A CHARACTERIZATION OF SOME MULTIPLY TRANSITIVE PERMUTATION GROUPS, I

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The objective of this paper is to give a proof of the following result:

Theorem A. *Let G be a finite simple group which contains an involution t such that the following conditions are satisfied*:

(I) *The centralizer $\mathbf{C}_G(t)$ of t in G is a splitting extension of an elementary abelian normal 2-subgroup of order at most 16 by S_4, the symmetric group of degree four*;

(II) *the centre of a Sylow 2-subgroup of $\mathbf{C}_G(t)$ is cyclic.*

Then G is isomorphic to one of the following groups A_8, A_9, A_{10} or M_{22}. Here A_n denotes the alternating group of degree n, and M_{22} is the Mathieu simple group on 22 letters.

This result is a consequence of the following

Theorem B. *Let π_0 be an involution contained in the centre of a Sylow 2-subgroup of A_{10}. Denote by H_0 the centralizer of π_0 in A_{10}.*

Let G be a finite group with the following two properties:

(a) *G has no subgroups of index 2, and*

(b) *G possesses an involution π such that the centralizer $\mathbf{C}_G(\pi)$ of π in G is isomorphic to H_0.*

Then G is isomorphic to A_{10}.

Remark. Let G be a group satisfying the assumptions of Theorem A. Then $\mathbf{C}_G(t)$ contains an elementary abelian normal 2-subgroup M of order at most 16 such that $\mathbf{C}_G(t)$ is a splitting extension of M by S_4. Hence $|M|$ is equal to 8 or 16. It is straightforward to check, that, if $|M| = 8$, then $\mathbf{C}_G(t)$ is uniquely determined. Application of the result in [8] yields that G is isomorphic to A_8 or A_9 if $|M| = 8$. However, if $|M| = 16$, there are precisely two possibilities for $\mathbf{C}_G(t)$ as has been observed in [10]. One of these possibilities is that $\mathbf{C}_G(t)$ is isomorphic to the centralizer H_1 of an involution of M_{22}, the other possibility is that $\mathbf{C}_G(t)$ is isomorphic to the centralizer of an involution of A_{10}. The theorem in [10] states that if $\mathbf{C}_G(t)$ is isomorphic to H_1 then G is isomorphic to M_{22}. Hence, in order to prove Theorem A, it suffices to prove Theorem B.

1. Some properties of H_0

The group H_0 is isomorphic to a group H generated by the elements π, μ,

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μ', τ, τ', ρ, λ, ξ subject to the following relations:

$$\pi^2 = \mu^2 = \mu'^2 = \tau^2 = \tau'^2 = \rho^3 = \lambda^2 = \xi^2 = 1,$$

$$\pi\mu = \mu\pi, \quad \pi\mu' = \mu'\pi, \quad \mu\mu' = \mu'\mu, \quad \tau\tau' = \tau'\tau,$$

$$\rho^{-1}\tau\rho = \tau\tau', \quad \rho^{-1}\tau'\rho = \tau, \quad \tau\lambda = \lambda\tau, \quad \lambda\tau'\lambda = \tau\tau',$$

$$\lambda\rho\lambda = \rho^{-1}, \quad \pi\tau = \tau\pi, \quad \tau'\pi = \pi\tau', \quad \rho\pi = \pi\rho, \quad \lambda\pi = \pi\lambda,$$

$$\tau\mu = \mu\tau, \quad \tau'\mu\tau' = \pi\mu, \quad \rho^{-1}\mu\rho = \mu\mu', \quad \lambda\mu = \mu\lambda,$$

$$\tau\mu'\tau = \pi\mu', \quad \tau'\mu' = \mu'\tau', \quad \rho^{-1}\mu'\rho = \mu, \quad \lambda\mu'\lambda = \mu\mu',$$

$$\pi\xi = \xi\pi, \quad \mu\xi = \xi\mu, \quad \mu'\xi = \xi\mu', \quad \xi\tau\xi = \mu\tau,$$

$$\xi\tau'\xi = \tau'\mu', \quad \xi\lambda\xi = \mu\lambda, \quad \xi\rho\xi = \rho\mu.$$

We put

$$D = \langle\pi, \mu, \mu', \tau, \tau', \lambda, \xi\rangle, \quad M = \langle\pi, \mu, \mu', \xi\rangle, \quad S = \langle\pi, \mu, \tau, \lambda\rangle,$$

$$L_1 = \langle\pi, \mu, \lambda, \mu'\xi\rangle \quad \text{and} \quad L_2 = \langle\pi, \mu, \tau\lambda, \xi\rangle.$$

M, S, L_1 and L_2 are the only elementary abelian subgroups of D of order 16. The groups M, S, L_1 and L_2 are all contained in $S\langle\mu', \xi\rangle$ which is equal to $\mathbf{C}_H(\mu)$ and $S\langle\mu', \xi\rangle$ is the only maximal subgroup of D with centre of order 4. The centres of all other maximal subgroups of D are equal to $\langle\pi\rangle$. We have that the elementary abelian subgroups of D of order 16 are self-centralizing in H. Further, $\mathbf{N}_H(M) = H$, $\mathbf{N}_H(S) = D$, $\mathbf{N}_H(L_1) = S\langle\mu', \xi\rangle$, $\mathbf{N}_H(L_2) = S\langle\mu', \xi\rangle$ and $L_1^{\tau'} = L_2$.

The group H is a semi-direct product of its normal subgroup M and its subgroup $\langle\tau, \tau'\rangle\langle\rho\rangle\langle\lambda\rangle$ which is isomorphic to S_4. There are eight classes of conjugate involutions of H with the representatives π, μ, τ, λ, $\pi\lambda$, ξ, $\pi\xi$ and $\tau\lambda\xi$. The orders of the centralizers of these involutions in H are $2^7 3$, 2^6, 2^5, 2^5, 2^5, $2^5 3$, $2^5 3$, 2^4, respectively.

