COMPATIBILITY OF IMPOSED DIFFERENTIABLE STRUCTURES

BY

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In an earlier paper [1], we constructed an obstruction theory for the problem of imposing a differentiable structure on a combinatorial manifold K. At the time, we did not know whether the structures obtained by means of this theory could be chosen to be compatible with the piecewise-linear structure of K. Our purpose here is to prove that they can be so chosen; the proof is a corollary of our recent work on the concordance problem [3]. For convenience in exposition, we restrict ourselves to the case of a non-bounded manifold M; extension to the case where M has a boundary is not difficult. Compactness is not assumed.

The proof involves the notion of a *smooth cell complex* in a differentiable n-manifold M. This is a collection C of closed cells imbedded in M with disjoint interiors such that

(1) For each cell c, Bd c is the union of finitely many cells of lower dimension.

(2) Each point of M has a neighborhood intersecting only finitely many cells of C.

(3) Each m-cell c is smooth, in the sense that it lies in a smooth m-dimensional non-bounded submanifold N_c of M.

(4) The submanifolds N_c can be chosen to intersect transversally.[2]

If the union $| C |$ of the cells of C equals M, we call C a *smooth cell decomposition* of M.

A *triangulation* of C is a triangulation $f : L \to | C |$ of $| C |$ which induces a triangulation of each cell of C; it is said to be *smooth* if f is a smooth imbedding of each simplex of L into M.

Now any smooth cell complex in M has a smooth triangulation, uniquely determined up to a piecewise linear homeomorphism (Lemma 1). In particular, the subcomplex of C consisting of a single cell c and its faces has a smooth triangulation $f : L \to | c |$; if L is a combinatorial ball (a piecewise-linear homeomorph of a simplex) then we call c a *smooth combinatorial cell.* (This is actually a restriction on c only in those dimensions where a triangulated topological ball need not be a combinatorial ball.) If every cell in C is combinatorial, we call C a *smooth combinatorial cell complex.*

For example, let us consider a manifold M with a smooth triangulation

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[2] This means that they form a locally-finite collection, and that for each x in M, there is a coordinate system $h : U \to R^n$ about x in M such that for each N_c containing x, $h(U \cap N_c)$ is an open subset of a plane in R^n.

$h : K \to M$. It is not hard to see that the collection of simplex images under h forms a smooth combinatorial cell decomposition of M. On the other hand, if we let K^* denote the usual dual cell decomposition of $|K|$, the images of the cells of K^* need not be smooth cells in M. Most of the hard work in [3] was to prove the existence of a smooth triangulation $h' : K \to M$ such that the collection $h'(K^*)$ does give a smooth cell decomposition of M.[3] We in fact constructed h' so that $h'(\sigma) = h(\sigma)$ for each simplex σ of K.

Roughly speaking, this theorem says that if one imposes the structure of combinatorial manifold on a smooth manifold M, then one may choose the combinatorial manifold K so that the dual cells of K^* form a smooth combinatorial cell decomposition of M. The main point of the present paper is to prove the converse: if one imposes the structure of smooth manifold on a combinatorial manifold K, then one may choose the smooth manifold M so that this same condition is satisfied (Theorem 3).

From this theorem, it follows that the differentiable structure may in fact be chosen so as to be compatible with the piecewise-linear structure of K: The identity triangulation $i : K \to M$ is not necessarily piecewise smooth, but it may be isotoped to a triangulation $h : K \to M$ which *is* piecewise smooth (Lemma 2). The differentiable structure on K induced by h is the desired compatible one.

LEMMA 1. *Let C be a smooth cell complex in M.*

(a) *There exists a smooth triangulation of M which restricts to a smooth triangulation of C.*

(b) *If C is smoothly triangulated by K and by L, then K and L are piecewise-linearly homeomorphic.*

Proof. Let N_c be the hypothesized collection of transversally intersecting non-bounded manifolds.

(a) For each c, choose a smooth triangulation of N_c. Restrict this triangulation to $h_c : L_c \to N_c$ where L_c is a *finite* complex whose image contains a neighborhood of c in N_c. Let L be the disjoint union of the complexes L_c; let $g : L \to M$ be the smooth mapping which equals h_c on L_c. Let $f : K \to M$ be an arbitrary smooth triangulation of M. Using 10.4 and 10.11 of [2], we may choose δ-approximations $f' : K' \to M$ and $g' : L' \to M$ to f and g which intersect in a subcomplex; we may require that g' carry the subcomplex L_c' of L' into N_c for each c;[4] choose δ small enough that f' is a triangulation of M and $g' \mid L_c$ is a triangulation of some neighborhood of c in N_c.

