ON SOME FORMAL IMBEDDINGS

BY

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In this paper, the reader will find the following theorem: *Let X be a smooth irreducible algebraic scheme over an algebraically closed field k. Assume $\dim X \geq 2$. Then an imbedding of X into a projective space P over k is uniquely determined by the formal scheme $\hat{P}$ which is obtained by completing P along X.* (See Th.V, §2, for its precise meaning.) We actually prove that the field of "formal-rational" functions on $\hat{P}$ coincides with the field of rational functions on P. If $k = C$ (the complex number field), this result implies that *for any connected open neighborhood U of X in P in the sense of the usual metric topology, every meromorphic function on U extends to a rational function on the entire P.* (This implication is proven, for instance, by applying the technique of GAGA, due to J. P. Serre, to the infinitesimal neighborhoods of X in P which are complex-analytic spaces.)

A general problem I have in mind may be posed as follows: *Let $\mathbf{Z}$ be a regular irreducible formal scheme over a field k, such that if $\mathbf{I}$ is a defining ideal sheaf of $\mathbf{Z}$ then the subschemes X_ν of $\mathbf{Z}$ defined by $\mathbf{I}^{\nu+1}$ are proper over k. Let $A = H^0(\mathbf{Z}, 0_{\mathbf{Z}})$ which is a k-algebra. We ask if there exists an A-morphism $f : \mathbf{Z} \to T$ with an integral scheme of finite type (or finite presentation) over A such that if $g : \mathbf{Z} \to W$ is any A-morphism into an A-scheme of finite type (or finite presentation), then there exists a unique rational map $h : T \to W$ with $g = hf$.*[2]

In this paper, my interest is confined strictly to the case of ample normal bundle (e.g. $X = X_0$ is smooth and the dual of I/I^2 as a sheaf of 0_X-modules is an ample locally free sheaf on X). We have a satisfactory answer to the above question only in the case of codimension one, i.e., when I/I^2 is an invertible sheaf on X. (See Theorems I, II, III, §1, and Theorems IV*, V*, §2.) The case of higher codimensions is still very little understood. Our result in this case is done only for a very special kind of imbeddings, i.e., imbeddings into a projective space. This seems, however, to throw some encouraging light onto the general problem of higher codimensions. (See Theorems IV, V, §2.) For a certain technical reason we assume $\dim X \geq 2$ throughout this paper. Some novel phenomena as well as generalizations for the case of $\dim X = 1$ will be investigated in a future joint paper with Matsumura.

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[2] The answer is expected to be affirmative in various interesting cases, but not in general. For instance, let $\mathbf{Z}$ be the completion of a line bundle Z over X along the zero section, where X is smooth and projective. The answer is negative if Z is associated with a non-torsion point of the Picard variety $\mathrm{Pic}^0(X)$.

The ample imbeddings of codimension one have been well studied at least in the complex-analytic framework, notably by Nierenberg-Spencer [4], Griffiths [1] and Rossi [5]. The results of this paper in that case are not new beyond their works, except for the uniqueness of "algebraic structure" in $\mathbf{Z}$. (cf. Th. II, §1.) But here the proofs are purely algebraic and, in a sense, simpler than the others.

An essentially new feature of the higher codimensional problem is that a given formal scheme $\mathbf{Z}$ of ample normal bundle can admit two different (i.e., birationally inequivalent) algebraic structures. Such an example can be constructed as follows: Take any finite morphism of projective schemes of dimension N, say $h : Z \to Z'$. Let $d > N/2$ and pick generically independent hypersurfaces H_i, $1 \leq i \leq d$, in an ambient projective space of Z. Let $X = Z \cap H_1 \cap \cdots \cap H_d$. Then h induces a closed imbedding $X \to Z'$ at least where h is étale. In particular, if Z is smooth irreducible and h is étale outside a closed subset of codimension $\geq N/2$, then h induces an isomorphism of the completions of Z and Z', along X and $h(X)$ respectively. In contrast, the uniqueness of algebraic structure is proven in this paper if the codimension of X is one, and seems plausible if the dimension of X is at least equal to its codimension.

Needless to say, this work was inspired by the works and their authors listed at the end of this paper.

1. The case of codimension 1

Let $\mathbf{Z}$ be a formal scheme over a field k, and $\mathbf{I}$ a defining ideal sheaf on $\mathbf{Z}$ such that

(1.0.1) $\mathbf{I}$ *is invertible as a sheaf of* $\mathcal{O}_{\mathbf{Z}}$*-modules*;

(1.0.2) *the subscheme* X *of* $\mathbf{Z}$, *defined by the ideal sheaf* $\mathbf{I}$, *is irreducible smooth and proper over* k;

(1.0.3) $\dim X \geqq 2$; *and*

(1.0.4) $\mathbf{I}/\mathbf{I}^2$, *viewed as a sheaf of* $\mathcal{O}_X$*-modules, is dual to an ample invertible sheaf on* X.

LEMMA (1.1) *For every locally free sheaf* F *on* $\mathbf{Z}$, $H^i(\mathbf{Z}, F)$ *is a finite* k*-module for* $i \leqq \dim X - 1$.

(For this assertion, (1.0.3) is not needed but it implies the finiteness of $H^1(\mathbf{Z}, F)$.)

Proof. We have an exact sequence

$$0 \to \mathbf{I}^{\nu+1}F \to \mathbf{I}^{\nu}F \to \bar{F} \otimes \bar{\mathbf{I}}^{\nu} \to 0,$$

where $\bar{F} = j^*(F)$ ($= F/IF$ restricted to X) and $\bar{\mathbf{I}} = j^*(\mathbf{I})$ with the inclusion map $j : X \to \mathbf{Z}$. ($\bar{\mathbf{I}}^{\nu}$ denotes the ν-times tensor product of $\bar{\mathbf{I}}$ with itself.) Hence we get an exact sequence

(1.1.1) $H^i(\mathbf{Z}, \mathbf{I}^{\nu+1}F) \to H^i(\mathbf{Z}, \mathbf{I}^{\nu}F) \to H^i(X, \bar{F} \otimes \bar{\mathbf{I}}^{\nu})$

Now let $i \leqq \dim X - 1$. Serre duality theorem on X shows that, by the ampleness of the dual of $\mathbf{I}/\mathbf{I}^2$,

$$H^i(X, \bar{F} \otimes \bar{\mathbf{I}}^\nu) = 0 \quad \text{for all } \nu \gg 0.$$

Hence, by (1.1.1), $H^i(\mathbf{Z}, \mathbf{I}^{\nu+1}F) \to H^i(\mathbf{Z}, \mathbf{I}^\nu F)$ is surjective for all $\nu \gg 0$. F being $\mathbf{I}$-adically complete, it follows that $H^i(\mathbf{Z}, \mathbf{I}^\nu F) = (0)$ for all $\nu \gg 0$. $H^i(X, \bar{F} \otimes \bar{\mathbf{I}}^\nu)$ being a finite k-module for all $\nu \geqq 0$, (1.1.1) implies that $H^i(\mathbf{Z}, \mathbf{I}^\nu F)$ is a finite k-module for all $\nu \geq 0$, especially for $\nu = 0$, Q.E.D.

LEMMA (1.2) *If F is a locally free sheaf on $\mathbf{Z}$, then we have canonical isomorphisms*

$$H^1(\mathbf{Z}, \mathbf{I}^{-\nu}F) \to_{\approx} H^1(\mathbf{Z}, \mathbf{I}^{-(\nu+1)}F)$$

for all $\nu \gg 0$.

