THE COHOMOLOGY OF THE CLASSIFYING SPACE FOR COMPLEX K-THEORY mod p

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1. Introduction

This paper is intended to be the first of several investigating the structure of K-theory with coefficients. Here we shall be concerned with the mod p cohomology of the classifying space for complex K-theory mod p. §1 is introductory and concludes with a definition, and elementary properties, of the space BU_p. In §2 we calculate the cohomology ring $H^*(\Omega BU_p; \mathbf{Z}_p)$, and in §3 the ring $H^*(BU_p; \mathbf{Z}_p)$. §4 is an appendix, and contains a technical lemma on Bockstein operations, which is used in the calculation of §2. These results formed a portion of the author's doctoral dissertation submitted to Columbia University, and the author would like to thank Professor Samuel Eilenberg for his support, Professor Donald Kahn for his encouragement, and Professor Eldon Dyer for his critical reading of this thesis.

If X and Y are topological spaces with basepoints x_0 and y_0 respectively, then we will denote by (X, Y) the space of (free) maps from X to Y, topologized by the compact open topology. $(X, Y)^\bullet$ will denote the space of basepoint preserving maps. The corresponding homotopy classes of maps will be denoted by $[X, Y]$ and $[X, Y]^\bullet$. $X \wedge Y$, the "smash" product of X and Y, is the space formed from $X \times Y$ by collapsing $X \times y_0 \cup x_0 \times Y$ to the basepoint. The adjointness relation $(X \wedge Y, Z)^\bullet \approx (X, (Y, Z)^\bullet)^\bullet$ is valid when Y is locally compact and regular, X is Hausdorff, and Z is arbitrary [6]. Most of our applications will have X a CW-complex, and Y a *finite* CW-complex. Given a sequence of spaces and basepoint preserving maps

$$\varepsilon : \cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots ,$$

we say ε is *exact* if the sequence

$$[Y, \varepsilon]^\bullet : \cdots \to [Y, X_{n+1}]^\bullet \to [Y, X_n]^\bullet \to [Y, X_{n-1}]^\bullet \to \cdots$$

is exact for any Y, and *co-exact* if the sequence

$$[\varepsilon, Y]^\bullet : \cdots \to [X_{n-1}, Y]^\bullet \to [X_n, Y]^\bullet \to [X_{n+1}, Y]^\bullet \to \cdots$$

is exact for any Y.

Let I denote the unit interval [0, 1], and define

$$\varepsilon Y : (I, Y)^\bullet \to Y \quad \text{and} \quad \eta Y : Y \to I \wedge Y$$

to be, respectively, the projection of a path to its endpoint, and the inclusion

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of Y to the base of the cone on Y. If $f : X \to Y$ is a basepoint-preserving map, Ef is defined by requiring that the following be a pullback diagram in the category of spaces and basepoint preserving maps:

$$\begin{tikzcd} Ef \arrow[r] \arrow[d, "\pi f"'] & (I, Y)^{\cdot} \arrow[d, "\varepsilon Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Dually, define Cf by requiring that the following be a pushout diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta Y"'] & Y \arrow[d, "if"] \\ I \wedge X \arrow[r] & Cf \end{tikzcd}$$

Then

$$EF \xrightarrow{\pi f} X$$

is a fibration (satisfies the ACHP) and

$$Y \xrightarrow{if} Cf$$

is a cofibration (satisfies the AHEP). Letting $\Omega Y \to Ef$ be the inclusion of the fibre, and $Cf \to SX$ the projection mapping Y to the basepoint, the sequence

$$\mathcal{E}f : \cdots \to \Omega Ef \to \Omega X \xrightarrow{\Omega f} \Omega Y \to Ef \to X \xrightarrow{f} Y$$

is exact, and the sequence

$$\mathcal{C}f : X \xrightarrow{f} Y \to Cf \to SX \xrightarrow{Sf} SY \to SCf \to \cdots$$

is co-exact. We shall call $\mathcal{E}f$ the *Puppe sequence of f*, and $\mathcal{C}f$ the *co-Puppe sequence* of f. In the following lemma, which is proved in [9], the symbol $\approx$ between two sequences means each term in one is naturally homeomorphic to the corresponding term in the other, and the homeomorphisms commute with the maps of the sequences.

LEMMA 1. *Let $f : X \to Y$ be a basepoint preserving map.*

(i) *If Z is a finite* CW*-complex, then*
$$(Z, \mathcal{E}f)^{\bullet} \approx \mathcal{E}(Z, f)^{\bullet}.$$

(ii) *If Z or X is a finite* CW*-complex, then*
$$\mathcal{C}f \wedge Z \approx \mathcal{C}(f \wedge Z).$$

(iii) *If X is a finite* CW*-Complex, then*
$$(\mathcal{C}f, Z)^{\bullet} \approx \mathcal{E}(f, Z)^{\bullet}.$$

Remark. The assumptions of Lemma 1 are clearly stronger than necessary, but since these conditions suffice for what follows, we use this formulation for convenience.

Now, let

$$S^1 \xrightarrow{m} S^1$$

be a map of degree m, and write $L(m, 2)$ for $\mathcal{C}m$. Then the co-Puppe sequence of m gives a co-exact sequence:

$$\mathcal{C}m : S^1 \xrightarrow{m} S^1 \to L(m, 2) \to S^2 \xrightarrow{m} S^2 \to L(m, 3) \to \cdots .$$

Note that

$$\begin{aligned} H^i(L(m, n) ; \mathbf{Z}) &\approx \mathbf{Z}_m \quad \text{if} \quad i = n \\ &\approx 0 \qquad \text{otherwise} \end{aligned}$$

Also, $L(2, 2) = RP^2$—the real projective plane. By Lemma 1 (ii), $\mathcal{C}m \wedge X \approx \mathcal{C}(m \wedge X)$, so we obtain another co-exact sequence:

$$\cdots \to S^{n-1} X \xrightarrow{m \wedge X} S^{n-1} X \to L(m, n) \wedge X \to S^n X \xrightarrow{m \wedge X} S^n X \to \cdots .$$

Thus, if t is any half exact functor in the sense of Dold [5], applying t gives an exact sequence:

$$\cdots \to t(S^n X) \xrightarrow{t(m \wedge X)} t(S^n X) \to t(L(m, n) \wedge X) \to t(S^{n-1} X) \xrightarrow{t(m \wedge X)} t(S^{n-1}) \to \cdots .$$