It follows that f' induces a triangulation of each cell c; so that it is the de-

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[3] It is automatically combinatorial. K is necessarily a combinatorial n-manifold (8.4 of [2]). This means the link of every m-simplex is piecewise linearly homeomorphic to the boundary of an $(n - m)$-simplex; hence each dual cell is a combinatorial ball.

[4] The theorem quoted is stated only for a finite collection of submanifolds N_c, but the proof holds for a locally-finite collection just as well.

sired triangulation of M: Assuming f' induces a triangulation of $\text{Bd}\, c$, we prove it induces a triangulation of c. By hypothesis, $(f')^{-1}g'$ is a linear isomorphism of L_c' with a subcomplex of K'. Since f' induces a triangulation of $\text{Bd}\, c$, so does $g' \mid L_c'$. Since $\text{Bd}\, c$ separates N_c, $g' \mid L_c'$ must induce a triangulation of c; then f' induces a triangulation of c also.

(b) Let $h : K \rightarrow |C|$ and $k : L \rightarrow |C|$ be smooth triangulations of C. Choose δ-approximations $h' : K' \rightarrow M$ and $k' : L' \rightarrow M$ to h and f which intersect in a subcomplex, such that for each cell c of C, h' carries the subcomplex $h^{-1}(c)$ of K into N_c and k' carries $k^{-1}(c)$ into N_c. Let δ be small enough that h' and k' are imbeddings. It follows from an inductive argument as in (a) that h' must carry $h^{-1}(c)$ onto c and k' must carry $k^{-1}(c)$ onto c. Then h' and k' are triangulations of C; since they intersect in a subcomplex, $(k')^{-1}h'$ is a piecewise-linear homeomorphism of K with L.

LEMMA 2. *Let M be a differentiable manifold; let $f : K \rightarrow M$ be a continuous triangulation of M by the combinatorial manifold K. If $f(K^*)$ is a smooth combinatorial cell decomposition C of M, then f is isotopic to a triangulation $h : K \rightarrow M$ which is piecewise-smooth; $h(K^*)$ defines the same cell decomposition C of M.*

Proof. Choose a smooth triangulation $g : L \rightarrow M$ which triangulates C. We construct by induction a piecewise-linear homeomorphism ϕ of K onto L which carries τ onto $g^{-1}f(\tau)$ for each cell τ of K^*:

Suppose ϕ is defined for all cells of K^* of dimension less than m. If τ is an m-cell, then τ is a subcomplex of the first barycentric subdivision K' of K, and ϕ is a piecewise-linear homeomorphism of $\text{Bd}\, \tau$ with $\text{Bd}\, g^{-1}f(\tau) \subset L$. Now τ, being a dual cell of K^*, is a combinatorial ball; by Lemma 1, $g^{-1}f(\tau)$ is also a combinatorial ball. Any piecewise-linear homeomorphism of the boundary of one combinatorial ball onto another may be extended to a piecewise-linear homeomorphism of the balls (radially).

Then $g\phi : K \rightarrow M$ is the desired piecewise-smooth triangulation h. Since $f^{-1}h$ carries each cell of K^* homeomorphically onto itself, the same sort of inductive process may be used to construct an isotopy of $f^{-1}h$ to the identity. Then h is isotopic to f.

THEOREM 3. *Let K be a non-bounded combinatorial n-manifold. If the obstruction theory of* [1] *suffices to construct a differentiable structure on $|K|$, then this differentiable structure may be chosen so that the cells of K^* form a smooth combinatorial cell decomposition of the resulting differentiable manifold M.*

Hence it may be chosen to be compatible with the piecewise-linear structure of K.

Proof. Recall how the differentiable structure was constructed, in §1 of [1]. We began with a collection of linear imbeddings $l_v : \overline{\text{St}}\, v \rightarrow R^n$, one for each vertex v of K. These constituted our first try at covering $|K|$ by coordinate systems. Since they did not overlap differentiably, we sought to modify them by a step-by-step "smoothing" procedure until the resulting maps did overlap differentiably.