Proof. If N denotes the dual of $\mathbf{I}/\mathbf{I}^2$ as a sheaf of 0_X-modules, then we get an exact sequence

$$0 \to \mathbf{I}^{-\nu}F \to \mathbf{I}^{-(\nu+1)}F \to \bar{F} \otimes N^{\nu+1} \to 0$$

where N^p denotes the p-times tensor product of N with itself. Since N is ample, $H^i(X, \bar{F} \otimes N^{\nu+1}) = (0)$ for all $\nu \gg 0$. Hence

$$H^1(\mathbf{Z}, \mathbf{I}^{-\nu}F) \to H^1(\mathbf{Z}, \mathbf{I}^{-(\nu+1)}F)$$

is surjective for all $\nu \gg 0$. But these cohomologies are finite k-modules by (1.1), so that $H^1(\mathbf{Z}, \mathbf{I}^{-\nu}F)$ should attain the minimal rank as k-module. Hence the surjective homomorphisms should become bijective for all $\nu \gg 0$, Q.E.D.

LEMMA (1.3) *F being the same as in (1.2), we have an exact sequence*

$$0 \to H^0(\mathbf{Z}, \mathbf{I}^{-\nu}F) \to H^0(\mathbf{Z}, \mathbf{I}^{-(\nu+1)}F) \to H^0(X, \bar{F} \otimes N^{\nu+1}) \to 0$$

for all $\nu \gg 0$, where $\bar{F} = j^*(F)$ with the inclusion map $j : X \to \mathbf{Z}$ and $N =$ the dual of $j^*(\mathbf{I})$.

Proof. Immediate from (1.2).

LEMMA (1.4) *The graded k-algebra $S = \oplus_{\nu=0}^{\infty} H^0(\mathbf{Z}, \mathbf{I}^{-\nu})$ is finitely generated.*

Proof. Since N is ample on X, $\bar{S} = \oplus_{\nu=0}^{\infty} H^0(X, N^\nu)$ is finitely generated as a graded k-algebra. This implies that for any integer $\nu' > 0$ there exists ν'' (depending upon ν') such that $k[\bar{S}_\mu, \nu' \leq \mu \leq \nu'']$ contains all the $\bar{S}_\gamma$ with $\gamma \geq \nu'$, where the suffixes indicate the homogeneous parts of those degrees. Choose and fix any ν' so large that the exact sequences of (1.3) hold for all $\nu \geq \nu'$, where $F = 0_Z$. Since the homomorphisms

$$H^0(\mathbf{Z}, \mathbf{I}^{-\nu}F) \to H^0(X, N^\nu)$$

(appearing in (1.3)) are compatible with the algebra structures of S and $\tilde{S}$, it then follows that the subalgebra $k[S_\mu, \nu' \leq \mu \leq \nu'']$ of S contains all the S_γ with $\gamma \geq \nu'$. Therefore $S = k[S_\mu, 0 \leq \mu \leq \nu'']$ which is finitely generated by (1.1), Q.E.D.

LEMMA (1.5) *Let* $Z = \text{Proj } (S)$, *where* S *is the graded* k*-algebra of* (1.4). *Then there exists a canonical morphism* $h : \mathbf{Z} \to Z$ *and this induces an imbedding* $X \to Z$.

Proof. For the existence of a morphism h, it suffices to show that $\mathbf{I}^{-\nu}$ is generated by $H^0(\mathbf{Z}, \mathbf{I}^{-\nu})$ for some $\nu \gg 0$. By if $\nu \gg 0$, then $H^0(X, N^\nu)$ generates N^ν by the ampleness of N and the canonical map $H^0(\mathbf{Z}, \mathbf{I}^{-\nu}) \to H^0(X, N^\nu)$ is surjective by (1.3). Since $\mathbf{I}$ is non-unit everywhere on $\mathbf{Z}$, it follows by Nakayama's lemma that $H^0(\mathbf{Z}, \mathbf{I}^{-\nu})$ generates $\mathbf{I}^{-\nu}$. Thus we have $h : \mathbf{Z} \to Z$. Since N is ample on X, (1.3) implies that h induces an imbedding $X \to Z$, Q.E.D.

LEMMA (1.6) *Let* $\hat{Z}$ *be the completion of* Z *along the image of* X, *and* $\hat{h} : \mathbf{Z} \to \hat{Z}$ *the morphism induced by* h. *Then* $\hat{h}$ *is an imbedding.*

Proof. Both $\mathbf{Z}$ and $\hat{Z}$ have the same underlying topological space as X, so that $\hat{h}$ is clearly a homeomorphism. I shall show that every point has an affine neighborhood U in X such that $\hat{h}$ induces an epimorphism of the ring of $\hat{Z} \mid U$ to the ring of $\mathbf{Z} \mid U$, where U is viewed as an open set in $\hat{Z}$ as well as in $\mathbf{Z}$. (Note that $\hat{Z} \mid U$ and $\mathbf{Z} \mid U$ are affine formal schemes.) This clearly suffices for (1.6). Now let us choose $\nu \gg 0$ so large that $H^0(\mathbf{Z}, \mathbf{I}^{-\mu})$ generates $\mathbf{I}^{-\mu}$ for all $\mu \geq \nu$. Pick any point $x \in X$. We have $\xi \in H^0(\mathbf{Z}, \mathbf{I}^{-\nu})$ and $\eta \in H^0(\mathbf{Z}, \mathbf{I}^{-(\nu+1)})$ which generate $\mathbf{I}^{-\nu}$ and $\mathbf{I}^{-(\nu+1)}$ respectively at the point x. Let

$$U = \{y \in X \mid \mathbf{I}_y^{-\nu} = \xi 0_{\mathbf{Z},y} \quad \text{and} \quad \mathbf{I}_y^{-(\nu+1)} = \eta 0_{\mathbf{Z},y}\},$$

where the suffix y indicates the stalk at y. Since $\mathbf{I}$ is invertible and N is ample on X, U is affine. Hence we have a topological k-algebra $\mathbf{B}$ such that $\mathbf{Z} \mid U = \text{Spf } (\mathbf{B})$. Let Spec (A) be the affine piece of $Z = \text{Proj } (S)$ associated with the homogeneous element $\xi\eta$ of degree $2\nu + 1$ in S. ($A = S[(\xi\eta)^{-1}]_0$.) Then $\hat{h}$ induces Spf $(\mathbf{B}) \to$ Spec (A). Let ε denote the constant $1 \in k$ viewed as an element of $S_1 = H^0(\mathbf{Z}, \mathbf{I}^{-1})$. Let $\zeta = \varepsilon\xi\eta^{-1}$. Then $\zeta \in A$ and generates $\mathbf{I}$ in Spf $(\mathbf{B})$. Hence $\mathbf{J} = \zeta\mathbf{B}$ is the ideal of X in $\mathbf{B}$. Let $\hat{A}$ be the $(\mathbf{J} \cap A)$-adic completion of A. Then Spf $(\hat{A})$ is the affine part of $\hat{Z}$ corresponding to Spec (A). Clearly $\hat{h}$ induces a morphism Spf $(\mathbf{B}) \to$ Spf $(\hat{A})$ which is a homeomorphism. By (1.5), $\hat{h}$ induces an isomorphism $\hat{A}/(\mathbf{J} \cap A)\hat{A} \to_{\approx} \mathbf{B}/\mathbf{J}$. Since $\mathbf{J} = \zeta\mathbf{B}$ and $\zeta \in \mathbf{J} \cap A$, it follows that $\hat{A} \to \mathbf{B}$ is surjective, Q.E.D.