By 3.5 of [5], $t(m \wedge X)$ is multiplication by m. Therefore, if we put

$$t^{-n}(X) = t(S^n X)$$

and

$$t^{-n}(X; \mathbf{Z}_m) = t(L(m, n) \wedge X)$$

the usual argument gives a universal coefficient sequence:

$$0 \to t^{-n}(X) \otimes \mathbf{Z}_m \to t^{-n}(X; \mathbf{Z}_m) \to \operatorname{Tor}(t^{-n+1}(X) ; \mathbf{Z}) \to 0.$$

Also, the above long exact sequence may be conveniently recorded as a singly graded exact couple:

$$\begin{array}{ccc} t^{-*}(X) & \xrightarrow{i} & t^{-*}(X) \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ & t^{-*}(X ; \mathbf{Z}_m) & \end{array}$$

where i = multiplication by m and is of degree 0, j is reduction mod m and is of degree 0, and k is an extraordinary Bockstein operation of degree $+1$ associated with the coefficient sequence

$$0 \to \mathbf{Z} \xrightarrow{m} \mathbf{Z} \xrightarrow{r} \mathbf{Z}_m \to 0$$

where m = multiplication by m and r = reduction mod m. This couple leads, of course, in the usual way to a Bockstein spectral sequence as in [4]. We hope to exploit this in a later paper.

Now let U be the infinite unitary group and BU its classifying space. Then SU, the infinite special unitary group, is the simply connected covering group of U, and BSU, its classifying space, is a 2-connected fibre space over BU. As is well known, [2], if we denote by $\Omega_0 X$ the component of the constant loop in X, by $\bar{X}$ a simply connected fibre space over X, and by $\bar{\bar{X}}$ a 2-connected fibre space over X, then

$$\Omega_0^2 BU \equiv BU, \quad \Omega^2 U \equiv U, \quad \overline{\Omega^2 SU} \equiv SU, \quad \text{and} \quad \overline{\overline{\Omega^2 BSU}} \equiv BSU.$$

In the above discussion, put $t = [\quad, BU]^\bullet$, and let

$$\tilde{K}^{-n}U(X\,;\mathbf{Z}_m) = t^{-n}(X\,;\mathbf{Z}_m).$$

Then

$$\begin{aligned} \tilde{K}^{-n}(X\,;\mathbf{Z}_m) &= [L(m,n) \wedge X, BU]^\bullet \\ &\approx [L(m,n) \wedge X, \Omega^2 BSU]^\bullet \\ &\approx [S^{n-2}L(m,2) \wedge X, \Omega^2 BSU]^\bullet \\ &\approx [S^n X, (L(m,2), BSU)^\bullet]^\bullet. \end{aligned}$$

These spaces $(L(m,2), BSU)^\bullet$ will then be the classifying spaces for complex K-theory mod m, and we will denote them by BU_m. Putting $X = S^0$ in the universal coefficient sequence above gives an exact sequence:

$$0 \to \pi_n(BU) \otimes \mathbf{Z}_m \to \pi_n(BU_m) \to \operatorname{Tor}(\pi_{n-1}(BU), \mathbf{Z}_m) \to 0.$$

Therefore, BU_m is connected and since

$$\begin{aligned} \pi_n(BU) &\approx \mathbf{Z}, \quad n \text{ even} \\ &\approx 0, \quad n \text{ odd} \end{aligned}$$

we have

$$\begin{aligned} \pi_n(BU_m) &\approx \mathbf{Z}_m \quad n \text{ even} \\ &\approx 0 \quad n \text{ odd} \end{aligned}$$

Moreover, BU_m has the homotopy type of the component of the constant map in $(L(m,2), BU)^\bullet$, and hence $\Omega_0^2 BU_m \equiv BU_m$.

In §2 and §3 respectively, we shall compute the ring structures of $H^*(\Omega BU_p\,;\mathbf{Z}_p)$ and $H^*(BU_p\,;\mathbf{Z}_p)$ for p a prime. In the integral case this was done by Borel [1], and the results are:

$$H^*(U;\mathbf{Z}) \approx E(\bar{y}_1, \bar{y}_3, \cdots)$$
$$H^*(BU;\mathbf{Z}) \approx P(c_2, c_4, \cdots)$$

where the c_{2i} are the universal Chern classes, and are images of the $\bar{y}_{2i-1}$ under transgression. Notice that we have deviated from the standard notation by indexing these classes by their dimensions. Thus, for example, our c_{2i} is ordinarily written c_i. This convention will make it easier to keep track of dimensions in the computations that follow. Similarly,

$$H^*(SU;\mathbf{Z}) \approx E(\bar{y}_3, \bar{y}_5, \cdots)$$
$$H^*(BSU;\mathbf{Z}) \approx P(\bar{c}_4, \bar{c}_6, \cdots),$$

where if τ denotes transgression in the universal fibration, $\tau\bar{y}_{2i-1} = \bar{c}_{2i}$, $i \geq 2$.

In order to use these results and the standard tool of algebraic topology for calculating cohomology, namely spectral sequences, we put BU_p as total space in a fibration whose base and fibre, up to homotopy type, are SU and BU respectively. So consider the co-Puppe sequence

$$S^1 \xrightarrow{p} S^1 \to L(p, 2) \to S^2 \xrightarrow{p} S^2 \to \cdots$$

defined above for p prime. Putting BSU in the second variable and applying Lemma 1, we obtain a fibration

$$\begin{array}{ccc} \Omega^2 BSU \to & BU_p & \\ & \downarrow & \\ & \Omega BSU & \xrightarrow{p} \Omega BSU, \end{array}$$

where we denote the map $(p, BSU)^\bullet$ again by p. Up to homotopy equivalence we may rewrite this as

$$\begin{array}{ccc} BU \to & BU_p & \\ & \downarrow & \\ & SU & \xrightarrow{p} SU. \end{array}$$

Again by Lemma 1, this fibration is induced by p from the universal fibration over SU with fibre BU. For technical reasons, which will become apparent later, we wish to deloop this fibration. So, consider the fibration

$$\begin{array}{ccc} K(\mathbf{Z}, 2) \to & E\bar{Y}_3 & \\ & \downarrow & \\ & SU & \xrightarrow{\bar{Y}_3} K(\mathbf{Z}, 3) \end{array}$$

and let $\sigma : H^n(SU : \mathbf{Z}) \to H^{n-1}(BU : \mathbf{Z})$ denote the suspension. Then clearly, $\sigma\bar{y}_3 = c_2$—the first universal Chern class. Thus, applying Ω to this fibration gives

$$\begin{array}{ccc} K(\mathbf{Z}, 1) \to & \Omega E\bar{Y}_3 & \\ & \downarrow & \\ & BU & \xrightarrow{c_2} K(\mathbf{Z}, 2) \end{array}$$

and $\Omega E\bar{Y}_3 \equiv BSU$. Now if we put our co-Puppe sequence for

$$S^1 \xrightarrow{p} S^1$$

in the first variable, and $E\bar{Y}_3$ in the second, we obtain a fibration

$$\begin{array}{ccc} \Omega^2 E\bar{Y}_3 \to & (L(p, 2), E\bar{Y}_3)^{\bullet} & \\ & \downarrow & \\ & \Omega E\bar{Y}_3 \xrightarrow{p} \Omega E\bar{Y}_3 . & \end{array}$$

Let us denote $(L(p, 2), E\bar{Y}_3)^{\bullet}$ by $BU_p^{\wedge}$. Then, again up to homotopy, we may rewrite this as

$$\begin{array}{ccc} SU \to & BU_p^{\wedge} & \\ & \downarrow & \\ & BSU \xrightarrow{p} BSU, & \end{array}$$

and this fibration has the property that Ω applied to it gives our original fibration for BU_p.