At the general step of the procedure, we had maps $f_v : \overline{\mathrm{St}}\, v \to R^n$ satisfying certain conditions (1.1 of [1]); these conditions formed the induction hypothesis for the procedure. What we do here is (a) to replace the original maps l_v by certain maps l'_v satisfying an additional condition. This condition is given in (b) below. We then strengthen the induction hypothesis by requiring the maps f_v to satisfy this additional condition. In (c) we show that the construction in the inductive step preserves the extra condition. Finally, in (d) we prove that at the final step, when the maps f_v overlap differentiably, the extra condition implies that the cells of K^* define a smooth combinatorial cell decomposition of the resulting differentiable manifold M.

(a) Consider $l_v : \overline{\mathrm{St}}\, v \to R^n$; denote the image complex by K_v. Extend K_v to a rectilinear complex L_v covering R^n.

Now for each n-m simplex s of L_v and each $\varepsilon > 0$, there is defined (1.2 of [3]) a certain open m-cell $c_\varepsilon(s)$ containing the closed m-cell τ dual to s; it is called a *transverse cell.* If $\delta < \varepsilon$, $c_\delta(s) \subset c_\varepsilon(s)$. By the basic result (1.4) of [3], the identity map $i : L_v \to R^n$ may be isotoped to a smooth triangulation $h_v : L_v \to R^n$ such that $h_v(\sigma) = \sigma$ for each simplex σ of L_v and such that for some $\varepsilon > 0$ (depending on v), each transverse cell c_ε of L_v is orthogonal to each simplex of L_v under the map h_v.[5] Let ε denote this function of v from now on. It follows that the cells $h_v(c_\varepsilon)$ are smooth transversally intersecting submanifolds of R^n, so that $h_v(L_v^*)$ is a smooth cell decomposition of R^n (1.5 of [3]). Now transverse cells are defined combinatorially, so that if $c_\delta(s)$ is a transverse cell of K, then $l_v(c_\delta(s) \cap \overline{\mathrm{St}}\, v) = c_\delta(l_v(s)) \cap |K_v|$. In particular, as s ranges over all simplices of K having v as a vertex, $\{h_v\, l_v(c_\delta(s))\}$ is a collection of smooth transversally intersecting submanifolds of R^n. We define

$$l'_v = h_v\, l_v : \overline{\mathrm{St}}\, v \to R^n.$$

(b) We now alter the induction hypothesis in 1.1 of [1] by replacing l_v by l'_v in condition (1) (this condition requires f_v to agree with l'_v on Lk v and on simplices of dimension $\leq m$). We also adjoin the following condition (trivially satisfied by l'_v) to the three given:

(4) *For each vertex v of K and each simplex s of K, there exists $\delta > 0$ such that*

$$f_v(c_\delta(s) \cap \overline{\mathrm{St}}\, v) \subset l'_v(c_\varepsilon(s) \cap \overline{\mathrm{St}}\, v).$$

For later use, we need this condition in the following equivalent form:

Let s and v be given. Then given $\gamma > 0$, there exists $\delta > 0$ such that

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[5] We really need only half of the theorem quoted. For we can work with the single coordinate system $i : R^n \to R^n$ throughout, setting $u_s = i$ for each i; Step 2 of the proof becomes unnecessary.

$$f_v(c_\delta(s) \cap \overline{\text{St}}\, v) \subset l_v'(c_\gamma(s) \cap \overline{\text{St}}\, v);$$

given $\delta > 0$, there exists $\gamma > 0$ such that

$$f_v(c_\delta(s) \cap \overline{\text{St}}\, v) \supset l_v'(c_\gamma(s) \cap \overline{\text{St}}\, v).$$

To prove this, we first verify that if τ is the m-cell of K^* dual to s, then $f_v(\tau \cap \overline{\text{St}}\, v) = l_v'(\tau \cap \overline{\text{St}}\, v)$. We proceed by induction on m : $\tau \cap \overline{\text{St}}\, v$ is an m-cell, and the images of its boundary under f_v and l_v' are the same set. For the boundary consists of pieces like $\tau' \cap \overline{\text{St}}\, v$ (where τ' is a dual $(m-1)$-cell) to which the induction hypothesis applies, and pieces like $\tau \cap \text{Lk}\, v$, on which f_v and l_v' agree. Since the images of $\tau \cap \overline{\text{St}}\, v$ under f_v and l_v' both lie in $l_v'(c_\varepsilon(s) \cap \overline{\text{St}}\, v) \subset h_v(c_\varepsilon(l_v(s)))$ and the latter is an open m-manifold, these two images must coincide.