LEMMA (1.7) *The ring* S *is normal.*

Proof. By assumption, X is connected. Moreover, X is regular and $\mathbf{I}$ is invertible. Hence $\mathbf{Z}$ is connected and regular. This means that if U is an affine open set in $\mathbf{Z}$, then $\mathbf{Z} \mid U = \mathrm{Spf}\,(\mathbf{B})$ with a regular integral domain $\mathbf{B}$. Pick any point $x \in \mathbf{Z}$. Then there exists an open affine neighborhood U of x in $\mathbf{Z}$ such that, if $\mathbf{Z} \mid U = \mathrm{Spf}\,(\mathbf{B})$, there exists $\zeta \in \mathbf{B}$ which generates $\mathbf{I}$ in U. It then follows that there exists a canonical homomorphism

$$H^0(\mathbf{Z}, \mathbf{I}^{-\nu}) \to \mathbf{B}\zeta^{-\nu} \subset \mathbf{B}[\zeta^{-1}]$$

which is injective for all $\nu \gg 0$. Hence $S = \oplus_{\nu=0}^{\infty} H^0(\mathbf{Z}, \mathbf{I}^{-\nu})$ is an integral domain. Let $\tilde{S}$ be the integral closure of S in its field of fractions. Then $\tilde{S}$ is naturally graded. To prove $\tilde{S} = S$, it suffices to show that every homogeneous element ξ of $\tilde{S}$ belongs to S. Since $\mathbf{B}$ is normal, $\oplus_{\nu=0}^{\infty} \mathbf{B}\zeta^{-\nu}$ is normal. Hence if $\xi \in \tilde{S}_\nu$, then $\xi \in \mathbf{B}\zeta^{-\nu}$ in the sense of the above $H^0(\mathbf{Z}, \mathbf{I}^{-\nu}) \to \mathbf{B}\zeta^{-\nu}$. Namely, $\xi \in H^0(U, \mathbf{I}^{-\nu})$. This is true for every (x, U) as above. Hence $\xi \in H^0(\mathbf{Z}, \mathbf{I}^{-\nu}) = S_\nu$, Q.E.D.

LEMMA (1.8) *The morphism $\hat{h}$ of (1.6) is an isomorphism.*

Proof. By (1.4) and (1.7), Z is a normal algebraic scheme over k. Hence the local rings of Z are analytically normal. Let x be any point of X. Then, in a canonical way, $0_{Z^\wedge,x}$ is identified with a subring of the completion of $0_{Z,x}$ and is faithfully flat over $0_{Z,x}$. Hence $0_{Z^\wedge,x}$ is normal and $\dim 0_{Z,x} = \dim 0_{Z^\wedge,x}$ for all x. The homomorphism $0_{Z^\wedge,x} \to 0_{\mathbf{Z},x}$ induced by $\hat{h}$ is surjective by (1.6). It is therefore bijective if $\dim 0_{\mathbf{Z},x} = \dim 0_{Z^\wedge,x}$, or $= \dim 0_{Z,x}$. We have $\dim 0_{\mathbf{Z},x} = \dim 0_{X,x} + 1$. I shall prove that $\dim 0_{Z,x} = \dim 0_{X,x} + 1$. Since both X and Z are irreducible algebraic schemes over k, it suffices to show that $\dim Z = \dim X + 1$, or $\dim S = \dim \bar{S} + 1$, where $\bar{S}$ is the same as in the proof of (1.4). Let ε be the element of $S_1 = H^0(\mathbf{Z}, \mathbf{I}^{-1})$ which is represented by $1 \in k$. Then, by (1.3), we get a homomorphism of graded k-algebras $S/\varepsilon S \to \bar{S}$ which is bijective in the homogeneous parts of sufficiently large degrees. Clearly ε is not zero. Hence, by (1.4) and (1.7), $\dim S = \dim (S/\varepsilon S) + 1$. But $\dim S/\varepsilon S = \dim \bar{S}$ by the above homomorphism. We have thus proven that $\hat{h}$ induces an isomorphism $0_{Z^\wedge,x} \to_{\approx} 0_{\mathbf{Z},x}$ for all x. By (1.6), $\hat{h}$ is an isomorphism, Q.E.D.

THEOREM I. *Let $\mathbf{Z}$ be any formal scheme over a field k and X a defining subcheme of $\mathbf{Z}$ satisfying the four conditions* (1.0.1)–(1.0.4). *Then there exists an imbedding of X into a projective scheme Z over k, say $f : X \to Z$, and an isomorphism of formal schemes $\mathbf{Z} \to_{\approx} \hat{Z}$ which induces f, where $\hat{Z}$ is the completion of Z along the subscheme $f(X)$.*

This existence theorem is included in (1.8). We shall next prove the uniqueness. For this purpose, let us take any imbedding $g : X \to Z_1$ such that

(1.9.1) *there exists an isomorphism $\varphi : \hat{Z}_1 \to_{\approx} \hat{Z}$ such that $\varphi\hat{g} = \hat{f}$, where $\hat{Z}_1$, $\hat{g} : X \to \hat{Z}_1$ and $\hat{f} : X \to \hat{Z}$ are respectively the completions along the images of X.*

We are interested in the *equivalence* of the two imbeddings f and g, i.e., the existence of an isomorphism $e : U \to V$ such that $ef = g$, where U (resp. V) is an open neighborhood of $f(X)$ (resp. $g(X)$) in Z (resp. Z_1). For this purpose, we may replace Z_1 by any open neighborhood of $g(X)$, so that Z_1 is regular irreducible, and then by its closure in some algebraic scheme proper over k. (For the existence of this proper algebraic scheme, we can refer to [3].) We may further replace Z_1 by its normalization for Z_1 is regular in a neighborhood of $g(X)$. Therefore we shall assume that

(1.9.2) *Z_1 is a normal algebraic scheme proper over k and the ideal sheaf I_1 of X on Z_1 is invertible as a sheaf of $\mathcal{O}_{Z_1}$-modules.*

LEMMA (1.9) *Under the assumptions (1.9.1) and (1.9.2) the dual N of I_1/I_1^2 (viewed as a sheaf of $\mathcal{O}_X$-modules) is ample on X and we have exact sequences*

$$0 \to H^0(Z_1, I_1^{-\nu}) \to H^0(Z_1, I_1^{-(\nu+1)}) \to H^0(X, N^{\nu+1}) \to 0$$

for all $\nu \gg 0$.

Proof. Note that N of (1.9) is the same as N of (1.3) by (1.9.1) and (1.8). Thus N is ample by (1.0.4) and hence $H^1(X, N^{\nu+1}) = (0)$ for $\nu \gg 0$, so that

$$H^1(Z_1, I_1^{-\nu}) \to H^1(Z_1, I_1^{-(\nu+1)})$$

is surjective for $\nu \gg 0$. Since $H^1(Z_1, I_1^{-(\nu+1)})$ is a finite k-module, it attains its minimal for $\nu \gg 0$. Hence

$$H^1(Z_1, I_1^{-\nu}) \to H^1(X, I_1^{-(\nu+1)})$$

is bijective for all $\nu \gg 0$. Then (1.9) follows, Q.E.D.

