ON THE ANTICENTER OF NILPOTENT GROUPS

BY

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The anticenter $AC(G)$ of a group, as defined by N. Levine [3] is the subgroup generated by the set RG of elements with trivial centralizer. Here an element x is said to have trivial centralizer if $\langle x, y\rangle$ is cyclic for all $y \in c_G\langle x\rangle$. Free groups and a class of groups investigated by Greendlinger [2] are examples of infinite groups where every element has trivial centralizer. In a finite p-group P we have $RP = P$ if and only if there is at most one subgroup of order p, i.e. P is cyclic or a generalized quaternion group. If G is any finite group it follows easily that $RG = G$ if and only if the Sylow subgroups are cyclic or generalized quaternion groups. These groups have been classified by Zassenhaus [6, Satz 7] and Suzuki [5, Theorem E]. Abelian groups with $RG \neq 1$ are easily determined:

Theorem A [1, Theorem 3]. *Assume $G \neq 1$ is an abelian group. $RG \neq 1$ if and only if G is either torsion free of rank 1 or G is a torsion group and at least one of the Sylow subgroups has rank 1.*

In all cases mentioned so far the anticenter coincides with the set of elements with trivial centralizer. Little is known about the structure and embedding of $AC(G)$ in G in the general case. For some groups the anticenter has been determined [1]. Finite groups with a cyclic Sylow subgroup have a nontrivial anticenter. But a suitable product of dihedral groups has nontrivial anticenter and noncyclic Sylow subgroups. So it seems unlikely that a classification of all finite groups with nontrivial anticenter can be given. We show in this paper that for nonabelian nilpotent groups the question reduces to finite p-groups having a self-centralizing element. The investigation of these groups seems to be of independent interest, and we give here some results for groups of low class.

Definition. $RG = \{x \in G \mid \text{for } g \in G, xg = gx \text{ implies the group generated by } x \text{ and } g \text{ is cyclic}\}$.

$R_0 G = \{x \in G \mid \text{for } g \in G, xg = gx \text{ implies } g \text{ is a power of } x\}$.

The elements of RG are said to have trivial centralizer, the elements of $R_0 G$ are called self-centralizing. The anticenter $AC(G)$ of G is the subgroup generated by RG.

Lemma 1. *$R_0 G \subseteq RG$. For a subgroup H of G we have $H \cap RG \subseteq RH$. The sets $R_0 G$ and RG are characteristic sets.*

Notation. $N_G H$ is the normalizer of H in G.

$c_G H$ is the centralizer of H in G.

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H_i is the i^{th} term of the lower central series of H, $H = H_1$.
Z_i is the i^{th} term of the upper central series of H.
H^G is the normal closure of H in G.
$\langle M \rangle$ is the subgroup generated by the set M.
$[a, b] = a^{-1}b^{-1}ab = a^{-1}a^b$.
$[A, B] =$ subgroup generated by the $[a, b]$ with $a \in A$, $b \in B$.
$[a, {}_1 b] = [a, b]$, $[a, {}_k b] = [[a, {}_{k-1} b], b]$ for $k > 1$.
$d(G)$ is the minimal number of generators for G.

The following commutator identities are used repeatedly:

$$[ab, c] = [a, c]^b [b, c]. \tag{1}$$

If G_2 is abelian and $a \in G_2$, then for all $b, c \in G$

$$[[a, b], c] = [[a, c], b]. \tag{2}$$

THEOREM 1. *If G is locally nilpotent and $RG \neq 1$, then G is periodic or abelian.*

Proof. Assume G is nonabelian. The proof is based on repeated applications of the following simple observation.

(i) If $1 \neq x \in RG$ and $[x, a] = 1$, then a and x both have either finite or infinite order.
(ii) If $H = \langle u, v \rangle$ is nilpotent with u an element of finite order, then H_2 is finite.

Let k be the order of u and n the class of H. Then $1 \equiv [u^k, v] \equiv [u, v]^k$ mod H_3; hence H_2/H_3 is cyclic and its order divides k. This implies that H_i/H_{i+1} has exponent dividing k for all i with $2 \leq i \leq n$. Hence $H_2 = H_2/H_{n+1}$ has finite exponent. Further H_2 is finitely generated since H is nilpotent and finitely generated. But a finitely generated nilpotent group of finite exponent is finite.

(iii) If $1 \neq x \in RG \cap Z_1 G$, then G is periodic.