The groups M, S, and L_2 split into D-conjugate classes in the following way:

M: $1; \pi; \mu, \pi\mu; \mu', \pi\mu', \mu\mu', \pi\mu\mu'; \xi, \mu'\xi, \mu\xi, \pi\mu\mu'\xi; \pi\xi, \pi\mu'\xi, \pi\mu\xi, \mu\mu'\xi.$

S: $1; \pi; \mu, \pi\mu; \tau, \pi\tau, \mu\tau, \pi\mu\tau; \lambda, \mu\lambda, \tau\lambda, \pi\mu\tau\lambda; \pi\lambda, \pi\mu\lambda, \pi\tau\lambda, \mu\tau\lambda.$

L_2 : $1; \pi; \mu, \pi\mu; \tau\lambda, \pi\mu\tau\lambda; \pi\tau\lambda, \mu\tau\lambda; \xi, \mu\xi; \pi\xi, \pi\mu\xi; \tau\lambda\xi, \pi\mu\tau\lambda\xi, \mu\tau\lambda\xi, \pi\tau\lambda\xi.$

The main problem in this paper is the fusion of the conjugate classes of involutions. Some properties of the alternating groups of low degree are needed for our proof; the character tables of [11] seem to be of some help.

In the whole paper, G denotes a group with properties (a) and (b) of the theorem. Thus we assume that H is embedded in G and that $\mathbf{C}_G(\pi) = H$. The notation $x \sim y$ means that x is conjugate to y. All other notation is standard.

2. Conjugacy classes of involutions of G

(2.1) LEMMA. *The involution π is contained in the centre of a Sylow 2-subgroup of G.*

Proof. Let R be a Sylow 2-subgroup of G containing D. Then $H \cap R = D$. We have $\pi \in D \subseteq R$, and if $y \in \mathbf{Z}(R)$, then $[y, \pi] = 1$. It follows $y \in R \cap = D$. Hence $\mathbf{Z}(R) \subseteq \mathbf{Z}(D) = \langle \pi \rangle$ and so $\mathbf{Z}(R) = \langle \pi \rangle$.

(2.2) LEMMA. *Each involution of G is conjugate to an involution of S.*

Proof. Put $\bar{H} = \langle \pi, \mu, \mu', \tau, \tau', \rho, \lambda \rangle$ and $\bar{D} = S\langle \mu', \tau' \rangle$. It is a consequence of [16; p. 361] that every conjugacy class of involutions of $\bar{H}$ intersects S nontrivially. Application of a lemma in [14] yields that each involution of G is conjugate to some involution in $\bar{D}$.

(2.3) LEMMA. *The involution π is conjugate in G to an involution $t \in H$ with $t \neq \pi$.*

Proof. If π were not conjugate to an involution $t \in H$ with $t \neq \pi$, then π would not be conjugate to any involution of D different from π. Application of [5; Corollary 1, p. 404] would yield $\pi \in \mathbf{Z}(G \bmod O(G))$, and the Frattini-argument of [1; Lemma 1, p. 117] would give $G = HO(G)$ against the assumption that G has no subgroups of index 2.

(2.4) LEMMA. *The involutions π, λ and $\pi\lambda$ do not lie in the same conjugate class of G.*

Proof. Assume the lemma to be false. We have

$$\mathbf{Z}(S\langle \mu'\xi \rangle) = \langle \pi, \mu, \lambda \rangle \quad \text{and} \quad \mathbf{C}_G(\langle \pi, \mu, \lambda \rangle) = S\langle \mu'\xi \rangle.$$

Call this group W. Denote by D_λ^1 a group of order 64 contained in $\mathbf{C}_G(\lambda)$ which contains $S\langle \mu'\xi \rangle$. Define $D_{\pi\lambda}^1$ similarly. It is $W' = \langle \pi\mu \rangle$ and therefore $\mathbf{Z}(D_\lambda^1) = \langle \lambda, \pi\mu \rangle$ and $\mathbf{Z}(D_{\pi\lambda}^1) = \langle \pi\lambda, \pi\mu \rangle$. Put $N = \langle W\langle \xi \rangle, D_\lambda^1, D_{\pi\lambda}^1 \rangle$. Obviously, $\langle \pi\mu \rangle = \mathbf{Z}(N)$. N cannot be a 2-group because otherwise $|N| = 2^7$ but D contains precisely one subgroup of order 64 with centre of order 4. Since N/W is isomorphic to a subgroup of $PSL(2, 7)$ we get that 3 divides $|N/W|$ but 7 does not. Hence $\pi\mu$ is centralized by an element x of order 3 in N. We know that $S \subseteq W\langle \xi \rangle \cap D_\lambda^1 \cap D_{\pi\lambda}^1$ and so since $|\mathbf{Z}(D_\lambda^1)| = |\mathbf{Z}(D_{\pi\lambda}^1)| = 4$ we must have $S \lhd \langle N, D \rangle$. The group S is elementary abelian of order 16. Hence $\mathcal{S} = \mathbf{N}_G(S)/S$ is isomorphic to a subgroup of A_8. The involution $\pi\mu$ of S cannot be conjugate to π under $\mathbf{N}_G(S)$ since $[x, \pi\mu] = 1$ and $H \not\subseteq \mathbf{N}_G(S)$. It follows that $3 \cdot 5$, $3 \cdot 7$ and $5 \cdot 7$ do not divide $|\mathcal{S}|$. But we know that 3 divides $|\mathcal{S}|$. Therefore, for $|\mathcal{S}|$ one obtaines the possibilities $8 \cdot 3$ and $8 \cdot 3^2$.

If N/W is of order $4 \cdot 3$ then $N/W \cong A_4$ and a Sylow 2-subgroup of G would be normalized by an element of order 3 which however is not the case. Hence $N/W \cong S_3$. —Now assume $|\mathcal{S}| = 8 \cdot 3$. In this case $N \lhd \mathbf{N}_G(S)$ and so $\langle \pi\mu \rangle = \mathbf{Z}(\mathbf{N}_G(S))$. But then we would have $\pi\mu = \pi$ which is not possible.