LEMMA (1.10) *There exists a morphism $e : Z_1 \to Z$ such that*
(1.10.1) $f = eg$,
(1.10.2) *e induces an isomorphism from a neighborhood of $g(X)$ in Z_1 to a neighborhood of $f(X)$ in Z.*
(1.10.3) *φ is the completion of e along the images of X.*

Proof. We have canonical commutative diagrams

$$\begin{array}{ccccccccc}
0 \to & H^0(Z_1, I_1^{-\nu}) & \to & H^0(Z_1, I_1^{-(\nu+1)}) & \to & H^0(X, N^{\nu+1}) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^0(\mathbf{Z}, \mathbf{I}^{-\nu}) & \to & H^0(\mathbf{Z}, \mathbf{I}^{-(\nu+1)}) & \to & H^0(X, N^{\nu+1}) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^0(Z, I^{-\nu}) & \to & H^0(Z, I^{-(\nu+1)}) & \to & H^0(X, N^{\nu+1}) & \to 0
\end{array}$$

where I denotes the ideal sheaf of $f(X)$ on Z. The top vertical arrows are injective because the sheaf maps $I_1^{-\nu} \to \mathbf{I}^{-\nu}$ are injective. The isomorphisms between the second row and the last are due to the definition of Z as Proj $(\oplus_{\nu=0}^{\infty} H^0(\mathbf{Z}, \mathbf{I}^{-\nu}))$ and to the obvious monomorphisms from the last row to the second. Now the horizontal sequences are all exact for $\nu \gg 0$

by (1.3) and (1.9). Say this is the case for all $\nu \geq \nu_0$. In dealing with the modules of the above diagram, $H^0(Z_1, I_1^{-\nu})$ and $H^0(Z, I^{-\nu})$ will be viewed as submodules of $S_\nu = H^0(\mathbf{Z}, \mathbf{I}^{-\nu})$ by the monomorphisms of the above diagram. It then follows from the diagram that, for all $\nu \geq \nu_0$,

$$(1.10.4) \quad H^0(\mathbf{Z}, I^{-\nu}) = H^0(Z_1, I_1^{-\nu}) + \varepsilon^{\nu-\nu_0} H^0(\mathbf{Z}, \mathbf{I}^{-\nu_0})$$

where ε is the identity $1 \in k$, which is viewed as an element of $H^0(\mathbf{Z}, \mathbf{I}^{-1})$ (as well as $H^0(Z_1, I_1^{-1})$). Let $T = \oplus_{\nu=0}^{\infty} H^0(Z_1, I_1^{-\nu})$, which will be viewed as a graded k-subalgebra of $S = \oplus_{\nu=0}^{\infty} H^0(\mathbf{Z}, \mathbf{I}^{-\nu})$. By (1.10.4), if ξ is any homogeneous element of degree $d > 0$ of S, then for every $\nu \geq \nu_0$ we get

$$\xi^\nu = \eta_\nu + \varepsilon^{d\nu-\nu_0} \zeta_{\nu_0}$$

with some $\eta_\nu \in T_{\nu d} (= H^0(Z_1, I_1^{-\nu d}))$ and $\zeta_{\nu_0} \in S_{\nu_0}$. Since S_{ν_0} is a finite k-module, we get an equation of the form

$$\xi^\nu + a_1 \varepsilon^d \xi^{\nu-1} + \cdots + a_{\nu-\nu_0} \varepsilon^{d(\nu-\nu_0)} \xi^{\nu_0} + \eta = 0$$

where $a_j \in k$ for all j and $\eta \in T_{\nu d}$. (For instance, this ν can be $\nu_0 + \operatorname{rank}_k S_{\nu_0}$.) Thus S is integral over T. Moreover (1.10.4) shows that S_ν/T_ν are k-modules of bounded ranks for all $\nu \geq 0$. Since S (and hence T) is an integral domain, this is possible only when S and T have the same field of fractions. In fact, if the rank of S as T-module is bigger than 1, then there exists a homogeneous element $\omega \neq 0$ in S such that $\omega T \cap T = (0)$, so that, if $b = \deg \omega$, then $\operatorname{rank}_k S_\nu/T_\nu \geq \operatorname{rank}_k (\omega T)_\nu = \operatorname{rank}_k T_{\nu-b}$ for all $\nu \geq b$. Here obviously $\operatorname{rank}_k T_{\nu-b}$ is not bounded for $\nu \gg 0$, by (1.9) and (1.0.3). Thus S is birational and integral over T. But T is normal because Z_1 is so. Therefore $S = T$. By (1.9) and the ampleness of N on X, $H^0(Z_1, I_1^{-\nu})$ generates $I_1^{-\nu}$ at every point of X for all $\nu \gg 0$. Moreover, ε^ν generates $I_1^{-\nu}$ at every point of $Z_1 - X$ for all $\nu > 0$. Thus we have a canonical morphism $e' : Z_1 \to \operatorname{Proj}(T)$. Let $e = e''e'$ with the obvious isomorphism $e'' : \operatorname{Proj}(T) \to_{\approx} \operatorname{Proj}(S) = Z$. Then (1.10.1) is clear by the diagrams in the beginning of the proof. The isomorphism φ of (1.9.1) is clearly compatible with the morphism e. Hence by (1.8) and (1.9.1), we get (1.10.3). As for (1.10.2), it suffices to show that $e' : Z_1 \to \operatorname{Proj}(T)$ induces an isomorphism in a neighborhood of $g(X)$. First of all, $\operatorname{Proj}(T)$ is integrally closed in the function field of Z_1. But (1.9.1) implies that $\dim Z_1 = \dim X + 1 = \dim Z$. Hence e' must be birational. Then (1.10.3) implies that e' is an isomorphism along $g(X)$ or in a neighborhood of $g(X)$, Q.E.D.

We have now established

THEOREM II. *Given $(\mathbf{Z}, X)$ as in Theorem I, the solution $f : X \to Z$ is unique up to an isomorphism of an open neighborhood of $f(X)$ in Z.*

As a matter of fact, the solution $f : X \to Z$ with

$$Z = \operatorname{Proj}(\oplus_{\nu=0}^{\infty} H^0(\mathbf{Z}, \mathbf{I}^{-\nu}))$$

is the *minimal* solution with normal Z. Here the minimality is meant in the sense of birational morphism (or birational domination). (See (1.10).)

LEMMA (1.11) *Let $h : \mathbf{Z} \to Z$ be the same as in* (1.5). *Let Z_0 be any open neighborhood of $f(X)$ in Z and F_0 a locally free sheaf on Z_0. Let $\mathbf{F} = h^*(F_0)$. Then h induces an isomorphism $H^0(Z_0, F_0) \to H^0(\mathbf{Z}, \mathbf{F})$.*

Proof. By (1.8), if I is the ideal sheaf of $f(X)$ in Z, then I^{-1} is an ample invertible sheaf on Z. Moreover, Z (and hence Z_0) is normal. Therefore, since $Z_0 \supset f(X)$ and F_0 is locally free, the canonical homomorphism

$$H^0(Z_0, F_0) \to H^0(X, F_0 \mid X)$$

is bijective, where (and later) $f(X)$ is identified with X. Hence it suffices to prove that $H^0(X, F_0 \mid X) \to H^0(\mathbf{Z}, \mathbf{F})$ is bijective. I^{-1} being ample as above, we have a monomorphism of sheaves

$$e : F_0 \to \oplus^m I_0^{-\nu}$$

for some positive integers m and ν, where $I_0 = I \mid Z_0$. By the definition (1.5) of $\mathbf{Z}$, we have a canonical isomorphism $H^0(\mathbf{Z}, \mathbf{I}^{-\nu}) \to H^0(Z, I^{-\nu})$ and hence an isomorphism

$$H^0(X, \oplus^m (I \mid X)^{-\nu}) \to_{\approx} H^0(\mathbf{Z}, \oplus^m \mathbf{I}^{-\nu}).$$

$\mathbf{Z}$ being identified with X as a topological space, we have

$$F_0 \mid X = \mathbf{F} \cap \oplus^m (I \mid X)^{-\nu}$$

in the sense of e and the other obvious monomorphisms into

$$\oplus^m \mathbf{I}^{-\nu}.$$

(In general, the intersection of two subsheaves of a sheaf in the presheaf sense is a sheaf.) Hence,

$$\begin{aligned} H^0(X, F_0 \mid X) &= H^0(X, \oplus^m (I \mid X)^{-\nu}) \cap H^0(\mathbf{Z},\mathbf{F}) = H^0(\mathbf{Z}, \oplus^m \mathbf{I}^{-\nu}) \cap H^0(\mathbf{Z},\mathbf{F}) \\ &= H^0(\mathbf{Z}, \mathbf{F}). \end{aligned}$$

Q.E.D.