There exist noncommuting elements $a, b \in G$, and if

$$y \in Z_2\langle a, b \rangle - Z_1\langle a, b \rangle,$$

then $\langle y, a \rangle$ and $\langle y, b \rangle$ have class not exceeding two, and one, say $\langle y, a \rangle$ is nonabelian. The subgroup $H = \langle x, y, a \rangle$ has class two, and $x \in RG \cap Z_1 G$ gives $\langle x, y \rangle = \langle c \rangle$, $x = c^i$ with $c \in H$. Since $[y, a] \neq 1$, we have $[c, a] \neq 1$; and $1 = [x, a] = [c^i, a] = [c, a]^i$ shows that $[c, a]$ has finite order. Thus (i) implies that x and all elements of G have finite order.

(iv) If $1 \neq x \in RG$ and $\langle x \rangle^G$ is abelian, then x has finite order.

From (iii) we may assume that there is an $a \in G$ with $[x, a] \neq 1$. If the nilpotent group $H = \langle x, a \rangle$ has class k, the subgroup

$$X = \langle x, [x, a], \cdots, [x, {}_k a] \rangle$$

is normal in H, and cyclic by Lemma 1 and Theorem A. Since $[x, a] \neq 1$ the element a induces a nontrivial automorphism in X; hence $x^a = x^{-1}$ if x has infinite order. But then $X \cap Z_1 H = 1$, contrary to the assumption that H is nilpotent.

(v) If $1 \neq x \in RG$ and $\langle x \rangle^G$ is nonabelian, then x has finite order.

Assume x has infinite order. Since $\langle x \rangle^G$ is nonabelian, there exist conjugates x', x'' of x with $[x', x''] \neq 1$. From Lemma 1 we have $x', x'' \in RG$. Let $H = \langle x', x'' \rangle$ and $y \in Z_2 H - Z_1 H$, and say $[y, x'] \neq 1$. Now $[y, x'] \in Z_1 H$, and $\langle [y, x'], x' \rangle$ is infinite cyclic since x' is a conjugate of x. If $\langle [y, x'], x' \rangle = \langle d \rangle$, $[y, x'] = d^i$, $x' = d^j$, then $\langle y, d \rangle$ has class two; hence

$$1 = [y, d^i] = [y, d]^i \quad \text{and} \quad 1 \neq [y, x'] = [y, d^j] = [y, d]^j.$$

The second equation shows $[y, d]$ has infinite order, which contradicts the first equation.

(vi) If $1 \neq x \in RG$ has finite order then G is periodic.

If $b \in G$ commutes with x, we see from (i) that b has finite order. If $a \in G$ does not commute with x the subgroup $H = \langle x, a \rangle$ satisfies the assumptions of (ii). Hence H_2, and also $\langle H_2, x \rangle$ is finite. In this finite group the element $a \in H$ induces an automorphism of finite order m. In particular $[a^m, x] = 1$; hence $x \in RG$ implies $\langle a^m, x \rangle$ is cyclic, so a^m and thus a has finite order.

Remark. Locally nilpotent periodic groups are direct products of their Sylow subgroups. Hence [1, Corollary 2.1] $RG \neq 1$ if and only if $RP \neq 1$ for at least one Sylow subgroup P of G.

It is therefore sufficient to consider p-groups. A group G is said to satisfy the normalizer condition if every proper subgroup U of G is a proper subgroup of its normalizer $N_G U$. Groups satisfying the normalizer condition are locally nilpotent [4, Theorem VI.7.e].

LEMMA 2. *For $p \neq 2$ the automorphism group of the quasicyclic group $C(p^\infty)$ has no element of order p. The only nontrivial automorphism of 2-power order of $C(2^\infty)$ is the inversion which maps each element into its inverse.*

Proof [4, Proposition III.2.r]. An automorphism σ of order p of the cyclic group $C(p^{n+1})$ induces the identity automorphism on the subgroup of order p^n unless σ is the inversion.