It remains to consider $|\mathcal{S}| = 8 \cdot 3^2$. A Sylow 2-subgroup of $\mathcal{S}$ is dihedral of order 8. [6; Theorem 1, p. 553] implies that $\mathcal{S}$ has a subgroup of index 2. Hence $\mathcal{S}$ is isomorphic either to a Sylow 3-normalizer of A_8 or to the group $(\langle y \rangle \times A)\langle z \rangle$ where $z^2 = y^3 = 1$, $\langle y, z \rangle \cong S_3$, $A \cong A_4$ and $A\langle z \rangle \cong S_4$. Suppose the second case holds. Let T_λ be a Sylow 2-subgroup of $\mathbf{N}_G(S)$ containing D_λ^1. $\mathbf{Z}(T_\lambda)$ is equal either to $\langle \lambda \rangle$, $\langle \pi\mu\lambda \rangle$ or $\langle \pi\mu \rangle$. Clearly $\mathbf{Z}(T_\lambda) = \langle \pi\mu \rangle$ is not possible because in this case we would have $\pi \sim \pi\mu$ in $\mathbf{N}_G(S)$. If $\mathbf{Z}(T_\lambda) = \langle \pi\mu\lambda \rangle$, then note that $\pi\mu\lambda \sim \pi\lambda$ under D, and we get $|D \cap T_\lambda| = 32$. On the other hand, $\mathcal{S}$ contains a normal 2-subgroup of order 4 which yields $|D \cap T_\lambda| = 64$ and gives a contradiction. If $\mathbf{Z}(T_\lambda) = \langle \lambda \rangle$ one argues similarly.

Finally, we have to consider the case that $\mathcal{S}$ is isomorphic to a Sylow 3-normalizer of A_8. The four-group $\langle \mu', \tau' \rangle S/S$ acts on $\mathfrak{M}$ where by $\mathfrak{M}$ we denote $\mathbf{O}(\mathcal{S})$. Put $\alpha_1 = \mu'S$, $\alpha_2 = \tau'S$, $\alpha_3 = \mu'\tau'S$. A result due to R. Brauer [15; p. 146] yields

$$|\mathfrak{M}| \cdot |\mathbf{C}_{\mathfrak{M}}(\langle \alpha_1, \alpha_2 \rangle)|^2 = |\mathbf{C}_{\mathfrak{M}}(\alpha_1)| \cdot |\mathbf{C}_{\mathfrak{M}}(\alpha_2)| \cdot |\mathbf{C}_{\mathfrak{M}}(\alpha_3)|.$$

It is $|\mathfrak{M}| = 9$ and for $i = 1, 2, 3$ the integer $|\mathbf{C}_{\mathfrak{M}}(\alpha_1)|$ is a divisor of 3. It follows that

$$\mathbf{C}_{\mathfrak{M}}(\langle \alpha_1, \alpha_2 \rangle) = 1 \quad \text{and} \quad |\mathbf{C}_{\mathfrak{M}}(\alpha_i)| = |\mathbf{C}_{\mathfrak{M}}(\alpha_j)| = 3$$

for certain two different involutions α_i and α_j in $\langle \alpha_1, \alpha_2 \rangle$. Therefore, in $\mathbf{N}_G(S)$, we have that

(1) $S\langle \mu' \rangle$ and $S\langle \tau' \rangle$

or

(2) $S\langle \mu' \rangle$ and $S\langle \mu'\tau' \rangle$

or

(3) $S\langle \tau' \rangle$ and $S\langle \mu'\tau' \rangle$

are normalized by elements of order 3. It is $\mathbf{Z}(S\langle \mu' \rangle) = \langle \pi, \mu \rangle$, $\mathbf{Z}(S\langle \tau' \rangle) = \langle \pi, \tau \rangle$ and $\mathbf{Z}(S\langle \mu'\tau' \rangle) = \langle \pi, \mu\tau \rangle$. The first two cases cannot happen because $\pi \not\sim \pi\mu$ in $\mathbf{N}_G(S)$ and $H \not\subseteq \mathbf{N}_G(S)$. In the third case conjugates of π in $\mathbf{N}_G(S)$ are π, τ, $\pi\tau$, $\mu\tau$, $\pi\mu\tau$. Denote by T_λ a Sylow 2-subgroup of $\mathbf{N}_G(S)$ with $D_\lambda^1 \subset T_\lambda$. The group $\langle \pi\mu \rangle$ cannot be the centre of T_λ. Hence $\mathbf{Z}(T_\lambda)$ is either $\langle \lambda \rangle$ or $\langle \pi\mu\lambda \rangle$. Consequently we get that π is conjugate to λ or to $\pi\lambda$ in $\mathbf{N}_G(S)$. If $|\mathbf{N}_G(L_2)| = 2^7 3^2$, then π would have 18 conjugates in L_2 under $\mathbf{N}_G(L_2)$ against $|L_2| = 16$. If $|\mathbf{N}_G(M)| = 2^7 3^2$, then π would have precisely 3 conjugates in M under $\mathbf{N}_G(M)$ which is not possible. We have proved that S is not conjugate to M and not conjugate to L_2 in G. If $\mathbf{Z}(T_\lambda) = \langle \lambda \rangle$, then $|T_\lambda \cap \mathbf{C}(\pi\mu\lambda)| = 64$ and so $\pi\mu\lambda$ is conjugate to μ in $\mathbf{N}_G(S)$. If $\mathbf{Z}(T_\lambda) = \langle \pi\mu\lambda \rangle$, then $|T_\lambda \cap \mathbf{C}(\lambda)| = 64$ and λ is conjugate to μ in $\mathbf{N}_G(S)$. In any case we obtain $\mu \sim \pi$ in G. Denote by D_μ a Sylow 2-subgroup of $\mathbf{C}_G(\mu)$ which contains $S\langle \mu', \xi \rangle$. Since all the elementary abelian subgroups of D and D_μ are contained in $S\langle \mu', \xi \rangle$ we get $S \lhd \langle D, D_\mu \rangle$. It follows $\pi \sim \mu \sim \pi\mu$ in $\mathbf{N}_G(S)$, a contradiction. The lemma is proved.

(2.5) LEMMA. *Interchanging* λ *and* $\pi\lambda$ *if necessary we may and shall assume that* π *is not conjugate to* λ *in* G.

(2.6) LEMMA. *The involutions* π *and* μ *are not conjugate in* G.