LEMMA (1.12) *Let $h : \mathbf{Z} \to Z$ be as above. Let $\mathbf{F}$ be a locally free sheaf on $\mathbf{Z}$. Then there exists a coherent sheaf F on Z such that $\mathbf{F}$ is isomorphic to $h^*(F)$.*

Proof. By (1.3) and the ampleness of N, $\mathbf{F}\mathbf{I}^{-\nu}$ is generated by $H^0(\mathbf{Z}, \mathbf{F}\mathbf{I}^{-\nu})$ for $\nu \gg 0$. Thus there exists an epimorphism

$$r : \oplus^m \mathbf{I}^n \to \mathbf{F}$$

for some positive integers m and n. Let $\mathbf{G} = \operatorname{Ker}(r)$, which is locally free. Let us take the graded S-modules

$$B = \oplus_{\nu=0}^{\infty} H^0(\mathbf{Z},(\oplus^m \mathbf{I}^n)\mathbf{I}^{-\nu}) \quad \text{and} \quad C = \oplus_{\nu=0}^{\infty} H^0(\mathbf{Z}, \mathbf{G}\mathbf{I}^{-\nu}).$$

Then r induces a monomorphism of graded S-modules $s : C \to B$. Let

$A = \text{Coker}(s)$. By (1.4), S is noetherian and the S-modules A, B and C are finite. Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be the coherent sheaves on $Z = \text{Proj}(S)$ respectively associated with A, B and C. We then have an exact sequence

$$0 \to \tilde{B} \to \tilde{C} \to \tilde{A} \to 0.$$

Since the morphism h is flat, it induces an exact sequence

$$0 \to h^*(\tilde{B}) \to h^*(\tilde{C}) \to h^*(\tilde{A}) \to 0.$$

But we have a natural isomorphism

$$h^*(\tilde{C}) \to_{\approx} \oplus^m \mathbf{I}^n$$

by (1.8), and a natural epimorphism $h^*(\tilde{B}) \to \mathbf{G}$ by (1.3). Comparing the two exact sequences of sheaves on $\mathbf{Z}$, we get an isomorphism $h^*(\tilde{A}) \to \mathbf{F}$, Q.E.D.

Remark (1.13) The homomorphisms of local rings induced by the morphism h are all faithfully flat. Thus the coherent sheaf F obtained in (1.12) is locally free at every point of $f(X)$ or in some neighborhood of $f(X)$ in Z.

In an algebraic scheme, the set of non-regular points is closed. The complement of this closed subset will be called the *regular part* of the scheme.

THEOREM III. *Let $h : \mathbf{Z} \to Z$ be the morphism of* (1.5). *Let Z' be the regular part of Z, and $h' : \mathbf{Z} \to Z'$ the morphism induced by h. Then h' induces an isomorphism* $\text{Pic}(Z') \to_{\approx} \text{Pic}(\mathbf{Z})$.

Proof. By (1.12), this homomorphism $\text{Pic}(h')$ is surjective, for if F is a coherent sheaf on Z' and induces an invertible sheaf on a neighborhood of $f(X)$ in Z', then the double dual of F on Z' is invertible and induces the same sheaf in the neighborhood as F itself. To see that $\text{Pic}(h')$ is injective, take any two invertible sheaves F and G on Z'. Let $\mathbf{F} = h'^*(F)$ and $\mathbf{G} = h'^*(G)$. Then

$$\text{Hom}_{0_{\mathbf{Z}}}(\mathbf{F}, \mathbf{G}) = h'^*(\text{Hom}_{0_{Z'}}(F, G)).$$

These Hom being invertible, (1.11) implies that $\mathbf{F}$ is isomorphic to $\mathbf{G}$ if and only if F is to G, Q.E.D.

Remark (1.14) Let us consider the case in which the base field is the complex number field C. Then, for each integer $\nu \geq 0$, the ideal sheaf $\mathbf{I}^{\nu+1}$ on $\mathbf{Z}$ defines an algebraic C-scheme X_ν (e.g., $X = X_0$) and hence a complex-analytic space X_ν^{an} associated with it. The complex-analytic spaces X_ν^{an} for all the non-negative integers ν form an inductive system in an obvious fashion, and this has a limit space $\mathbf{Z}^{an}$ in the category of local-ringed spaces, which is uniquely determined by $\mathbf{Z}$ (i.e., independent of the choice of a defining ideal I). In short, we have the "*complex-analytic formal*" space $\mathbf{Z}^{an}$ associated with a given "*algebraic formal*" scheme $\mathbf{Z}$. Conversely, if a complex-analytic formal space $\mathbf{Z}^{\wedge}$ is given and satisfied the conditions analoguous to (1.0.1)–

(1.0.4), then there exists a unique algebraic formal scheme $\mathbf{Z}$ such that $\mathbf{Z}^{an}$ is isomorphic to $\mathbf{Z}^\wedge$. Thus all the lemmas and theorems of §1, preceding this remark, make sense and can be proven essentially in the same way, except for the finiteness of the k-module $H^1(Z_1, I_1^{-v})$ which is needed in the proof of (1.9). In the complex-analytic situation, we are not a priori able to reduce the proof of Th. II to the case of a proper (i.e., compact) Z_1 as was done in (1.9.1)–(1.9.2). But, instead, with no loss of generality we may assume that Z_1 is strongly pseudoconcave, so that $H^1(Z_1, F)$ for a locally free sheaf F is a finite C-module by a theorem of Andreotti-Grauert (cf. [5, Lemma 4, §5]).

Remark (1.15) Suppose we are given a complex-analytic formal space $\mathbf{Z}^\wedge$ in the same way as in (1.14). By a complex-analytic structure in $\mathbf{Z}^\wedge$, we mean a complex-analytic space $\tilde{Z}$ containing X^{an} such that its completion along X^{an} is isomorphic to $\mathbf{Z}^\wedge$. The uniqueness of $\tilde{Z}$ (within a neighborhood of X^{an}) for such $\mathbf{Z}^\wedge$ has been proven by Nirenberg-Spencer [4] and Griffiths [1]. Rossi proved that a complex-analytic manifold $\tilde{Z}$ containing X^{an} and satisfying the conditions analogous to (1.0.1)–(1.0.4) admits an algebraic structure [5, Th. 3, §5]. Using this theorem of Rossi, we may deduce by a standard GAGA-technique the uniqueness of complex-analytic structure in $\mathbf{Z}^\wedge$ from the same of algebraic structure in the corresponding $\mathbf{Z}$, i.e., Theorem II. However, it should be noted that in the case of higher codimensions, the latter uniqueness is definitely false in general as was remarked in the introduction while the former uniqueness may still hold. The existence theorem of Rossi can be rather easily deduced from Theorems I and II (cf. Remark (1.14)).