THEOREM 2. *Assume G is a nonabelian p-group satisfying the normalizer condition and $1 \neq x \in RG$. Let A be a maximal abelian subgroup of G containing x. Then*

(i) *A is cyclic unless G is the infinite quaternion group or the in-*

finite (periodic) dihedral group;

(ii) *$AC(G)$ is generated by the set $R_0 G$ of self-centralizing elements unless G is the infinite (periodic) dihedral group;*

(iii) *if G is nilpotent, A is cyclic and $R_0 G$ generates $AC(G)$.*

Proof. To show (i), assume A is not cyclic, and let $H = N_G A$. Theorem A implies $A \cong C(p^{\infty})$, and, by assumption, $A \neq G$ and $A \neq H$. Since A is maximal abelian, $A = c_H A$, so the group of induced automorphisms $H/c_H A$ is a nontrivial p-group. For $p \neq 2$, this contradicts Lemma 2. For $p = 2$, Lemma 2 gives $[H:A] = 2$, and $a^s = a^{-1}$ for all $a \in A$ and $s \in H - A$. Since $s^2 \in A$ is fixed by s, the order of s^2 is at most two. If $s^2 = 1$, the group H is the infinite (periodic) dihedral group D_{2^∞}. If s^2 is the unique element of order two in A, the group H is the infinite quaternion group Q_{2^∞}. In either case the elements in $H - A$ have order at most four, so A is the characteristic subgroup generated by the elements of order greater than four in H. Hence $H = N_G H$, and the normalizer condition implies $H = G$.

The set $R_0 G$ and the anticenter are easily determined for the two exceptional groups D_{2^∞} and Q_{2^∞}. It follows from [1, Theorem 7] that $AC(D_{2^\infty}) = A$. There is no self-centralizing element in D_{2^∞}. On the other hand every element in $Q_{2^\infty} - A$ is self-centralizing, and these elements generate Q_{2^∞}. Hence $AC(Q_{2^\infty}) = Q_{2^\infty}$, and together with (i) this gives (ii).

Finally (iii) follows from (i) since the exceptional groups are not nilpotent.

THEOREM 3. *If G is a periodic group with a self-centralizing element $x \in R_0 G$ and $\langle x \rangle^G$ is nilpotent, then G is finite.*

Proof. Let A be a finite subgroup containing x, and assume A is normal in some subgroup B of G. Since A is finite, the group of induced automorphisms $B/c_B A$ is finite. But

$$c_B A \subseteq c_B\langle x \rangle \subseteq \langle x \rangle \subseteq A,$$

so B is finite. Assume $X = \langle x \rangle^G$ has class k. Then $X = \langle x, Z_k X\rangle$ is normal in G, and $\langle x, Z_i X\rangle$ is normal in $\langle x, Z_{i+1} X\rangle$, and Theorem 3 follows.

LEMMA 3. *If the group generated by a and b is metabelian, then*

(a) $$[b, a^m] = \prod_{i=1}^{m} [b, {}_i\, a]^{C(m,i)}$$

(b) $$(ba^{-1})^m = b^m \prod_{0<i+j<m} [[b, {}_i\, a], {}_j\, b]^{C(m,i+j+1)}\, a^{-m},$$

(*where $C(m, i)$ means $\binom{m}{i}$.*)

Proof. Identity (a) follows immediately by induction from (1). To prove (b), we observe that $a^{-m}ba^{-1} = b[b, a^m]a^{-m-1}$, so (b) follows from (a), (2) and induction on m.

Example. The following example of a metabelian p-group with $AC(G) = G$ shows that the assumption $AC(G) \neq 1$ imposes no restriction on the class or the minimum number of generators.

Let p be an arbitrary prime, $d \geq 0$ and $n \geq 2$ integers, and, for $p = 2$, assume $d > 0$ or $n > 2$. Let A be an abelian group with generators $x_1, \cdots, x_n$ and defining relations $x_i^{d_i} = 1$, where $d_n = d_{n-1}$, and $d_i = p^{d+i}$ for $i < n$. The mapping σ defined by

$$x_n^\sigma = x_n, \quad x_{n-1}^\sigma = x_{n-1} x_n \quad \text{and} \quad x_i^\sigma = x_i x_{i+1}^p \quad \text{for} \quad 1 \leq i < n - 1,$$

preserves the defining relations of A and is clearly onto. Hence σ is an automorphism of the finite group A.

Lemma 3(a) applied to the subgroups $\langle \sigma, x_k \rangle$ of the holomorph of A shows that σ has order d_n. Let G be the cyclic extension of A by $\langle x \rangle$, where x induces the automorphism σ and $x^{d_n} = x_n$. The set of elements of A fixed by σ is $\langle x_n \rangle$, so x is self-centralizing, and $[x_k, {}_i x] = x_{i+k}^{p^i}$ for $i + k < n$ which implies that the class of G is precisely n. The elements $x, x_1, \cdots, x_{n-1}$ generate G, and in the abelian quotient group $G/\langle A^p, x_n \rangle$ these elements are independent generators; hence $d(G) = n$.