Proof. By way of contradiction assume $\pi \sim \mu$ in G. Suppose first that neither τ, $\pi\lambda$, ξ, $\pi\xi$ nor $\tau\lambda\xi$ is conjugate to π in G. Each of the groups S and L_2 contains only 3 involutions conjugate to π in G whereas M contains 7 involutions conjugate to π. It follows that M is not conjugate to L_2 and not conjugate to S. If D_μ denotes a Sylow 2-subgroup of $\mathbf{C}_G(\mu)$ which contains $S\langle\mu', \xi\rangle$, then all the elementary abelian subgroups of order 16 of D_μ are contained in $S\langle\mu', \xi\rangle$. It follows $M \lhd \langle H, D_\mu\rangle$ and $\langle\pi, \mu\rangle \lhd \langle D, D_\mu\rangle$. Clearly, $\langle D, D_\mu\rangle$ is not a 2-group and therefore $\langle D, D_\mu\rangle$ contains an element v of order 3 with $\pi^v = \mu$, $\mu^v = \pi\mu$. Hence π has precisely 7 conjugates in M under $\mathbf{N}_G(M)$. It follows $|\mathbf{N}_G(M)| = 2^7\cdot 3\cdot 7$. $\mathbf{N}_G(M)/M$ acts faithfully on $\langle\pi, \mu, \mu'\rangle$ and so $\mathbf{N}_G(M)/M = PSL(2, 7)$. The involution ξ possesses 4 or 8 conjugates under $\mathbf{N}_G(M)$. Since $|\mathbf{C}_H(\xi)| = 2^5\cdot 3$ we obtain $|\mathbf{C}(\xi) \cap \mathbf{N}_G(M)| = 2^5\cdot 3\cdot 7$. Denote by γ an element of order 7 in $\mathbf{C}(\xi) \cap \mathbf{N}_G(M)$. γ acts transitively on $\{\mu\xi, \mu'\xi, \pi\mu\mu'\xi, \pi\xi, \pi\mu\xi, \pi\mu'\xi, \mu\mu'\xi\}$. Hence ξ possesses precisely 8 conjugates under $\mathbf{N}_G(M)$ against $|\mathbf{C}(\xi) \cap \mathbf{N}_G(M)| = 2^5\cdot 3\cdot 7$.

We have shown that at least one of the involutions τ, $\pi\lambda$, ξ, $\pi\xi$ and $\tau\lambda\xi$ is conjugate to π in G.

Suppose that $\pi \sim \tau$ or $\pi \sim \pi\lambda$ holds in G. Assume first $\pi \sim \tau$ in G. Denote by D_τ^1 a group of order 64 with $S\langle\tau'\rangle \subset D_\tau^1 \subset \mathbf{C}_G(\tau)$. Then $S \lhd \langle D, D_\tau^1\rangle$ since S char $S\langle\tau'\rangle$. Further, $\langle D, D_\tau^1\rangle$ is not a 2-group because $|\mathbf{C}_H(\tau)| = 32$. Since $\lambda \not\sim \pi$ in G we get the following possibilities for $|\mathbf{N}_G(S)| : 2^7\cdot 3$, $2^7\cdot 7$, $2^7\cdot 5$, $2^7\cdot 3^2$. The case $|\mathbf{N}_G(S)| = 2^7\cdot 7$ or $2^7\cdot 5$ cannot happen because A_8 has no subgroups of order $2^3\cdot 7$ or $2^3\cdot 5$ with dihedral Sylow 2-subgroups. If $|\mathbf{N}_G(S)| = 2^7\cdot 3$, then π, μ and $\pi\mu$ are the only conjugates of π under $\mathbf{N}_G(S)$. Denote by X a Sylow 2-subgroup of $\mathbf{N}_G(S)$ with $D_\tau^1 \subset X$. It follows that $\mathbf{Z}(X)$ is equal to $\langle\mu\rangle$ or to $\langle\pi\mu\rangle$. It is $|X \cap \mathbf{C}(\tau)| = 64$ and so $\tau \sim \mu$ in $\mathbf{N}_G(S)$ since μ and $\pi\mu$ are the only elements of D such that their centralizers intersect D in a group of order 64. This contradicts the fact that π, μ and $\pi\mu$ are the only conjugates of π under $\mathbf{N}_G(S)$. We are in the case $|\mathbf{N}_G(S)/S| = 2^3 3^2$ and so $\pi \sim \tau \sim \pi\lambda$ under $\mathbf{N}_G(S)$. —Assume now $\pi \sim \pi\lambda$ in G. Denote by $D_{\pi\lambda}^1$ a group of order 64 with $S\langle\mu'\xi\rangle \subset D_{\pi\lambda}^1 \subset \mathbf{C}_G(\pi\lambda)$. It is $\mathbf{Z}(D_{\pi\lambda}^1) = \langle\pi\lambda, \pi\mu\rangle$ and so $S \lhd \langle D, D_{\pi\lambda}^1\rangle$. Further, $\langle D, D_{\pi\lambda}^1\rangle$ is not a 2-group. $|\mathbf{N}_G(S)/S|$ is equal to either $2^3 3$ or $2^3 3^2$. If $|\mathbf{N}_G(S)/S| = 2^3 3$, denote by X a Sylow 2-subgroup of $\mathbf{N}_G(S)$ which contains $D_{\pi\lambda}^1$. $\mathbf{Z}(X)$ is equal to $\langle\mu\rangle$ or to $\langle\pi\mu\rangle$ and $|X \cap D_{\pi\lambda}^1| = 64$. We obtain $\pi\lambda \sim \mu$ in $\mathbf{N}_G(S)$ which is a contradiction. Hence $|\mathbf{N}_G(S)/S| = 2^3 3^2$ and $\pi \sim \tau \sim \pi\lambda$ in $\mathbf{N}_G(S)$ also in this case. So, if $\pi \sim \tau$ or $\pi \sim \pi\lambda$ in G, then the conjugate class of μ in $\mathbf{N}_G(S)$ consists of μ and $\pi\mu$ because $\mu \sim \pi \not\sim \lambda$ and the fact that both τ and $\pi\lambda$ have 4 conjugates under D. It follows that 3^2 divides $|\mathbf{C}(\mu) \cap \mathbf{N}_G(S)|$ against $\mu \sim \pi$ in G.