2.

Here we shall begin the calculations by computing $H^*(\Omega BU_p\ ;\ \mathbf{Z}_p)$. Towards the end of the calculation we shall have to make a distinction between the cases p odd and p even, but for now, p will denote any prime. So, consider the fundamental fibration

$$\begin{array}{ccc} BU \xrightarrow{i} & BU_p & \\ & \downarrow \pi & \\ & SU \xrightarrow{p} SU & \end{array}$$

and apply Ω.

We obtain

$$\begin{array}{cccc} U \xrightarrow{i'} & \Omega\, BU_p & EU \leftarrow U \\ & \downarrow \pi' & \downarrow \\ & BU \xrightarrow{p} & BU \end{array}$$

where the fibration on the right is the universal one (EU contractible), and the fibration on the left is induced from it by p. Since p is just the p-fold diagonal composed with p-fold loop multiplication and this last is compatible under homotopy equivalence with the given H-space structure of BU, we may apply the Whitney duality theorem to obtain:

$$p^* c_{2i} = \sum\nolimits_{j_1 + \cdots + j_p = i} c_{2j_1} \cdot \cdots \cdot c_{2j_p} .$$

Now if we collect terms in this sum, we obtain a sum of terms of the form $c_{2k_1}^{q_1} \cdots c_{2k_r}^{q_r}$ corresponding to terms $c_{2j_1} \cdots c_{2j_p}$ where q_1 of the c's are $c_{2k_1}, \cdots, q_r$ of the c's are c_{2k_r}. For each choice of $r, q_1, \cdots, q_r$, and

$k_1, \cdots, k_r$, satisfying the conditions listed below, there are $p!/(q_1! \cdots q_r!)$ such terms. Thus,

$$p^* c_{2i} = \sum \frac{p!}{q_1 ! \cdots q_r !} c_{2k_1}^{q_1} \cdots c_{2k_r}^{q_r}$$

where the sum is over integers $q_1, \cdots, q_r, k_1, \cdots, k_r$ for $1 \leq r \leq p$ such that $1 \leq q_\mu \leq p$, $\mu = 1, \cdots, r$, $q_1 + \cdots + q_r = p$, $q_1 k_1 + \cdots + q_r k_r = i$, and $0 \leq k_1 < k_2 < \cdots < k_r$. Except for a non-vacuous occurrence of the case $r = 1$, which implies $i \equiv 0\ (p)$, all of these coefficients are divisible by p. Thus, if $i \not\equiv 0\ (p)$, we have

$$p^* c_{2i} = pc_{2i} + p\, d_{2i}$$

where d_{2i} is decomposable.

If $i \equiv 0\ (p)$, say $i = pj$, then

$$p^* c_{2pj} = c_{2j}^p + pb_{2pj}$$

Therefore, mod p we have

$$p^* c_{2i} = 0 \quad \text{if} \quad i \not\equiv 0\ (p)$$
$$p^* c_{2pj} = c_{2j}^p$$

The naturality of transgression shows that in the fibration

$$\begin{array}{ccc} U & \xrightarrow{i'} & \Omega BU_p \\ & & \downarrow \pi' \\ & & BU \end{array}$$

we have, in mod p cohomology, y_{2i-1} is transgressive, and

$$\tau y_{2i-1} = 0 \quad \text{when } i \not\equiv 0\ (p)$$
$$\tau y_{2pj-1} = c_{2j}^p .$$

An application of the comparison theorem for spectral sequences [10], shows that in the Serre spectral sequence of the above fibration,

$$E_\infty \approx P(c_2, \cdots, c_{2j}, \cdots)/\{c_2^p, \cdots, c_{2j}^p \cdots\}$$
$$\otimes E(y_1, \cdots, y_{2i-1}, \cdots), \quad i \not\equiv 0\ (p).$$

Let α'_{2j} and α'_{2i-1}, $i \not\equiv 0\ (p)$, be classes in $H^*(\Omega BU_p : \mathbf{Z}_p)$ defined by $\alpha'_{2j} = \pi'^* c_{2j}$ and $i'^* \alpha'_{2i-1} = y_{2i-1}$. Then by Proposition 4.3 of the appendix,

$$H^*(\Omega BU_p ; \mathbf{Z}_p) \approx P(\alpha'_2, \cdots, \alpha'_{2j}, \cdots)/\{\alpha_2'^p, \cdots, \alpha_{2j}'^p, \cdots\}$$
$$\otimes E(\alpha'_1, \cdots, \alpha'_{2i-1}, \cdots) \quad i \not\equiv 0\ (p)$$

as a left $P(\alpha'_2, \cdots, \alpha'_{2j}, \cdots)/\{\alpha_2'^p, \cdots, \alpha_{2j}'^p, \cdots\}$ module.

When p is odd, this isomorphism is valid as algebras by 4.4, since then

$E(y_1, \cdots, y_{2i-1}, \cdots)$ is free as an algebra. For any p, let

$$\beta_p : H^n(\Omega BU_p ; \mathbf{Z}_p) \to H^{n+1}(\Omega BU_p ; \mathbf{Z}_p)$$

be the Bockstein operation associated with the coefficient sequence

$$0 \to \mathbf{Z}_p \xrightarrow{p} \mathbf{Z}_{p^2} \xrightarrow{r} \mathbf{Z}_p \to 0.$$

Then, Corollary 4.2 of the appendix gives

$$\beta_p \alpha'_{2i-1} = \pi'^*(c_{2i} + d_{2i}) \bmod \beta_p(\ker i'^*) = \alpha'_{2i} + d,$$

where d is decomposable, and in $\{\alpha'_2, \cdots, \alpha'_{2j}, \cdots\}$—the ideal in $H^*(\Omega BU_p : \mathbf{Z}_p)$ generated by the α'_{2j}, $j > 0$.