Now both $f_v(c_\delta(s) \cap \overline{\text{St}}\, v)$ and $l_v'(c_\gamma(s) \cap \overline{\text{St}}\, v)$ are neighborhoods of $f_v(\tau \cap \overline{\text{St}}\, v)$ in the manifold with boundary $h_v(c_\varepsilon(l_v(s))) \cap |K_v|$; if δ is small, the first is contained in the second, if γ is small, the second is contained in the first.

(c) We now check that the inductive step preserves the extra condition. The inductive step is given in 1.3 of [1]. We are given maps f_v satisfying conditions (1)–(4); we are given a simplex σ with vertices $v_0, \cdots, v_m$; for convenience we denote f_{v_i} by f_i. For each $i = 1, \cdots, m$, we modify the map $f_i f_0^{-1}$ in a neighborhood of $f_0(\sigma)$, obtaining a smoothed map H_i; then we define the new maps g_i which are to replace the maps f_i by the equations $g_0 = f_0$ and $g_i = H_i f_0$ for $i = 1, \cdots, m$. The g_i satisfy conditions (1)–(3); we must check that they satisfy (4).

Choose $\gamma < \delta < \varepsilon$ so that for $i = 1, \cdots, m$ and each s,

$$f_i(c_\gamma(s) \cap \overline{\text{St}}\, v_i) \subset l_i'(c_\delta(s) \cap \overline{\text{St}}\, v_i).$$

Choose β so that

$$f_0(c_\gamma(s) \cap \overline{\text{St}}\, v_0) \supset l_0'(c_\beta(s) \cap \overline{\text{St}}\, v_0).$$

Then

$$(^*) \qquad f_i f_0^{-1} : l_0'(c_\beta(s) \cap \overline{\text{St}}\, v_0 \cap \overline{\text{St}}\, v_i) \to l_i'(c_\delta(s) \cap \overline{\text{St}}\, v_0 \cap \overline{\text{St}}\, v_i).$$

Given $\alpha < \beta$, we may choose the smoothing H_i of $f_i f_0^{-1}$ so that it satisfies this same condition $(^*)$, with β replaced by α and δ by ε. This is the crux of the entire proof; the reasons it holds are first, that for each j, $l_j'(c_\beta(s) \cap \overline{\text{St}}\, v_j)$ is orthogonal to the simplex $f_j(\sigma) = l_j'(\sigma)$, and secondly, that the smoothing process from its very definition preserves planes which are locally orthogonal to the simplex in question. (See 2.1 of [3] for the precise construction.)

Choose η so that $f_0(c_\eta(s) \cap \overline{\text{St}}\, v_0) \subset l_0'(c_\alpha(s) \cap \overline{\text{St}}\, v_0)$. Then $g_i = H_i f_0$ satisfies condition (4) with δ replaced by η.

(d) Suppose now that the smoothing process is completed, so that we have maps $g_v : \text{St}\, v \to R^n$ which overlap differentiably. Let M be the differentiable manifold for which the g_v are coordinate systems.

Let s be a simplex of K; let τ be the cell of K^* dual to s; let v be a vertex of s, so that $\tau \subset \operatorname{St} v$. By condition (4) there is a δ for which $g_v(c_\delta(s))$ is an (open) subset of the smooth submanifold $l'_v(c_\varepsilon(s))$ of R^n. Since g_v is a coordinate system on M, $c_\delta(s)$ is a smooth submanifold of M. If $s_1, \cdots, s_k$ are simplices having a vertex v in common, then the manifolds $c_{\delta_i}(s_i)$ must intersect transversally, because the manifolds $l'_v(c_\varepsilon(s_i))$ do. Thus the collection of cells of K^* forms a smooth cell decomposition of M.

Finally, note that $g_v^{-1} l'_v$ maps the combinatorial ball τ in a piecewise-smooth fashion into M, since l'_v is smooth on K and g_v is smooth by definition. As proved in (b), $g_v(\tau) = l'_v(\tau)$; hence $g_v^{-1} l'_v$ carries τ onto itself. Hence each cell in M is a combinatorial cell.

References

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