We have proved so far that at least one of the involutions ξ, $\pi\xi$ and $\tau\lambda\xi$ is conjugate to π in G and that neither τ nor $\pi\lambda$ are conjugate to π in G. Denote by D_μ a Sylow 2-subgroup of $\mathbf{C}_G(\mu)$ which contains $S\langle\mu', \xi\rangle$. Then $|\langle D, D_\mu\rangle| = 2^7 3$ since $\langle\pi, \mu\rangle \lhd \langle D, D_\mu\rangle$, and $S\langle\mu', \xi\rangle$ contains all elementary abelian subgroups of order 16 of D_μ. Since M and L_2 contain at least 4 conjugates of π in G and S contains only 3 conjugates of π, we conclude that S is normal in $\langle D, D_\mu\rangle$. The element λ has at least 4 conjugates under $\langle D, D_\mu\rangle$. If 3 divides $|\mathbf{C}(\lambda) \cap \langle D, D_\mu\rangle|$ then denote by v an element of order 3 in $\mathbf{C}(\lambda) \cap \langle D, D_\mu\rangle$. We may choose v so that $\pi^v = \mu$, $\mu^v = \pi\mu$. It follows $(\mu\lambda)^v = \pi\mu\lambda$, and so, λ would have more than 4 conjugates in $\langle D, D_\mu\rangle$. This is a contradiction since $2^5 3$ divides $|\mathbf{C}(\lambda) \cap \langle D, D_\mu\rangle|$ in this case. Hence 3 does not divide $|\mathbf{C}(\lambda) \cap \langle D, D_\mu\rangle|$. Because of $\pi \not\sim \lambda$ we have that λ has precisely 12 conjugates in $\langle D, D_\mu\rangle$. Therefore $\lambda \sim \tau$ in $\langle D, D_\mu\rangle$ and so $S\langle\mu'\xi\rangle$ would be conjugate to $S\langle\tau'\rangle$ against $|\mathbf{Z}(S\langle\tau'\rangle)| = 4$ and $|\mathbf{Z}(S\langle\mu'\xi\rangle)| = 8$. This contradiction proves the lemma.

(2.7) LEMMA. *The involutions π, ξ and $\pi\xi$ do not lie in the same conjugate class of G.*

Proof. Assume that $\pi \sim \xi \sim \pi\xi$ in G. Denote by D_ξ^1 a group of order 64 with $L_2\langle\mu'\rangle \subset D_\xi^1 \subset \mathbf{C}_G(\xi)$. Since $\mathbf{Z}(D_\xi^1) = \langle\xi, \pi\mu\rangle$ we have $M \lhd \langle H, D_\xi^1\rangle$ and $H \subset \langle H, D_\xi^1\rangle$. The involution π has 5 or 9 conjugates in M under $\mathbf{N}_G(M)$. Since $\mathbf{N}_G(M)/M$ is isomorphic to a subgroup of A_8, it follows that π has precisely 5 conjugates in M under $\mathbf{N}_G(M)$. An element of order 5 in $\mathbf{N}_G(M)$ must operate fixed-point-free on M, and so, either $\mu \sim \pi\xi$ or $\mu \sim \xi$ since μ has 6 conjugates in M under H. This contradicts (2.6).

(2.8) LEMMA. *Interchanging ξ and $\pi\xi$ if necessary, we may and shall assume that π is not conjugate to ξ in G.*

(2.9). LEMMA. *The involution π is conjugate to τ or to $\pi\lambda$ in G.*

Proof. Assume by way of contradiction that the lemma is false. By (2.3), (2.5), (2.6) and (2.8) follows that $\pi \sim \pi\xi$ or $\pi \sim \tau\lambda\xi$ in G and $[\mathbf{N}_G(S):D] = 1$.

Suppose first that $\pi \sim \pi\xi$ in G. Denote by $D_{\pi\xi}^1$ a group of order 64 with $L_2\langle\mu'\rangle \subset D_{\pi\xi}^1 \subset \mathbf{C}_G(\pi\xi)$. Since $\mathbf{Z}(D_{\pi\xi}^1) = \langle\pi\xi, \pi\mu\rangle$, we get $L_2 \lhd \langle S\langle\mu', \xi\rangle, D_{\pi\xi}^1\rangle = V$. Clearly, V is not a 2-group and V normalizes $\langle\pi, \mu, \xi\rangle$ since $\mathbf{Z}(L_2\langle\mu'\rangle) = \langle\pi, \mu, \xi\rangle$. Not all involutions of $\langle\pi, \mu, \xi\rangle$ lie in the same conjugate class of G. Hence V contains an element x of order 3 such that $\pi^x = \pi\xi$, $(\pi\xi)^x = \pi\mu\xi$, $\mu^x = \mu\xi$, $(\mu\xi)^x = \xi$ and $[x, \pi\mu] = 1$. From a lemma in [14] we conclude that $\pi\lambda$ is conjugate to an involution of $M\langle\tau, \tau'\rangle\langle\rho\rangle$. It follows that $\pi\lambda$ is conjugate to μ or τ in G. Assume that $\pi\lambda \sim \mu$ in G. Denote by $T_{\pi\lambda}$ a Sylow 2-subgroup of $\mathbf{C}_G(\pi\lambda)$ which contains S. Clearly, $S \lhd \langle D, T_{\pi\lambda}\rangle$ and $\langle D, T_{\pi\lambda}\rangle$ is not a 2-group. It follows $[\mathbf{N}_G(S):D] > 1$ which is not possible. Now assume that $\pi\lambda \sim \tau$ in G. Then 64 divides $|\mathbf{C}_G(\pi\lambda)|$ since $S\langle\mu'\xi\rangle$ and $S\langle\tau'\rangle$ are not isomorphic. Denote by $T_{\pi\lambda}$ a subgroup of $\mathbf{C}_G(\pi\lambda)$ of order 64 which contains

$S\langle\mu'\xi\rangle$. Since $\mathbf{Z}(T_{\pi\lambda}) = \langle\pi\lambda, \pi\mu\rangle$ we have $S \lhd \langle D, T_{\pi\lambda}\rangle$ and $[\mathbf{N}_G(S):D] > 1$ which again cannot happen. We have shown that π is not conjugate to $\pi\xi$ and that π must be conjugate to $\tau\lambda\xi$.

Denote by $D^1_{\tau\lambda\xi}$ a group of order 64 with centre of order 4 and $L_2 \subset D^1_{\tau\lambda\xi} \subset \mathbf{C}_G(\tau\lambda\xi)$. Then $L_2 \lhd \langle S\langle\mu', \xi\rangle, D^1_{\tau\lambda\xi}\rangle = V$. Clearly, V is not a 2-group. It follows $[\mathbf{N}_G(L_2):S\langle\mu', \xi\rangle] = 5$. An element of order 5 in $\mathbf{N}_G(L_2)$ must act fixed-point-free on L_2. Hence, $\mu \sim \tau\lambda$ or $\mu \sim \pi\tau\lambda$ in G. If $\mu \sim \pi\tau\lambda$ then $\mu \sim \lambda$ in G. Denote by T_λ a Sylow 2-subgroup of $\mathbf{C}_G(\lambda)$ which contains S. Then $S \lhd \langle D, T_\lambda\rangle$ and $[\mathbf{N}_G(S):D] > 1$ which is not possible. If $\mu \sim \tau\lambda$ then $\mu \sim \pi\lambda$ in G and again one gets a contradiction. The lemma is proved.