To obtain the algebra structure of $H^*(\Omega BU_2 ; \mathbf{Z}_2)$ we must argue further. By the above we have

$$H^*(\Omega BU_2 ; \mathbf{Z}_2) \approx E(\alpha'_2, \cdots, \alpha'_{2j}, \cdots) \otimes E(\alpha'_1, \cdots, \alpha'_{4i+1}, \cdots)$$

as a left $E(\alpha'_2, \cdots, \alpha'_{2j}, \cdots)$ module. Also $Sq^1\alpha'_{4i+1} = \alpha'_{4i+2} + d$ where d is a decomposable element in $\{\alpha'_2, \cdots, \alpha'_{2j}, \cdots\}$. Hence, since $Sq^{4i}y_{4i+1} = y_{8i+1}$, we have

$$\alpha'^2_{4i+1} = Sq^{4i+1}\alpha'_{4i+1} = Sq^1Sq^{4i}\alpha'_{4i+1} = Sq^1(\alpha'_{8i+1} + d'),$$

where $d' \in \ker i'^*$ in dimension $8i + 1$, and is thus decomposable and in $\{\alpha'_2, \cdots, \alpha'_{2j}, \cdots\}$. Then,

$$\alpha'^2_{4i+1} = Sq^1\alpha'_{8i+1} + Sq^1 d' = \alpha'_{8i+2} + (d + Sq^1 d').$$

Since Sq^1 is an $E(\alpha'_2, \cdots, \alpha'_{2j}, \cdots)$ module homomorphism (Sq^1 is zero in $H^*(BU : \mathbf{Z}_2)$), we see that $d'' = d + Sq^1 d'$ is decomposable, and in $\{\alpha'_2, \cdots, \alpha'_{2j}, \cdots\}$. Therefore,

$$\alpha'^2_{4i+1} = \alpha'_{8i+2} + d'' \neq 0, \quad \text{but} \quad \alpha'^4_{4i+1} = \alpha'^2_{8i+1} + d''^2 = 0.$$

(Note that $\alpha'_{8i+2} + d'' = 0$ would give an indecomposable element in $\ker \pi'^*$.) Now we can define an obvious map of algebras

$$E(\alpha'_4, \cdots, \alpha'_{2j}, \cdots) \otimes P(\alpha'_1, \cdots, \alpha'_{4i+1}, \cdots)/\{\alpha'^4_1, \cdots, \alpha'^4_{4i+1}, \cdots\} \to H^*(\Omega BU_2 ; \mathbf{Z}_2), \quad j \not\equiv 1 \ (4)$$

Clearly, in each dimension this map takes a vector space basis into a basis, and hence is an isomorphism. Therefore, we have finally,

$$H^*(\Omega BU_2 : \mathbf{Z}_2) \approx E(\alpha'_4, \cdots, \alpha'_{2j}, \cdots) \otimes P(\alpha'_1, \cdots, \alpha'_{4i+1}, \cdots)/\{\alpha'^4_1, \cdots, \alpha'^4_{4i+1}, \cdots\} \quad j \not\equiv 1 \ (4)$$

as an algebra.

Recall that the fibration

$$\begin{array}{ccc} BU & \xrightarrow{i} & BU_p \\ & & \downarrow \pi \\ & & SU \end{array}$$

may be delooped to give

$$\begin{array}{ccc} SU & \xrightarrow{\bar{i}} & BU_p^{\wedge} \\ & & \downarrow \bar{\pi} \\ & & BSU \end{array}$$

where $\Omega BU_p^{\wedge} \equiv BU_p$. Then the same argument as above gives

$$H^*(BU_p^{\wedge}; \mathbf{Z}_p) \approx P(\bar{\alpha}_4, \cdots, \bar{\alpha}_{2j}, \cdots)/\{\bar{\alpha}_4^p, \cdots . \bar{\alpha}_{2j}^p, \cdots\}$$
$$\otimes E(\bar{\alpha}_{2p-1}, \bar{\alpha}_1, \cdots, \bar{\alpha}_{2i-1}, \cdots), \quad i \not\equiv 0 \ (p)$$

as an algebra for p odd. (The generator $\bar{\alpha}_{2p-1}$ arises because there is no class $\bar{c}_2^p$ for $\bar{y}_{2p-1}$ to kill.) For $p = 2$, we have

$$H^*(BU_2^{\wedge}; \mathbf{Z}_2) \approx E(\bar{\alpha}_4, \cdots, \bar{\alpha}_{2j}, \cdots)$$
$$\otimes P(\bar{\alpha}_3, \cdots, \bar{\alpha}_{4i+1}, \cdots)/\{\bar{\alpha}_3^4, \cdots, \bar{\alpha}_{4i+1}^4, \cdots\}, \quad j \neq 3, \quad j \not\equiv 1 \ (4)$$

3.

To compute $H^*(BU_p; \mathbf{Z}_p)$ one would like to use a similar technique· Namely, analyze the universal fibration

$$\begin{array}{ccc} BU & \rightarrow & E \\ & & \downarrow \\ & & SU \end{array}$$

and transfer the results to the fundamental fibration via p. Unfortunately, this does not seem to work here. One can indeed analyze the above universal fibration, and by changing the polynomial generators of BU, determine all the differentials in the spectral sequence. However, since only some of the new generators are transgressive, and the ones that are not pass further into the spectral sequence upon transference by p, this does not seem very helpful, and hence we proceed somewhat differently. To begin with, we recall two theorems of Browder [3]. First

THEOREM 5.14. *Let X be an arcwise connected, simply connected H-space; then*

$$\sigma ; Q(H^i(X ; \mathbf{Z}_p)) \rightarrow P(H^{i-1}(\Omega X ; \mathbf{Z}_p))$$

is a monomorphism if $i \not\equiv 2 \ (2p)$ and an epimorphism if $i \not\equiv -1 \ (2p)$.

Secondly,

THEOREM 5.8. *Let $\pi : E \rightarrow B$ be a fibre map such that E and B are H-spaces and π is multiplicative. Let the fibre of π be F. Then in the spectral se-*

quence of π,

$$E_r \approx E_r^{*,0} \otimes E_r^{0,*} \otimes E(\cdots x_i \cdots) \otimes E(\cdots w_j \cdots) \qquad \textit{for} \quad r \geq 2,$$

where the filtration degree $x_i < r$, *and the complementary degree* $w_j < r - 1$. *Further,* $\dim x_i = p^{q_i}(2m_i) + 1$ *where* $2m_i$ *is the dimension of some generator of* $H^*(F : \mathbf{Z}_p)$ *and* $\dim w_j = p^{t_j}(2n_j) - 1$, *where* $2n_j$ *is the dimension of some generator of* $H^*(B\,;\mathbf{Z}_p)$.