2. Imbeddings into a projective space

Throughout this section, k will denote an algebraically closed field, $P(=P^N)$ a projective space of dimension N over k, and X a closed connected reduced subscheme of dimension $n \geq 1$ of P. Let us pick an arbitrary pair of *linear subspaces* L and Y of P such that

(2.1) $\dim L = N - n - 1, \dim Y = n,$ *and* $L \cap X = L \cap Y = \emptyset$ *(empty)*.

Let $V = P - L$. For each geometric point y of V, M_y will denote the unique linear subspace of dimension $N - n$ of P, over the residue field $k(y)$ of y, such L and y are contained in M_y.

LEMMA (2.2) *There exists a unique morphism $\pi: V \to Y$ such that, for every geometric point y of Y, the fibre $\pi^{-1}(y)$ is equal to $M_y - L$. Moreover, there exists a unique structure of vector bundle for $\pi: V \to Y$ such that the given inclusion $s: Y \to V$ is the zero section.*

Proof. The uniqueness assertions are clear. To see the existence of π, we shall first construct the closure of $\pi: V \to Y$ into a projective fibre bundle $\bar{\pi}: \bar{V} \to Y$ whose fibres are isomorphic to those M_y with $y \in Y$. This can be done by means of Grassmanian variety, but here is a direct construction. Let $\beta : \bar{V} \to P$ be the monoidal transformation of P with center L. Let M_y^* be the strict transform of M_y by β. Then β induces an isomorphism $M_y^* \to_{\approx} M_y$. Then there exists a morphism $\bar{\pi} : \bar{V} \to Y$ such that $M_y^* = \bar{\pi}^{-1}(y)$ for all $y \in Y$. To be explicit, let $k[z_0, z_1, \cdots, z_N]$ be the homogeneous coordinate ring of P such that the ideal of L is generated by $(z_0, z_1, \cdots, z_n)$ and the ideal of Y by $(z_{n+1}, \cdots, z_N)$. Then we have

$$\bar{V} = \text{Proj}\ (k[z_j z_i, 0 \leq j \leq N \textit{ and } 0 \leq i \leq n]),$$

and we may identify Y with Proj $(k[z_0, z_1, \cdots, z_n])$. This gives rise to an obvious morphism $\bar{V} \to Y$ which is $\bar{\pi}$. The fibres of $\bar{\pi}$ are obviously projective spaces of dimension $N - n - 1$, which are those M_y^*. Moreover, $\beta^{-1}(L)$ intersects each M_y^* in a linear subspace of codimension 1 (= the image of L by $M_y \to_{\approx} M_y^*$). $\bar{\pi}$ has a section $Y \to \bar{V}$, the inclusion through V. Lemma (2.2) is now clear, Q.E.D.

The projective space Y has *the fundamental line bundle*, whose sections form an ample invertible sheaf on Y denoted by $0_Y(1)$. This is the sheaf associated with a hyperplane in Y. The fundamental line bundle of Y can be obtained as the normal bundle of an imbedding of Y into a projective space of dimension $n + 1$ as a hyperplane. In our situation as above, if $(E_1, \cdots, E_{N-n})$ is a system of linear subspaces of dimension $n + 1$ of P whose intersection is equal to Y and whose join is P, then the normal bundle of Y in P is in a natural way the direct sum of the normal bundles of Y in the E_j, $1 \leq j \leq N - n$. We thus obtain:

LEMMA (2.3) *The vector bundle $\pi : V \to Y$ is isomorphic to the $(N - n)$-fold fibre product of the fundamental line bundle of Y with itself. In particular, it is ample.*

Let $\pi_1 : W \to X$ be the vector bundle obtained from $\pi : V \to Y$ by base extension $q : X \to Y$, where q is induced by π and the inclusion $X \to V$. Namely $W = V \times_Y X$. (2.3) implies

LEMMA (2.4) *$\pi_1 : W \to X$ is a vector bundle over X such that the sheaf of its sections is isomorphic to $(N - n)$ times direct sum of $0_X(1)$ with itself.*

Proof. $0_X(1)$ is induced by the fundamental line bundle on P, or by $0_P(1)$. If G is a hyperplane in Y, then the closure of $\pi^{-1}(G)$ in P is a hyperplane. In other words, $0_X(1) = q^*(0_Y(1))$. (2.4) then follows from (2.3) and $W = V \times_Y X$, Q.E.D.

Let $t : W \to V$ be the projection. The inclusion map $r : X \to V$ induces an imbedding $r_1 : X \to W$ by (r, id_X), which is a section of the vector bundle π_1. Moreover, we have $r = tr_1$. Let $\hat{V}$ (resp. $\hat{W}$) be the completion of V (resp. W) along X with respect to r (resp. r_1). Let $\hat{t} : \hat{W} \to \hat{V}$ be the morphism induced by t, or the completion of t along X.

LEMMA (2.5) *The local homomorphism* $0_{V^\wedge,y} \to 0_{W^\wedge,x}$ *induced by* $\hat{t}$ *is injective* for every $y = \hat{t}(x)$ *with* $x \in \hat{W}$ *such that* X *has dimension* n *in any neighborhood of the point* x_0 *with* $r_1(x_0) = x$.

Proof. To prove that $0_{V^\wedge,y} \to 0_{W^\wedge,x}$ is injective, it is enough that the local homomorphism induces a monomorphism of their completions. $0_{V^\wedge,y}$ (resp. $0_{W^\wedge,x}$) has the same completion as $0_{V,y}$ (resp. $0_{W,x}$). Moreover, $W = V \times_Y X$. Therefore, it is enough to prove that the local homomorphism $0_{Y,y_0} \to 0_{X,x_0}$ induced by q induces a monomorphism of their completions. By (2.1), q is a finite morphism. Y is regular, so that the completion of $0_{Y,y_0}$ has no zero-divisor. It follows that the completion of $0_{Y,y_0} \to 0_{X,x_0}$ is injective, because by assumption, X has dimension n locally at x_0, Q.E.D.

I am indebted to R. Hartshorne for giving me the proof of the lemma below, which is much simpler than my original proof by induction.

LEMMA (2.6) *Let* $p : U \to X$ *be a vector bundle on* X, *which is a direct sum of ample line bundles on* X. *Let* F_0 *be a locally free sheaf on* X, *and* $F = p^*(F_0)$. *Let* $\hat{U}$ (resp. $\hat{F}$) *be the completion of* U (resp. F) *along the zero section of the bundle* p. *Then we have a canonical isomorphism*

$$H^0(U, F) \to_{\approx} H^0(\hat{U}, \hat{F}).$$

Proof. Let L_i, $1 \le i \le r$, be the ample invertible sheaves on X such that $L_1 \oplus \cdots \oplus L_r$ is isomorphic to the sheaf of sections of $p : U \to X$. Call this sheaf L. Let L' be the dual of L on X. Then we have a canonical isomorphism $U \to_{\approx} \operatorname{Spec}(S(L'))$, where $S(L')$ denotes the sheaf of symmetric tensor algebras of L' over 0_X. Let S_ν be the homogeneous part of degree ν of $S(L')$. Since p is affine and F_0 is coherent, we have

$$H^0(U, F) = H^0(X, F_0 \otimes (\oplus_{\nu=0}^{\infty} S_\nu)) = \oplus_{\nu=0}^{\infty} H^0(X, F_0 \otimes S_\nu)$$

and

$$H^0(\hat{U}, \hat{F}) = H^0(X, F_0 \otimes (\textstyle\prod_{\nu=0}^{\infty} S_\nu)) = \textstyle\prod_{\nu=0}^{\infty} H^0(X, F_0 \otimes S_\nu).$$

Hence, for (2.6), it suffices that $H^0(X, F_0 \otimes S_\nu) = 0$ for all $\nu \gg 0$. Now, S_ν is obviously a direct sum of invertible sheaves which are tensor products of the form $L'_{i_1} \otimes \cdots \otimes L'_{i_\nu}$, where L'_j are dual to L_j for all j. Since each L'_j is negative, there exists ν_0, depending on F_0, such that $F_0 \otimes L'_{i_1} \otimes \cdots \otimes L'_{i_\nu}$ is negative for every $\nu \ge \nu_0$ and for every choice of $(i_1, \cdots, i_r)$. This implies that $H^0(X, F_0 \otimes S_\nu) = (0)$ for all $\nu \ge \nu_0$, Q.E.D.