(2.10) LEMMA. *$\mathbf{N}_G(S)/S$ is isomorphic to a Sylow 3-normalizer in A_8. Further $\pi \sim \pi\lambda \sim \tau$ in $\mathbf{N}_G(S)$.*

Proof. From (2.9) we conclude that $\pi \sim \pi\lambda$ or $\pi \sim \tau$ in G. Assume first $\pi \sim \pi\lambda$ in G. Denote by $D^1_{\pi\lambda}$ a subgroup of order 64 of $\mathbf{C}_G(\pi\lambda)$ with $S\langle\mu'\xi\rangle \subset D^1_{\pi\lambda}$. Since $\mathbf{Z}(D^1_{\pi\lambda}) = \langle\pi\lambda, \pi\mu\rangle$ we get $S \lhd \langle D, D^1_{\pi\lambda}\rangle$. Hence $n = [\mathbf{N}_G(S):D]$ is equal to 5 or to 9. Since $\mathbf{N}_G(S)/S$ is isomorphic to a subgroup of A_8, we obtain $n = 9$ and so $\pi \sim \pi\lambda \sim \tau$ in $\mathbf{N}_G(S)$. Assume now that $\pi \sim \tau$ in G. Denote by D^1_τ a subgroup of order 64 of $\mathbf{C}_G(\tau)$ with $S\langle\tau'\rangle \subset D^1_\tau$. Since S char $S\langle\tau'\rangle$, we get $S \lhd \langle D, D^1_\tau\rangle$. Hence $[\mathbf{N}_G(S):D] = 9$ and $\pi \sim \tau \sim \pi\lambda$ in $\mathbf{N}_G(S)$. In any case $|\mathbf{N}_G(S)/S| = 2^3 9$ and $\pi \sim \tau \sim \pi\lambda$ in $\mathbf{N}_G(S)$. A Sylow 2-subgroup of $\mathbf{N}_G(S)/S = \mathcal{S}$ is dihedral of order 8. From [6; Theorem 1, p. 553] we conclude that $\mathcal{S}$ must have a subgroup of index 2. If $\mathcal{S}$ has no normal subgroups of index 4, then $\mathcal{S} = (\langle x\rangle \times A)\langle y\rangle$ where $x^3 = y^2 = 1$, $A \cong A_4$, $\langle x, y\rangle \cong S_3$ and $A\langle y\rangle \cong S_4$. Then either $S\langle\tau', \mu'\rangle \lhd \mathbf{N}_G(S)$ or $S\langle\mu', \xi\rangle \lhd \mathbf{N}_G(S)$. In the first case an element of order 3 in $\mathbf{N}_G(S)$ would normalize $\mathbf{Z}(S\langle\tau', \mu'\rangle)$ against $H \not\subseteq \mathbf{N}(S)$ and in the second case we would get $\pi \sim \mu$ in $\mathbf{N}_G(S)$ which is not possible because of (2.6). We have proved that $\mathcal{S}$ must have a normal subgroup of index 4. The lemma is proved.

(2.11) LEMMA. *There is an element u of order 3 in $\mathbf{N}_G(S)$ with $\pi^u = \tau$, $\tau^u = \pi\tau$. Further, $|\mathbf{C}(\mu) \cap \mathbf{N}_G(S)| = 64\cdot 3$ and μ is conjugate to λ in $\mathbf{N}_G(S)$. G has precisely two conjugacy classes of involutions.*

Proof. Denote by D^1_τ a subgroup of order 64 of $\mathbf{C}_G(\tau) \cap \mathbf{N}_G(S)$ which contains $S\langle\tau'\rangle$. It is $\langle\pi, \tau\rangle \lhd \langle S\langle\mu', \tau'\rangle, D^1_\tau\rangle = X$. Suppose X is a 2-group. Then $|X| = 2^7$ and $\mathbf{Z}(X) \subseteq \langle\pi, \tau\rangle$. It is $S\langle\mu', \tau'\rangle \lhd X$ and so $\mathbf{Z}(X) = \langle\pi\rangle$ against $|\mathbf{C}_H(\tau)| = 32$. Hence X is not a 2-group. If follows the existence of an element u of order 3 in X with $\pi^u = \tau$ and $\tau^u = \pi\tau$ since $u \in \mathbf{N}_G(S)$ and $H \not\subseteq \mathbf{N}_G(S)$. Assume that 9 divides $|\mathbf{C}(\mu) \cap \mathbf{N}_G(S)|$. Then $\{\mu, \pi\mu\}$ is the conjugate class of μ in $\mathbf{N}_G(S)$. Since $\mathbf{C}(\mu) \cap \mathbf{N}_G(S) \lhd \mathbf{N}_G(S)$ it follows $u \in \mathbf{C}(\mu)$. Then $(\pi\mu)^u = \tau\mu$ yields a contradiction. Hence $|\mathbf{C}(\mu) \cap \mathbf{N}_G(S)| = 64\cdot 3$ and $\mu \sim \lambda$ in $\mathbf{N}_G(S)$. Since by (2.2) each involution of G is conjugate to an involution in S, we get that G has precisely two conjugate classes of involutions.

(2.12) LEMMA. *The involution π is conjugate to $\pi\xi$ in $\mathbf{N}_G(M)$.*

Proof. It is a consequence of (2.8), (2 6) and (2.11) that $\mu \sim \xi$ in G. Denote by T_ξ a Sylow 2-subgroup of $\mathbf{C}_G(\xi)$ which contains $M\langle\tau\lambda\rangle$. It follows $M \lhd \langle H, T_\xi\rangle$ and $T_\xi \not\subseteq H$ since $|\mathbf{C}_H(\xi)| = 2^5 3$. Hence $[\mathbf{N}_G(M):H] > 1$ and π must be conjugate to $\pi\xi$ under $\mathbf{N}_G(M)$.