Now consider the previous universal fibration, and for $i \not\equiv 0\ (p)$, let x_{2i} in $H^{2i}(BU\,;\mathbf{Z}_p)$ be defined by $x_{2i} = \sigma\bar{y}_{2i+1}$. Then by the first theorem above, $x_{2i} \neq 0$, and furthermore must be indecomposable. Otherwise, being primitive, it would be a p^{th} power, which for dimensional reasons is impossible.

Remark. If $i \equiv 0\ (p)$, say $i = p^j k$ where $k \not\equiv 0\ (p)$, then

$$\mathcal{P}^{p^{j-1}k}(\bar{y}_{2p^{j-1}k+1}) = \bar{y}_{2p^j k+1}\,.$$

Thus by induction on j we obtain

$$\sigma\bar{y}_{2i+1} = \sigma\mathcal{P}^{p^{j-1}k}(\bar{y}_{2p^{j-1}k+1}) = \mathcal{P}^{p^{j-1}k}(\sigma\bar{y}_{2p^{j-1}k+1}) = (x_{2k}^{p^{j-1}})^p = x_{2k}^{p^j}$$

where for $p = 2$, interpret $\mathcal{P}^1$ as Sq^{21}.

Since x_{2i}, $i \not\equiv 0\ (p)$, is indecomposable, we may write

$$H^*(BU\,;\mathbf{Z}_p) \approx P(\cdots, x_{2i}, \cdots, c_{2pj}, \cdots), \qquad i \not\equiv 0\ (p), \quad j \geq 1.$$

By the above remark, the c_{2pj} are *not* transgressive, nor in fact is any indecomposable element of dimension $2pj$.

Now consider the fibrations

$$\begin{array}{ccc} BU \xrightarrow{\ i\ } BU_p & & E \leftarrow BU \\ \downarrow \pi & & \downarrow \\ SU & \xrightarrow{\ p\ } & SU. \end{array}$$

Then on the right, $H^*(SU\,;\mathbf{Z}_p) \approx E(\bar{y}_3, \cdots, \bar{y}_{2i+1}, \cdots)$, and the $\bar{y}_{2i+1}$ are primitive. So by the preceding, and the naturality of transgression, we have on the left: x_{2i} is transgressive for $i \not\equiv 0\ (p)$ and $\tau x_{2i} = p\bar{y}_{2i+1} = 0$. Thus, for $i \not\equiv 0\ (p)$ let α_{2i} in $H^{2i}(BU_p : \mathbf{Z}_p)$ be defined by $i^*\alpha_{2i} = x_{2i}$. Then in the spectral sequence for π, $P(\cdots, x_{2i}, \cdots)$, $i \not\equiv 0\ (p)$, is contained in $\operatorname{im} i^* = E_\infty^{0,*}$. We want to show the above inclusion is also onto.

For this consider the following piece of the long exact sequence of spaces:

$$\begin{array}{ccc} BU & \xrightarrow{\ p\ } & BU \\ & & \downarrow i \\ & & BU_p \xrightarrow{\ \pi p\ } SU\,. \end{array}$$

Note that we have previously written simply π for the more cumbersome πp. Now, as is well known, if $f : E \to B$ is a map, then Ef is the fibre of the fibration obtained by converting f to a fibration. Furthermore, if f was already a fibre map with fibre F, then $F \equiv Ef$. Thus in the above sequence we may assume $BU = E(\pi p)$, $i = \pi^2 p$ and p is the inclusion of the fibre. Also, if in the above discussion E and B are H-spaces and f is an H-map, then Ef is an H-space in a natural way, and πf is a multiplicative fibre map. Giving all spaces above the H-space structure of pointwise multiplication of maps, p is multiplicative. Thus, π is a multiplicative fibre map, and so, we may assume, is i. Therefore, Browder's second theorem on the structure of terms in the spectral sequence of a multiplicative fibre map applies to both π and i.

Consider the spectral sequence of i. Then $\operatorname{im} p^* \approx E_\infty^{0,*} = P(c_2^p, \cdots, c_{2i}^p, \cdots)$, and $\operatorname{im} i^* \approx E_\infty^{*,0}$. Now Browder's theorem says

$$E_\infty \approx E_\infty^{*,0} \otimes E_\infty^{0,*} \otimes E$$

where E is an exterior algebra on odd-dimensional generators. Here, however, $H^*(BU\,; \mathbf{Z}_p)$ has only even-dimensional elements, so that

$$E_\infty \approx E_\infty^{*,0} \otimes E_\infty^{0,*}$$

and 4.4 of the appendix gives $H^*(BU\,; \mathbf{Z}_p) \approx \operatorname{im} i^* \otimes \operatorname{im} p^*$ as an algebra, since $\operatorname{im} p^*$ is a polynomial algebra. Let the map

$$q : \operatorname{im} p^* \to H^*(BU\,; \mathbf{Z}_p)$$

be the unique extension of the function $c_{2j}^p \to c_{2pj}$. Denote the inclusion $\operatorname{im} i^* \to H^*(BU\,; \mathbf{Z}_p)$ by η. Then the above isomorphism is given by

$$\operatorname{im} i^* \otimes \operatorname{im} p^* \xrightarrow{\eta \otimes q} H^*(BU\,; \mathbf{Z}_p).$$

Now write

$$H^*(BU\,;\mathbf{Z}_p) \approx P(\cdots, x_{2i}, \cdots) \otimes P(\cdots, c_{2pj}, \cdots), \qquad i \not\equiv 0\,(p), \quad j \geq 1$$

and denote the inclusion $P(\cdots, x_{2i}, \cdots) \to \operatorname{im} i^*$ by η'. Map

$$P(\cdots, x_{2i}, \cdots) \otimes P(\cdots, c_{2pj}, \cdots) \to \operatorname{im} i^* \otimes \operatorname{im} p^*, \qquad i \not\equiv 0\,(p) \quad j \geq 1$$

by $\eta' \otimes p^*$. Then $(\eta \otimes q)(\eta' \otimes p^*)$ is an isomorphism, since

$$(\eta \otimes q)(\eta' \otimes p^*) = (\eta\eta') \otimes 1$$

is a monomorphism, and both domain and range have the same finite dimension in each degree. Thus $\eta' \otimes p^*$ is an isomorphism also. Since

$$\eta' \otimes p^* = (\eta' \otimes 1)(1 \otimes p^*),$$

and $p^* \mid P(\cdots, c_{2pj}, \cdots)$ is an isomorphism (hence so is $1 \otimes p^*$), it follows that $\eta' \otimes 1$ is also an isomorphism. Now, considering all algebras simply as graded vector spaces over $\mathbf{Z}_p$, let K be the cokernel of η'. Then the following is exact (and splits):

$$0 \to P(\cdots, x_{2i}, \cdots) \xrightarrow{\eta'} \operatorname{im} i^* \to K \to 0.$$