COROLLARY (2.6.1) (Hartshorne) *Let* $\hat{P}$ *be the completion of* P *along* X. *Then we have* $H^0(\hat{P}, 0_{P^\wedge}) = k$.

Proof. Let X_i be the irreducible components of X, and $\hat{P}(i)$ the completions of P along X_i. Then there is a natural homomorphism

$$H^0(\hat{P}, 0_{P^\wedge}) \to \oplus_i H^0(\hat{P}(i), 0_{P^\wedge(i)}),$$

which is clearly injective. If (2.6.1) holds for X_i, then $H^0(\hat{P}(i), 0_{P^\wedge(i)}) = k$

for every i. Hence the natural map

$$H^0(\hat{P}, 0_{P^\wedge}) \to \oplus_i H^0(X_i, 0_{X_i})$$

is also injective. This map clearly factors through $H^0(X, 0_X)$, which is k because X is connected and reduced by assumption. Thus we have only to prove (2.6.1) for the case of irreducible X. Now, $\hat{P} = \hat{V}$ and (2.5) shows that $H^0(\hat{V}, 0_{V^\wedge}) \to H^0(\hat{W}, 0_{W^\wedge})$ is injective. By (2.6), $H^0(\hat{W}, 0_{W^\wedge})$ is isomorphic to $H^0(W, 0_W)$. Let L' be the dual of the sheaf of sections of $\pi_1 : W \to X$. Then, by (2.4), L' is isomorphic to $(N - n)$ times direct sum of $0_X(-1)$ with itself. It follows that if S_ν is the ν-th symmetric tensor power of L', then $H^0(X, S_\nu) = 0$ for all $\nu > 0$. We have $W = \operatorname{Spec}(\oplus_{\nu=0}^{\infty} S_\nu)$. Therefore

$$H^0(W, 0_W) = H^0(X, \oplus_{\nu=0}^{\infty} S_\nu) = \oplus_{\nu=0}^{\infty} H^0(X, S_\nu) = H^0(X, 0_X) = k,$$

Q.E.D.

LEMMA (2.7) *Let $p : U \to X$ and U be the same as in* (2.6). *If X is smooth and has dimension ≥ 2, then there exists a commutative diagram of canonical homomorphisms*

$$\begin{array}{ccc} \operatorname{Pic}(X) & \xrightarrow{\gamma} & \operatorname{Pic}(U) \\ & {}_{\beta}\searrow \quad \swarrow_{\alpha} & \\ & \operatorname{Pic}(\hat{U}) & \end{array}$$

where γ is an isomorphism, β is injective and Coker (β) *is a torsion group. More precisely,* Coker (β) *is reduced to the identity if* char $(k) = 0$, *and it is a p^ν-torsion group with an integer $\nu > 0$ if* char $(k) = p > 0$.

Proof. We shall prove that there exists a canonical map

$$\beta^* : \operatorname{Pic}(\hat{U}) \to \operatorname{Pic}(X)$$

such that $\beta^* \circ \beta = \mathrm{id}$ and Ker (β^*) $(\to_{\approx}$ Coker $(\beta))$ has the torsion property. Naturally β (resp. β^*) is induced by the completed projection $\hat{p} : \hat{U} \to X$ (resp. the inclusion $X \to \hat{U}$ by the zero section of $p : U \to X$). Let U_ν be the subscheme of U defined by the ideal sheaf $J^{\nu+1}$, where J is the ideal sheaf of the zero section. In particular, $U_0 = X$. Let 0_ν be the structure sheaf of U_ν and 0_ν^* the sheaf of the groups of invertible elements in 0_ν. Then there exists a natural exact sequence

$$(2.7.1) \qquad 0 \to J^\nu/J^{\nu+1} \xrightarrow{f} 0_\nu^* \to 0_{\nu-1}^* \to 0$$

where $f(\xi) = 1 + \xi$. Here $J^\nu/J^{\nu+1}$, viewed as a sheaf on X, is isomorphic to the symmetric ν-th power S_ν of the proof of (2.6). S_ν is a direct sum of negative invertible sheaves on X. Therefore, if char $(k) = 0$, then Kodaira's vanishing theorem implies $H^i(U, J^\nu/J^{\nu+1}) = (0)$ for all $\nu > 0$ and all $i \leq \dim X - 1$. (See Theorem 2, [7].) In particular, this is the case for

$i = 1$. Thus $H^1(U_\nu, 0_\nu^*) \to H^1(U_{\nu-1}, 0_{\nu-1}^*)$ is injective for all $\nu > 0$, and, passing to their limit,

$$H^1(\hat{U}, 0_{\hat{U}}^*) \to H^1(X, 0_X^*)$$

is injective. Namely, β^* is injective. Clearly $\beta^*\beta = \mathrm{id}$, and hence β is an isomorphism. Next consider the case of char $(k) = p > 0$. In this case, $H^1(U, J^\nu/J^{\nu+1})$ may not be zero but is obviously a p-torsion group as every vector space over k is. Moreover, since X is smooth, Serre's duality and vanishing theorems show that $H^i(X, S_\nu) = (0)$ for all sufficiently large ν and all $i \leq \dim X - 1$. In particular, for $i = 1$. Thus the kernel of $H^1(U_\nu, 0_\nu^*) \to H^1(U_{\nu-1}, 0_{\nu-1}^*)$ is a p-torsion group for all $\nu > 0$ and vanishes for $\nu \gg 0$. Therefore, the kernel of $H^1(\hat{U}, 0_{\hat{U}}^*) \to H^1(X, 0_X^*)$ is a p^ν-torsion group for some integer $\nu > 0$. $\beta^*\beta = \mathrm{id}$ is clear, and the assertion on β follows. It is now enough to prove that γ is surjective, as the equality $\beta = \alpha\gamma$ is clear. (The canonical map γ (resp. α) is induced by the map $p : U \to X$ (resp. the completion $\hat{U} \to U$).) But the bijectivity of γ is easily proven, because $p : U \to X$ is a vector bundle, Q.E.D.

If Q is a formal or ordinary scheme, then K_Q will denote the sheaf of "total ring of fractions" of 0_Q. Namely, K_Q is the associated sheaf of the presheaf K' defined by: $K'(U) =$ the total ring of fractions of $0_Q(U)$, where U is any open subset of Q. If Q is a reduced irreducible scheme (ordinary), then $H^0(Q, K_Q)$ is *the field of rational functions* on Q, which is isomorphic to the field of fractions of $0_{Q,x}$ for every point x of Q.