(2.13) LEMMA. *Let T_ξ be a Sylow 2-subgroup of $\mathbf{C}_G(\xi)$ with $L_2\langle\mu'\rangle \subset T_\xi$. Put $L = \langle S\langle\mu', \xi\rangle, T_\xi\rangle$. Then $|L| = 2^6 3$. There exists an element α in L of order 3 such that $\pi^\alpha = \pi\xi$, $(\pi\xi)^\alpha = \pi\mu\xi$, $\mu^\alpha = \mu\xi$, $(\mu\xi)^\alpha = \xi$ and $[\alpha, \pi\mu] = 1$. $|\mathbf{N}_G(L_2)|$ is equal to $2^6 3$ or $2^6 3^2$. $\mathbf{Z}(L) = \langle\pi\mu\rangle$ and $L \subseteq \mathbf{N}_G(L_2)$.*

Proof. We know that $\mu \sim \xi$ in G from (2.11) and (2.9). Denote by T_ξ a Sylow 2-subgroup of $\mathbf{C}_G(\xi)$ which contains $L_2\langle\mu'\rangle$. Since $(L_2\langle\mu'\rangle)' = \langle\pi\mu\rangle$ one gets $\mathbf{Z}(T_\xi) = \langle\xi, \pi\mu\rangle$. Also $\mathbf{Z}(L_2\langle\mu'\rangle) = \langle\pi, \mu, \xi\rangle$ and $L_2 \lhd T_\xi$. Put $\langle S\langle\mu', \xi\rangle, T_\xi\rangle = L$. We have $\langle\pi, \mu, \xi\rangle \lhd L$ and $\langle\pi\mu\rangle = \mathbf{Z}(L)$. Clearly, L is not a 2-group since $\pi\mu \not\sim \pi$. $L/L_2\langle\mu'\rangle$ is isomorphic to a subgroup of $PSL(2, 7)$. Because of $\pi\mu \in \mathbf{Z}(L)$ we get $|L| = 2^6 3$. Since $H \cap L = S\langle\mu', \xi\rangle$, no element conjugate to π under L can be centralized by an element of order 3 of L. Considering the elements of $\langle\pi, \mu, \xi\rangle$ one gets the existence of an element α of order 3 in L such that $\pi^\alpha = \pi\xi$, $(\pi\xi)^\alpha = \pi\mu\xi$, $\mu^\alpha = \mu\xi$, $(\mu\xi)^\alpha = \xi$ and $[\pi\mu, \alpha] = 1$. For $[\mathbf{N}_G(L_2):S\langle\mu', \xi\rangle]$ we get the following possibilities: 3, 5, 3^2, 7. If $|\mathbf{N}_G(L_2)| = 2^6 5$ or $2^6 7$, then $\mathbf{N}_G(L_2) = \langle S\langle\mu', \xi\rangle, T_\xi\rangle$ which is not possible. The lemma is proved.

(2.14) LEMMA. *The involution π is conjugate to $\tau\lambda\xi$ in G.*

Proof. Assume the lemma to be false. Then $\tau\lambda\xi \sim \mu$ in G. Denote by $T_{\tau\lambda\xi}$ a Sylow 2-subgroup of $\mathbf{C}_G(\tau\lambda\xi)$ which contains L_2. Because of $\mathbf{Z}(T_{\tau\lambda\xi}) = \langle\tau\lambda\xi, x\rangle$ is a four-group we get $L_2 \lhd \langle S\langle\mu', \xi\rangle, T_{\tau\lambda\xi}\rangle = X$. Clearly, X cannot be a 2-group since $S\langle\mu', \xi\rangle \not\subseteq T_{\tau\lambda\xi}$. Application of (2.13) yields $\mathbf{N}_G(L_2) = X$ and X is of order $2^6 3$. Thus $X = L$. We may put $x = \pi\mu$. Obviously, $\langle\pi, \mu\rangle$ is conjugate to $\langle\tau\lambda\xi, \pi\mu\rangle$ in L, and so $\pi \sim \pi\mu\tau\lambda\xi$ in L. But $(\pi\mu\tau\lambda\xi)\mu' = \tau\lambda\xi$ against our assumption. The proof is complete.

(2.15) LEMMA. *We have $[\alpha, \tau\lambda] = 1$.*

Proof. There are nine elements in L_2 which are conjugate to π in G. From (2.13) follows that α acts transitively on $\{\mu, \mu\xi, \xi\}$. Also $[\alpha, \pi\mu] = 1$. There remain the elements $\tau\lambda$ and $\pi\mu\tau\lambda$ which α must centralize.

3. Simplicity of G

(3.1) LEMMA. *G is a simple group.*

Proof. Since $\mathbf{O}(H) = 1$ and $\pi \sim \tau \sim \pi\tau$ in G we get from [15; p. 146] that $\mathbf{O}(G) = 1$. The fact that $\mathbf{N}_G(D) = D$ together with [1; Lemma 1, p. 117] yields that G possesses no non-trivial odd order factor group. If G were not a simple group then G has a normal subgroup Y with $1 \subset Y \subset G$. Since

$|Y| \equiv 0 \pmod 2$ and $|G/Y| \equiv 0 \pmod 2$ we get that π or μ is contained in Y because G has precisely two classes of involutions. Hence, $\langle \pi, \mu \rangle \subseteq Y$ and since D is generated by involutions, we get $D \subseteq Y$ against $|G/Y| \equiv 0 \pmod 2$. The lemma is proved.

4. The centralizer of μ in G

(4.1) LEMMA. $\mathbf{C}(\mu) \cap \mathbf{N}_G(S)$ *is generated by the elements* $\pi, \mu, \tau, \lambda, \mu', \xi, \nu$ *subject to the following relations*: $\nu^3 = 1$, $[\nu, \mu] = [\nu, \lambda] = [\nu, \xi] = 1$, $\pi^\nu = \pi\tau\lambda$, $\tau^\nu = \pi\mu\lambda$, $\mu'\nu\mu' = \nu^{-1}$.