Tensoring with $P(\cdots, c_{2pj}, \cdots)$ gives the exact sequence

$$0 \to P(\cdots, x_{2i}, \cdots) \otimes P(\cdots, c_{2pj}, \cdots)$$
$$\xrightarrow[\approx]{\eta' \otimes 1} \cdot \mathrm{im}\, i^* \otimes P(\cdots, c_{2pj}, \cdots) \to K \otimes P(\cdots, c_{2pj}, \cdots) \to 0.$$

Thus $K \otimes P(\cdots, c_{2pj}, \cdots) = 0$, giving $K = 0$. Therefore,

$$\mathrm{im}\, i^* = P(\cdots, x_{2i}, \cdots), \qquad i \not\equiv 0\ (p).$$

We return now to our original fibration for BU_p and determine the kernel of π^*. We remark first, that for $i \not\equiv 0\ (p)$, we have $\pi^* \bar{y}_{2i+1} \neq 0$ (in fact is indecomposable). If it were zero, then $0 = \sigma \pi^* \bar{y}_{2i+1} = \pi'^* \sigma \bar{y}_{2i+1} = \pi^* x_{2i}$, which is false. Consider $\bar{y}_{2pj+1}$, however. Since $\bar{y}_{2pj+1}$ is primitive, and π is multiplicative, it follows that $\pi^* \bar{y}_{2pj+1}$ must also be primitive. Suppose it were indecomposable. Then by Browder's first theorem we would have (if $j = p^{q-1}k$ where $k \not\equiv 0\ (p)$)

$$0 \neq \sigma \pi^* \bar{y}_{2pj+1} = \pi'^* \sigma \bar{y}_{2pj+1} = \pi'^* x_{2k}^{p^q} = 0.$$

Thus, $\pi^* \bar{y}_{2pj+1}$ is decomposable, and if it is not zero, it must be a p^{th} power of a non-zero element of dimension h. But then we would have $2pj + 1 = ph$ or $p(h - 2j) = 1$, which is absurd. Therefore $\pi^* \bar{y}_{2pj+1} = 0$, and $\bar{y}_{2pj+1}$ is in $\ker \pi^*$. Thus, $\ker \pi^* =$ ideal in $H^*(SU; \mathbf{Z}_p)$ generated by the $\bar{y}_{2pj+1}$, and $\mathrm{im}\, \pi^* \approx E_\infty^{*,0} \approx E(\cdots, \bar{y}_{2i+1}, \cdots)$ where $i \not\equiv 0\ (p)$. For these values of i, define $\alpha_{2i+1} \in H^{2i+1}(BU_p; \mathbf{Z}_p)$ by $\alpha_{2i+1} = \pi^* \bar{y}_{2i+1}$.

We will now show that

$$E_\infty \approx E_\infty^{*,0} \otimes E_\infty^{0,*}.$$

Using the second quoted theorem of Browder we have

$$E_\infty \approx E_\infty^{*,0} \otimes E_\infty^{0,*} \otimes E(\cdots, x_i, \cdots).$$

(There are never any classes of type w, since these arise from even-dimensional generators in the base, which do not exist here.) Let the x_i be arranged by total degree, so that x_1 has minimal dimension, and recall that dimension $x_i \equiv 1\ (2p)$ for all i. Let μ_n be a class in $H^n(BU_p; \mathbf{Z}_p)$ such that $\chi\mu_n = x_1$. In this case the technique of 4.3 gives

$$\eta \otimes q : E_\infty^{*,0} \otimes E_\infty^{0,*} \to H^*(BU_p; \mathbf{Z}_p)$$

is an algebra monomorphism, and an *isomorphism* through dimension $n - 1$. Since μ_n is least dimensional, if it were decomposable it would lie in the image of $\eta \otimes q$. But then, passing to E_∞, x_1 would be in the image of $\overline{\eta \otimes q}$, which is false. Thus μ_n is indecomposable. Suppose $n = 2pj + 1$. Then μ_{2p+1} is in $Q(H^*(BU_p; \mathbf{Z}_p))$. Since BU_p is a homotopy commutative, homotopy associative H-space, Milnor-Moore [7], Proposition 4.23 shows

$$P(H^*(BU_p; \mathbf{Z}_p))_n \approx Q(H^*(BU_p; \mathbf{Z}_p))_n$$

for odd n. Thus we obtain a class μ'_{2pj+1} in $P(H^*(BU_p;\mathbf{Z}_p))$. But by the suspension theorem of Browder quoted earlier, σ is an epimorphism in this dimension, so we get a class $\bar{\mu}_{2pj+2}$ in $Q(H^*(BU_p^\wedge:\mathbf{Z}_p))$ with the property that $\sigma\bar{\mu}_{2pj+2} = \mu'_{2pj+1}$. Now there is only one indecomposable generator in this dimension, namely $\bar{\alpha}_{2pj+2}$. Since up to a non-zero field coefficient, $\sigma\bar{\alpha}_{2pj+2} = \sigma\bar{\mu}_{2pj+2}$, we may assume $\bar{\mu}_{2pj+2} = \bar{\alpha}_{2pj+2}$. Then, however, we get

$$\mu'_{2pj+1} = \sigma\bar{\alpha}_{2pj+2} = \sigma\bar{\pi}^*\bar{c}_{2pj+2} = \pi^*\sigma\bar{c}_{2pj+2} = \pi^*\bar{y}_{2pj+1} = 0,$$

which is a contradiction. Thus there are no classes x_i and

$$E_\infty \approx E_\infty^{*,0} \otimes E_\infty^{0,*}.$$

A final application of 4.4 gives

$$H^*(BU_p;\mathbf{Z}_p) \approx E(\cdots, \alpha_{2i+1}, \cdots) \otimes P(\cdots, \alpha_{2i}, \cdots), \qquad i \not\equiv 0 \ (p)$$

as an algebra.

