occurs when there is a significant area loss from two cutoffs from points in adjacent sextants. Property (v) (Section 3) insures that at most two cutoffs come from points with m-distance less than m^*.

We prove that

$$(3) \quad \text{area } T_i + \text{area } (K_i^* - K_i) \leqq \text{area } (D^*F^*FD)_{i+2} + \text{area } (C^*F^*WD)_{i-2}.$$

Then from (1), (2) and (3) we will have

$$(4) \qquad I_i^- \leqq I_{i+2}^+ + I_{i-2}^+.$$

$$(5) \quad \begin{aligned} &\text{Area } T_i + \text{Area } (K_i^* - K_i) \\ &\quad = 1/2\,|\,\Delta M_i\,|\,[(\sqrt{3}/3)m_i + (\sqrt{3}/3)m^*] + (\sqrt{3}/12)[t^{*2} - t_i^2] \end{aligned}$$

$$
\begin{aligned}
&\text{Area } (D^*F^*FD)_{i+2} + \text{Area } (C^*F^*WD)_{i-2} \\
&\quad \geqq (1/2)\Delta M_{i+2}[4\sqrt{3}/3 - (\sqrt{3}/3)m^* - (\sqrt{3}/3)m_{i+2} - t^* - t_{i+2}] \\
&\qquad + (1/2)\Delta M_{i-2}[(\sqrt{3}/2)m^* - 2t^* + \sqrt{3}/3 - (\sqrt{3}/3)m_{i-2}] \\
&\qquad + (1/2\Delta M_{i-2}(\sqrt{3}/3) \\
&\quad \geqq (1/2)\Delta M[5\sqrt{3}/3 - (\sqrt{3}/6)m^* - 3t^* - t_{i+2} - (\sqrt{3}/2)m_{i-2} \\
&\qquad - (\sqrt{3}/3)m_{i+2}] + (\sqrt{3}/6)\Delta M_{i-2} \\
&\quad = (6/5)\Delta M[25\sqrt{3}/36 - (5\sqrt{3}/72)m^* - (5/4)t^* - (5/12)t_{i+2} \\
&\qquad - (5\sqrt{3}/24)m_{i-2} - (5\sqrt{3}/36)m_{i+2}] + (\sqrt{3}/6)\Delta M_{i-2} \\
&\quad > (6/5)\Delta M[2\sqrt{3}/3 - (\sqrt{3}/18)m^* - (7/4)t^* - (\sqrt{3}/4)m_{i-2}] \\
&\qquad + (\sqrt{3}/6)\Delta M_{i-2} \\
&\quad > (6/5)\Delta M(.689) + (\sqrt{3}/6)\Delta M_{i-2}\,.
\end{aligned}
$$

But

$$
\begin{aligned}
(6/5)\Delta M(.689) + (\sqrt{3}/6)\Delta M_{i-2} &> (1/2)\,|\,\Delta M_i\,|\,(2\sqrt{3}/3)m^* \\
&\quad + (\sqrt{3}/12)(t^{*2} - t_i^2) \\
&> (1/2)\,|\,\Delta M_i\,|\,[(\sqrt{3}/3)m_i \\
&\quad + (\sqrt{3}/3)m^*] + (\sqrt{3}/12) \\
&\quad \cdot (t^{*2} - t_i^2)
\end{aligned}
\tag{6}
$$

by Propositions 2 and 1.

Equations (5) and (6) establish (3).

Hence $I_i^- \leqq I_{i+2}^+ + I_{i-2}^+$.

10. Proofs of Lemmas 3 and 4

LEMMA 3. *If $m_i < m^*$ for only one value of i, then $|\,D(0)\,| \geqq \Delta(\mathcal{S})$.*

Proof. The proof follows immediately from Proposition 3.

LEMMA 4. *If $m_i < m^*$ for two values of i, then $|\,D(0)\,| = \Delta(\mathcal{S})$.*

Proof. The proof follows upon application of Proposition 3 to both sets of three alternate sextants.

11. Proof of Theorem 2

THEOREM 2. *The point 0 is an element of an $\mathcal{S}$-admissible point set $\mathcal{P}$. Then $|\,D(0)\,| = \Delta(\mathcal{S})$ if and only if 0 together with the points of $\mathcal{P}$ contributing to $D(0)$ are points of a critical lattice of $\mathcal{S}$.*

Proof. Suppose $0 \in \mathcal{L}^*$ or $\mathcal{L}^{**}$. Computation of $|\,D(0)\,|$ shows $|\,D(0)\,| = \Delta(\mathcal{S})$.

Now suppose that $| D(0) | = \Delta(\mathcal{S})$. Theorem 1 says that $| D(0) | \geqq \Delta(\mathcal{S})$. In the proof of Theorem 1 we saw that if P_i is not a point of a critical lattice, then $| D(0) | > | D^*(0) |$. Hence $D(0) = D^*(0)$ implies that $P_i \in \mathcal{L}^*$ or $\mathcal{L}^{**}$. Hence 0 and the points of $\mathcal{P}$ contributing to $D(0)$ are points of an $\mathcal{S}$-critical lattice.

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