Finally we show that the elements $x_k x^{-1}$ are self-centralizing, and hence $AC(G) = G$. For $c \in A$ the element cx^j commutes with $x_k x^{-1}$ if and only if $cx^j(x_k x^{-1})^j \in A$ commutes with $x_k x^{-1}$. Since $[d, x_k x^{-1}] = 1$ for $d \in A$ if and only if $[d, x] = 1$, we have $c_G(x_k x^{-1}) = \langle x_k x^{-1}, x_n \rangle$. It remains to show that x_n is a power of $x_k x^{-1}$ for $k < n$. Applying Lemma 3(b) with $m = d_n$, we have $x_k^{d_n} = 1$, and $[x_k, {}_i x] = 1$ for $i + k > n$, since the class of G is n. Next the terms with $i + k < n$ are trivial since $[x_k, {}_i x] = x_{i+k}^{p^i}$, and p^s divides $\binom{p^{i+s}}{i+1}$. For $i + k = n$, $[x_k, {}_i x] = x_n^{p^{i-1}}$ and p^{s+1} divides $\binom{p^{i+s}}{i+1}$, unless $p = 2$ and $i = 1$. Since $s + 1 = d + n - i$, these terms are also trivial. If $p = 2$ and $i = 1$, $s = d + n - 2$, so $s > 0$ by the assumptions in case $p = 2$. Hence this term is in $\langle x_n^p \rangle$, and $x^{-d_n} = x_n$ shows $\langle (x_k x^{-1})^{d_n} \rangle = \langle x_n \rangle$.

THEOREM 4. *Assume G is a nonabelian finite p-group, $p \neq 2$, and $x \in R_0 G$. If $\langle x \rangle$ is normal in G, then $AC(G) = G$ and G has generators x, a and defining relations*

$$x^{p^n} = a^{p^k} = 1, \qquad [x, a] = x^{p^{n-k}}, \qquad 0 < k < n.$$

Proof. The induced automorphism group $G/c_G\langle x \rangle$ is a cyclic group of order p^k, $k < n$. But $c_G\langle x \rangle = \langle x \rangle$ since $x \in R_0 G$, so $G = \langle x, b \rangle$ for some $b \in G$ with $x^b = x^{1+p^{n-k}}$. Further $b^{p^k} \in \langle x \rangle$ commutes with b, hence $b^{p^k} = x^{\mu p^k}$ for some integer μ. Since $G_2 = \langle x^{p^{n-k}} \rangle$ has order p^k, and, for $p \neq 2$, a group with cyclic commutator subgroup is regular, then the element $a = x^{-\mu} b$ has order p^k. By the same reasoning $(xa)^{p^k} = x^{p^k}$. Since $\langle x \rangle \cap c_G(xa) = \langle x^{p r^k} \rangle$, the element xa is also self-centralizing, hence $AC(G) = G$.

COROLLARY 1. *If $x \in R_0 G$ and either $\langle x \rangle^G$ is abelian or $x \in Z_2 G$, then*

$$d(G) \leq 2.$$

Proof. $x \in R_0 G$ implies $\langle x \rangle^G = \langle x \rangle$ in the first case, and $[x, G] \subseteq Z_1 G \subseteq \langle x \rangle$ shows $\langle x \rangle$ normal in G in the second case.

LEMMA 4. *Let $t \equiv 1 \bmod p$ for $p \neq 2$ and $t \equiv 1 \bmod 4$ for $p = 2$. For given n, each integer y has a representation*

$$y \equiv 1 + t + t^2 + \cdots + t^k \mod p^n.$$

Proof. This is obvious for $n = 1$, so we proceed by induction on n, and assume $y = 1 + t + \cdots + t^k + \mu p^{n-1}$, $t \not\equiv 1 \bmod p^n$. If t has order p^τ in the group of prime residues mod p^n, then p^n does not divide

$$f_\tau = 1 + t + \cdots + t^{p^\tau - 1}.$$

But then

$$t^{-k-1}\mu p^{n-1} \equiv f_\tau(1 + t^{p^\tau} + \cdots + t^{p^\tau \alpha}) \mod p^n$$

for a suitable α, which proves Lemma 4.