As an application of these results we may prove: in a Postnikov system for BU_p we have $k^{2n-2} = 0$ for $n < p$, is unequal to 0 for $n \geq p$, and $k^{2p-2} = \mathcal{P}_p^1\beta_p$. The first statement follows from dimensional considerations. For the last, consider the $2p^{\text{th}}$ stage in a Postnikov system for BU_p.

$$\begin{array}{lll} K(\mathbf{Z}_p, 2p) \to & (BU_p)_{2p} & \\ & \downarrow & \\ & (BU_p)_{2p-2} \xrightarrow{k^{2p-2}} & K(\mathbf{Z}_p, 2p+1) \\ & \downarrow & \\ & \vdots & \\ & \downarrow & \\ & K(\mathbf{Z}_p, 2). & \end{array}$$

Let ι^{2p} denote the fundamental class of $K(\mathbf{Z}_p, 2p)$. Then since $\tau\iota^{2p} = k^{2p-2}$, if $k^{2p-2} = 0$ ι^{2p} appears as an indecomposable element of $H^{2p}((BU_p)_{2p};\mathbf{Z}_p)$, but this gives an indecomposable element of $H^{2p}(BU_p;\mathbf{Z}_p)$, which is impossible by the above. Thus $k^{2p-2} \neq 0$. Since

$$(BU_p)_{2p-2} = \textstyle\prod_{j=1}^{p-1} K(\mathbf{Z}_p, 2j)$$

it follows by dimension that $k^{2p-2} = \mathcal{P}_p^1\beta_p(\iota^2)$, where ι^2 is the fundamental class of $K(\mathbf{Z}_p, 2)$. Since higher k-invariants must suspend to k^{2p-2} by periodicity, $k^{2n-2} \neq 0$ for $n \geq p$.

Having this, one may also compute the primary cohomology operation that appears as d_{2p-1} in the spectral sequence converging to $KU^*(\ :\mathbf{Z}_p)$. Namely, the fact that it *is* a differential, i.e. has square zero, and must suspend to k^{2p-2} by periodicity, shows that it must be $\mathcal{P}_p^1\beta_p - \beta_p\mathcal{P}_p^1$.

4. Appendix

Here we prove a general theorem on connecting homomorphisms in chain complexes, and then, under some very special conditions, give a procedure for finding the Bockstein operator mod p in the total space of a fibration. We shall also give a very mild generalization of a theorem of Serre [8], which we used in §2 and §3.

Let δ denote the connecting homomorphism associated with the following exact sequence of torsion free integral cochain complexes:

$$0 \to A \xrightarrow{j} B \xrightarrow{i} C \to 0.$$

We will denote the coboundary in either A, B or C by d. Then consider the exact coefficient sequence:

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \xrightarrow{r} \mathbf{Z}_p \to 0$$

where p is multiplication by p, and r is reduction mod p, and denote by β the connecting homomorphism associated with any one of the exact sequences:

$$0 \to A \xrightarrow{p} A \xrightarrow{r} A \otimes \mathbf{Z}_p \to 0$$

$$0 \to B \xrightarrow{p} B \xrightarrow{r} B \otimes \mathbf{Z}_p \to 0$$

$$0 \to C \xrightarrow{p} C \xrightarrow{r} C \otimes \mathbf{Z}_p \to 0.$$

Note that we identify $p = 1 \otimes p$, $r = 1 \otimes r$. We shall also use the same symbols for the induced homomorphisms in cohomology. We write also j_p for $j \otimes 1 : A \otimes \mathbf{Z}_p \to B \otimes \mathbf{Z}_p$ etc.

Now $i_p^* : H^q(B \otimes \mathbf{Z}_p) \to H^q(C \otimes \mathbf{Z}_p)$, and suppose $x \in H^q(B \otimes \mathbf{Z}_p)$ is such that $i_p^* x \in \operatorname{im} r$. Let $x_1 \in H^q(C)$ satisfy $rx_1 = i_p^* x$. Consider $\delta x_1 \in H^{q+1}(A)$. Then $r\delta x_1 = \delta_p r x_1 = \delta_p i_p^* x = 0$, so there is $x_2 \in H^{q+1}(A)$ such that $px_2 = \delta x_1$. Consider $j^* x_2 \in H^{q+1}(B)$. Suppose we had chosen x_1' instead of x_1. Then $r(x_1 - x_1') = 0$ so $x_1 - x_1' = p\bar{x}_1$ and $\delta x_1 - \delta x_1' = p\delta\bar{x}_1$, and $j^*\delta\bar{x}_1 = 0$, so the choice of x_1 is immaterial. Suppose, however, we had chosen x_2' instead of x_2. Then $p(x_2 - x_2') = 0$ and $x_2 - x_2' = \beta\bar{x}_2$. Hence

$$j^* x_2 - j^* x_2' = j^* \beta \bar{x}_2 = \beta j_p^* \bar{x}_2 .$$

Thus, we have that $j^* p^{-1} \delta r^{-1} i_p^*(x)$ is well defined modulo $\beta \operatorname{im} j_p^*$, or since $\operatorname{im} j_p^* = \ker i_p^*$, modulo $\beta \ker i_p^*$.

THEOREM 4.1. $\beta x = j^* p^{-1} \delta r^{-1} i_p^*(x) \bmod \beta \ker i_p^*$.

Proof. Let $x = [b_p]$—the cohomology class of $b_p \in B^q \otimes \mathbf{Z}_p$. Let $b^1 \in B^q$ be such that $rb^1 = b_p$, and let $b^2 \in B^{q+1}$ be such that $pb^2 = db^1$. Then, by definition, $\beta x = [b^2]$. Now consider the diagram:

$$\begin{tikzcd}
B \arrow[r, "r"] \arrow[d, "i"'] & B \otimes \mathbf{Z}_p \arrow[r] \arrow[d, "i_p"] & 0 \\
C \arrow[r, "r"] \arrow[d] & C \otimes \mathbf{Z}_p \arrow[r] \arrow[d] & 0 \\
0 & 0. &
\end{tikzcd}$$