THEOREM IV. *Let X be a smooth irreducible subscheme of dimension ≥ 2 of a projective space P over an algebraically closed field. Let $\hat{P}$ be the completion of P along X. Then we have a natural isomorphism*

$$H^0(P, K_P) \to_{\approx} H^0(\hat{P}, K_{\hat{P}}).$$

Proof. The homomorphism $d^* : H^0(P, K_P) \to H^0(\hat{P}, K_{\hat{P}})$ is induced by the completion morphism $d : \hat{P} \to P$. Let $x \in X$, which may be viewed as point of both $\hat{P}$ and P. Then the local homomorphism $0_{P,x} \to 0_{\hat{P},x}$ induces an isomorphism of their completions and hence it is injective. $H^0(P, K_P)$ being the field of fractions of $0_{P,x}$, it follows that d^* is injective. Take any pair (L, Y) satisfying (2.1). This defines $\pi : V \to Y$, $\pi_1 : W \to X$, $t : W \to V$, $\hat{V} = \hat{P}$, $\hat{W}$ and $\hat{t}$ as before. $\hat{t}$ induces a homomorphism

$$\hat{t}^* : H^0(\hat{P}, K_{\hat{P}}) \to H^0(\hat{W}, K_{\hat{W}}),$$

which is injective by (2.5). By (2.4), Lemmas (2.6) and (2.7) are applicable to the vector bundle $\pi_1 : W \to X$. I claim that the canonical map $H^0(W, K_W) \to H^0(\hat{W}, K_{\hat{W}})$ is bijective. For the same reason as above, this map is injective. Take any $h^* \in H^0(\hat{W}, K_{\hat{W}})$. Let L^* be the sheaf associated with the pole of h^*, i.e., the dual of L^* is defined as the ideal sheaf $M^* = \lambda^{-1}(0_{\hat{W}})$ where $\lambda : 0_{\hat{W}} \to K_{\hat{W}}$ is the multiplication by h^*. (Note that we have a natural inclusion $0_{\hat{W}} \subset K_{\hat{W}}$.) $\hat{W}$ being regular, M^* and hence L^* are invertible. By (2.7), there exists a positive integer m such that L^{*m} is the completion of an invertible sheaf $F = \pi_1^*(F_0)$ on W, where F_0 is an invertible sheaf on X. By (2.6), $H^0(W, F)$ is isomorphic to $H^0(\hat{W}, L^{*m})$. We have natural $M^* \subset 0_{W^\wedge}$. This induces $0_{W^\wedge} \subset L^*$ and $L^* \subset L^{*m}$. In this sense, $h^* \in H^0(\hat{W}, L^{*m})$ and also $k \subset H^0(\hat{W}, L^{*m})$, where k is the base field of W. Let h' and h'' be the image of h^* and $1 \in k$ by the isomorphism $H^0(W, F) \to H^0(\hat{W}, L^{*m})$. Then there exists a *unique* $h \in H^0(W, K_W)$ such that $h' = hh''$, for F is an invertible sheaf on W. It follows that h^* is the image of h by the canonical map $H^0(W, K_W) \to H^0(\hat{W}, K_{W^\wedge})$. This map is thus bijective as was claimed. We now have a commutative diagram of monomorphisms:

$$\begin{array}{ccc} H^0(P, K_P) & \xrightarrow{d^*} & H^0(\hat{P}, K_{P^\wedge}) \\ t^* \downarrow & & \downarrow \hat{t}^* \\ H^0(W, K_W) & \to_{\approx} & H^0(\hat{W}, K_{W^\wedge})\,. \end{array}$$

To prove that d^* is surjective (and hence bijective), pick any element $g^* \in H^0(\hat{P}, K_{P^\wedge})$. Then g^* corresponds to an element $h \in H^0(W, K_W)$. Since W is obtained by the base extension $q : X \to Y$, the branch locus $B(t)$ of $t : W \to V$ is equal to $\pi^{-1}(B(q))$ where $\pi : V \to Y$ is the projection. The field extension $H^0(P, K_P) \to H^0(P, K_P)(g^*)$ is isomorphic to the subextension of $H^0(P, K_P) \to H^0(W, K_W)$, which is generated by the element h. Therefore it is an algebraic extension and its branch locus in V is contained in $\pi^{-1}(B(q))$. For the same element g^*, we can choose various (L, Y) satisfying (2.1). The intersection of $\pi^{-1}(B(q))$ for all such (L, Y) has codimension ≥ 2 in P. To see this, it is enough to take two pairs (L, Y) and (L', Y') such that either $L \cap L'$ or $Y \cap Y'$ is empty. $H^0(P, K_P)$ is a purely transcendental extension of k and any non-trivial extension ramifies. By the purity of branch locus, $H^0(P, K_P)(g^*) = H^0(P, K_P)$, Q.E.D.

Theorem V. *Let X be a smooth and irreducible subvariety of dimension ≥ 2 of a projective space P over an algebraically closed field k. Let $\hat{P}$ be the completion of P along X, and let $c : \hat{P} \to P$ be the natural morphism inducing the identity of X. Let Z be any algebraic scheme over k. Then every morphism $\varphi : \hat{P} \to Z$ over k is of the form fc with a rational map $f : P \to Z$ which induces a morphism within a neighborhood of X. Moreover f is uniquely determined by φ.*

Proof. With no loss of generality we may assume that Z is reduced and irreducible. The reason for this is that $\hat{P}$ is irreducible and locally integral. Now, take any rational function h on Z which is regular at least at one point of $\varphi(X)$. Then $\varphi^*(h) \in H^0(\hat{P}, K_{P^\wedge})$ is well defined, and, by Theorem III of §2, there is a unique rational function h' on P such that $c^*(h') = \varphi^*(h)$. For every point $x \in X$, the function field $H^0(P, K_P)$ being identified with the field of fractions of the local ring $0_{P,x}$, the intersection $H^0(P, K_P) \cap 0_{P^\wedge,x}$ makes sense and is equal to $0_{P,x}$. (In fact, the same is true even if $0_{P^\wedge,x}$ is replaced by its completion, which is canonically isomorphic to the completion of $0_{P,x}$.) Thus the existence of h' for every given h implies that the local homomorphism $0_{Z,\varphi(x)} \to 0_{P^\wedge,x}$ induced by φ for each $x \in X$ has its image in $0_{P,x}$. This immediately implies the existence of $f : P \to Z$ as is stated in Theorem V. The uniqueness of f is obvious, Q.E.D.

The analogues of the above Theorems IV and V have been proven implicitly in §1 for the case of codimension one in the situation as general as was described there. To be explicit, let us state them as follows:

THEOREM IV*. *Let X be a smooth irreducible subscheme of a smooth irreducible algebraic scheme Z over an algebraically closed field k. Assume that X is proper over k, that the codimension of X in Z is one and that the normal bundle of X in Z (or the dual of I/I^2 as a sheaf of 0_X-modules, where I is the ideal sheaf of X in Z) is ample. Let $\mathbf{Z}$ be the completion of Z along X. Then we have a natural isomorphism*

$$H^0(Z, K_Z) \to_{\approx} H^0(\mathbf{Z}, H_{\mathbf{Z}})$$

Proof. Immediate from (1.11) and (1.12), (cf. Proof of Th. IV).

THEOREM V*. *Let X, Z and $\mathbf{Z}$ be the same as in Theorem IV*. Let $c : \mathbf{Z} \to Z$ be the natural morphism. Let W be any algebraic scheme over k, and $\varphi : \mathbf{Z} \to W$ be any morphism over k. Then there exists a unique rational map $f : Z \to W$ such that f induces a morphism within a neighborhood of X in Z and $fc = \varphi$.*

Proof. Immediate from Th. IV* (cf. Proof of Th. V).

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