THEOREM 5. *Assume G is a finite p-group, $p \neq 2$, and $x \in R_0 G$. If N is a normal subgroup of G, and $\langle x \rangle$ is normal in N, then there exists an element $h \in G$ such that $G = \langle h, N_G\langle x \rangle\rangle$, and $d(G) \leq 3$.*

Proof. N satisfies the assumptions of Theorem 4, hence $N = \langle x, a \rangle$ with x, a satisfying the defining relations listed in Theorem 4. In the abelian quotient group N/N_2 the elements $N_2 x$ and $N_2 a$ are independent generators. The subgroup $\langle x^{p^{n-k}}, a \rangle = \langle N_2, a \rangle$ contains all elements of order p^k in N. Hence $(N_2 a)^g = (N_2 a)^\gamma$ and $(N_2 x)^g = (N_2 x)^\alpha (N_2 a)^\beta$, and the automorphism of N/N_2 induced by $g \in G$ is described by the triangular matrix

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}.$$

Since g has p-power order, $\alpha \equiv \gamma \equiv 1 \bmod p$. Select $h \in G$ such that in the matrix

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \gamma_1 \end{pmatrix}$$

corresponding to h the integer β_1 is divisible by the least power of p. The matrix equation

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \gamma_1 \end{pmatrix}^i \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}$$

is equivalent to

(i) $\alpha \equiv \alpha_1^i \sigma \bmod p^{n-k}$,
(ii) $\beta \equiv \beta_1 \tau(\alpha_1^{i-1} + \cdots + \gamma_1^{i-1}) \bmod p^k$,
(iii) $\gamma \equiv \gamma_1^i \tau \bmod p^k$.

Then (ii) and (iii) combined give

(ii′) $\beta \equiv \beta_1 \gamma \gamma_1^{-1}(t^{i-1} + \cdots + t + 1) \bmod p^k$ with $t = \alpha_1 \gamma_1^{-1}$.

By choice of β_1 and Lemma 4 there is an integer i such that (ii′) holds, and

σ and τ are determined from (i) and (iii). The matrix equation implies

$$x^{h^{-i}g} \equiv x^{\sigma} \mod N_2,$$

in particular $h^{-i}g \in N_G\langle x\rangle$, since $N_2 \subseteq \langle x\rangle$. Hence $G = \langle h, N_G\langle x\rangle\rangle$. But $N_G\langle x\rangle$ satisfies the assumptions of Theorem 4; hence

$$d(N_G\langle x\rangle) \leq 2, \quad \text{and} \quad d(G) \leq 3.$$

COROLLARY 2. *Let G be a finite p-group, $p \neq 2$, and $x \in R_0 G$. Then $d(G) \leq 3$ if $\langle x\rangle^G$ satisfies one of the following conditions*:

(i) $x \in Z_3 G$,

(ii) $\langle x\rangle^G$ *is of class two*,

(iii) $\langle x\rangle^G$ *satisfies the Engel condition* $[[u, v], v] = 1$ *for* $u, v \in \langle x\rangle^G$.

Proof. If N is normal in G, $x \in N$, and N satisfies $[[u, x], x] = 1$ for all $u \in N$, the $x \in R_0 G$ implies $[N, x] \subseteq \langle x\rangle$; hence $\langle x\rangle$ is normal in N. Thus (ii) and (iii) follow immediately from Theorem 5. To prove (i) let $N = \langle x, Z_2 G\rangle$, and observe that N has class two.

REFERENCES

1. J. DOMBEK, *Anticenters of several classes of groups*, Amer. Math. Monthly, vol. 69 (1962), pp. 738–741.
2. M. GREENDLINGER, *A class of groups all of whose elements have trivial centralizers*, Math. Zeitschr., 78 (1962), pp. 91–96.
3. N. LEVINE, *The anticenter of a group*, Amer. Math. Monthly, vol. 67 (1960), pp. 61–63.
4. E. SCHENKMAN, *Group theory*, Van Nostrand, New York, 1965.
5. M. SUZUKI, *On finite groups with cyclic Sylow subgroups for all odd primes*, Amer. J. Math., vol. 77 (1955), pp. 657–691.
6. H. ZASSENHAUS, *Über endliche Fastkörper*, Abh. Math. Sem. Univ. Hamburg, vol. 11 (1936), pp. 187–220.

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