Let $c^1 \in C^q$ satisfy $rc^1 = i_p b_p$. Since $i_p^* x = [i_p b_p] \in \operatorname{im} r$ in cohomology, c^1 may be chosen to be a cocycle. Choose $b^3 \in B^q$ such that $ib^3 = c^1$, and $a^1 \in A^{q+1}$ such that $ja^1 = db^3$. Then, by definition, $[a^1] = \delta r^{-1} i_p^*(x)$. Then

$$ri(b^3 - b^1) = rib^3 - rib^1 = rc^1 - i_p rb^1 = i_p b_p - i_p b_p = 0.$$

Thus, there is $c^2 \in C^q$ such that $pc^2 = i(b^3 - b^1) = c^1 - ib^1$ and $b^4 \in B^q$ such that $ib^4 = c^2$. Now $pdc^2 = d(c^1 - ib^1) = -dib^1 = -idb^1 = -ipb^2 = -pib^2$. So since p is a monomorphism, we have $dc^2 = -ib^2$, and hence

$$i(b^2 + db^4) = ib^2 + i\,db^4 = -dc^2 + dc^2 = 0.$$

So, let $a^2 \in A^{q+1}$ be such that $ja^2 = b^2 + db^4$, and consider pa^2. pa^2 is a cocycle, and $[pa^2] = pj^{*-1}\beta(x)$. Now

$$\begin{aligned} j(a^1 - pa^2) = db^3 - pja^2 = db^3 - p(b^2 + db^4) &= db^3 - pb^2 - pdb^4 \\ &= db^3 - db^1 - dpb^4 = d(b^3 - b^1 - pb^4). \end{aligned}$$

But

$$\begin{aligned} i(b^3 - b^1 - pb^4) = c^1 - ib^1 - ipb^4 &= c^1 - ib^1 - pib^4 \\ &= c^1 - ib^1 - pc^2 = c^1 - ib^1 - (c^1 - ib^1) = 0. \end{aligned}$$

Therefore, there is $a^3 \in A^q$ such that $ja^3 = b^3 - b^1 - pb^4$. Thus, $j(a^1 - pa^2) = dja_3 = jda^3$, and since j is a monomorphism, we have finally, $a^1 - pa^2 = da^3$. Therefore,

$$pj^{*-1}\beta(x) = \delta r^{-1} i_p^*(x) \quad \text{or} \quad \beta(x) = j^* p^{-1}\, \delta r^{-1} i_p^*(x)$$

modulo indeterminacy in $\beta \ker i_p^*$.

COROLLARY 4.2. *Let (X, A) be a pair of spaces, and let*

$$0 \to C(X, A) \xrightarrow{j} C(X) \xrightarrow{i} C(A) \to 0$$

denote the short exact sequence of singular integral cochains. Then β is the Bockstein operator associated with the exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_p \to 0.$$

δ is the connecting homomorphism of the pair (X, A) with integer coefficients and assuming $i_p^*(x) \in \operatorname{im} r$, we have

$$\beta x = j^* p^{-1} \delta r^{-1} i_p^*(x) \quad \operatorname{mod} \beta \ker i_p^*.$$

Of course, if the above formula is reduced mod p, one obtains the same formula for the Bockstein β_p associated with

$$0 \to \mathbf{Z}_p \xrightarrow{p} \mathbf{Z}_{p^2} \xrightarrow{r} \mathbf{Z}_p \to 0,$$

with δ replaced by the connecting homomorphism of (X, A) with $\mathbf{Z}_{p^2}$ coefficients.

Now consider a fibration

$$F \xrightarrow{i} E \xrightarrow{\pi} B.$$

We have a commutative diagram

$$\begin{tikzcd} H^*(E, F) \arrow[rr, "j^*"] & & H^*(E) \\ & H^*(B). \arrow[ul, "\pi'^*"] \arrow[ur, "\pi^*"'] & \end{tikzcd}$$

Suppose $x \in H^q(E\,;\mathbf{Z}_p)$ is such that $i_p^* x \in \operatorname{im} r$, where

$$i_p^* : H^*(E\,;\mathbf{Z}_p) \to H^*(F\,;\mathbf{Z}_p)$$

and suppose further that there is $z_x \in r^{-1} i_p^* x$ such that z_x is transgressive. Then we have $y_x \in H^{q+1}(B)$ with $\pi'^* y_x = \delta z_x$. Assume now $y_x = p y_x'$. Then $p\pi'^* y_x' = \pi'^* p y_x' = \pi'^* y_x = \delta z_x$. Hence for an element in $p^{-1}\delta r^{-1} i_p^* x$ we may choose $\pi'^* y_x'$. Then

$$j^* \pi'^* y_x' = \pi^* y_x' = j^* p^{-1} \delta r^{-1} i_p^* x = \beta x \mod \beta \ker i_p^* .$$

Therefore we obtain finally the following procedure. Let $x \in H^q(E\,;\mathbf{Z}_p)$ satisfy the following properties:

(i) $i_p^* x \in \operatorname{im} r$.
(ii) There is a z_x in $r^{-1} i_p^* x$ which is transgressive.
(iii) There is y_x in τz_x such that $y_x = p y_x'$.

Then, mod $\beta \ker i_p^*$ we have $\beta x = \pi^* y_x'$.

For the theorem of Serre, let

$$F \xrightarrow{i} E \xrightarrow{\pi} B$$

be a fibration, and suppose that in the spectral sequence of π (with coefficients in a field K) we have

$$E_\infty \approx E_\infty^{*,0} \otimes E_\infty^{0,*}$$

the isomorphism being the one given by inclusion. Then

Proposition 4.3. *$H^*(E) \approx E_\infty^{*,0} \otimes E_\infty^{0,*}$ as a left $E_\infty^{*,0}$ module.*

Proof. We have the following two standard commutative diagrams:

$$\begin{tikzcd} H^q(E) \arrow[rr, "i^*"] \arrow[dr, "\chi"'] & & H^q(F) \\ & E_\infty^{0,q} \arrow[ur, "\eta"'] & \end{tikzcd} \qquad \begin{tikzcd} H^p(B) \arrow[rr, "\pi^*"] \arrow[dr, "\chi"'] & & H^p(E) \\ & E_\infty^{p,0} \arrow[ur, "\eta"'] & \end{tikzcd}$$

where, in each case, χ is an epimorphism and η is a monomorphism. So, let

$q : E_\infty^{0,q} \to H^q(E)$ be an additive right inverse for χ, i.e. $\chi q = 1$. Consider the map

$$\eta \otimes q : E_\infty^{*,0} \otimes E^{0,*} \to H^*(E)$$

and filter the left hand side by

$$F^p(E_\infty^{*,0} \otimes E_\infty^{0,*}) = \sum_{i \geq p} E_\infty^{i,0} \otimes E_\infty^{0,*}.$$

Then $\eta \otimes q$ is filtration preserving, and induces

$$\overline{\eta \otimes q} : E_\infty^{p,0} \otimes E_\infty^{0,*} \to E_\infty^{p,*},$$

which is an isomorphism by assumption. Thus $\eta \otimes q$ is an isomorphism. Since $\eta \otimes q$ is clearly an $E_\infty^{*,0}$ module homomorphism, the proposition is proved.

COROLLARY 4.4. *The above isomorphism is valid as algebras iff q may be taken to be an algebra homomorphism, for example, when $E_\infty^{0,*}$ is free as an algebra.*

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