Proof. We are going to use the results of (2.10) and (2.11). It is $|\mathbf{C}(\mu) \cap \mathbf{N}_G(S)| = 64 \cdot 3$. Let ν be an element of order 3 in $\mathbf{C}(\mu) \cap \mathbf{N}_G(S)$. Denote by $\bar{N}$ the subgroup of $\mathbf{N}_G(S)$ of order $64 \cdot 9$ which has $S\langle \tau', \mu' \rangle$ as a Sylow 2-subgroup. We consider $N = \bar{N} \cap \mathbf{C}(\mu)$. Clearly, $\nu \in N$. Since the conjugate class of μ in $\mathbf{N}_G(S)$ consists of 6 elements, since $H \not\subseteq \mathbf{N}_G(S)$ and since $\pi \sim \pi\lambda \sim \tau$ in $\mathbf{N}_G(S)$ we get $[\nu, \lambda] = 1$. It follows $\mathbf{C}_S(\nu) = \langle \mu, \lambda \rangle$ and no element in $S \setminus \langle \mu, \lambda \rangle$ normalizes $\langle \nu \rangle$. The case $\mathbf{N}(\langle \nu \rangle) \cap N = \mathbf{C}(\nu) \cap N$ is not possible since otherwise $S\langle \mu' \rangle$ would be normal in N against $\pi \not\sim \mu$ and $H \not\subseteq \mathbf{N}_G(S)$. N contains precisely three Sylow 2-subgroups which one obtains from $S\langle \mu' \rangle$ by transforming with ν and ν^{-1}. Hence a Sylow 2-subgroup of $\mathbf{N}(\langle \nu \rangle) \cap N$ is contained in $S\langle \mu' \rangle$ and so an element in $S\langle \mu' \rangle \setminus S$ must invert ν. Elements in $S\langle \mu' \rangle \setminus S$ are the four elements of order 4 with square equal to π which cannot invert ν since $[\pi, \nu] \neq 1$, the four elements with square equal to $\pi\mu$ which cannot invert ν since $[\pi\mu, \nu] = [\pi, \nu] \neq 1$, the sets of elements $K_1 = \{\mu', \mu\mu', \pi\mu\mu', \pi\mu'\}$ and $K_2 = \{\mu'\lambda, \mu\mu'\lambda, \pi\mu\mu'\lambda, \pi\mu'\lambda\}$. If $x \in K_1$ with $x^{-1}\nu x = \nu^{-1}$, then by conjugating with an element in S we obtain an element ν' of order 3 in $\langle S\langle \mu' \rangle, \nu \rangle$ with $\mu'\nu'\mu' = \nu'^{-1}$. The same can be done if an element in K_2 inverts ν because $[\lambda, \nu] = 1$. Hence we may assume that $\mu'\nu\mu' = \nu^{-1}$. Considering the conjugate class of μ in $\mathbf{N}_G(S)$ and noting that $|\mathbf{C}_S(\nu)| = 4$, we get $(\pi\mu)^\nu = \pi\mu\tau\lambda$ or $\tau\lambda$. Interchanging ν and ν^{-1} if necessary we may and shall assume that $\pi^\nu = \pi\tau\lambda$ and $\tau^\nu = \pi\mu\lambda$.

Finally, we consider the subgroup $\bar{U}$ of $\mathbf{N}_G(S)$ of order $32 \cdot 9$ with Sylow 2-subgroup $S\langle \mu'\xi \rangle$. Put $U = \mathbf{C}(\mu) \cap \bar{U}$. Clearly, $U = \langle S\langle \mu'\xi \rangle, \nu \rangle$. From [17; Theorem 4, p. 169] we conclude that ν is inverted by an element in U since $(S\langle \mu'\xi \rangle)' = \langle \pi\mu \rangle$ and $[\pi\mu, \nu] \neq 1$. Such an element can be found in $S\langle \mu'\xi \rangle \setminus S$. All elements of order 4 in $S\langle \mu'\xi \rangle \setminus S$ have square equal to $\pi\mu$, and so, they cannot invert ν. There remain the eight involutions of $S\langle \mu'\xi \rangle \setminus S$: $\mu'\xi$, $\pi\mu'\xi$, $\mu\mu'\xi$, $\pi\mu\mu'\xi$, $\lambda\mu'\xi$, $\pi\lambda\mu'\xi$, $\mu\lambda\mu'\xi$, $\pi\mu\lambda\mu'\xi$. Since $[\mu, \nu] = [\nu, \lambda] = 1$ we have that either $\mu'\xi$ or $\pi\mu'\xi$ inverts ν. If $\pi\mu'\xi$ inverts ν then $\pi\xi$ centralizes ν and so $(\pi\xi)^\nu = \pi\lambda\tau\xi^\nu = \pi\xi$. It follows $\xi^\nu = \tau\lambda\xi$ against (2.14) and (2.8). We have proved that $\mu'\xi$ inverts ν and therefore $[\nu, \xi] = 1$. The proof is complete.

(4.2) LEMMA. $\mathbf{C}_G(\mu) = (\langle \mu, \lambda \rangle \times A)\langle \mu' \rangle$, *where* $A \cong A_6$, $A\langle \mu' \rangle \cong S_6$ *and* $\langle \pi\mu, \tau\lambda, \nu, \mu'\xi, \alpha^{\tau'} \rangle \subseteq A$. *Further*, $[u, \tau'] = 1$, $\mu^u = \lambda$, $\lambda^u = \mu\lambda$ *and* $\mu' u \mu' = u^{-1}$.

Proof. First we shall consider the normalizer of $\langle\pi, \tau\rangle$ in $\mathbf{N}_G(S)$. It is $\mathbf{C}_G(\langle\pi, \tau\rangle) = S\langle\tau'\rangle$. Hence, by (2.11), $\mathbf{N}_G(\langle\pi, \tau\rangle) = S\langle\tau'\rangle\langle u, \mu'\rangle = X$ and $|X| = 64\cdot 3$.

If 3 divides $\mathbf{C}_X(\mu)$, then $\{\mu, \pi\mu\}$ is the conjugate class of μ in X. Denote by v an element of order 3 in $\mathbf{C}_X(\mu)$. Since no element of order 3 in $\mathbf{N}_G(S)$ centralizes π, we get $(\pi\mu)^v = \tau\mu$ or $\pi\tau\mu$ which is not possible. It follows $|\mathbf{C}_X(\mu)| = 32$. In a similar way one proves $|\mathbf{C}_X(\mu\tau)| = 32$, because $\mu\tau$ is not in the centre of a Sylow 2-subgroup of X. It follows that $\mu \sim \lambda$ in X and $\mu\tau \sim \pi\lambda$ in X. The conjugate class of μ in X is $\{\mu, \pi\mu, \lambda, \mu\lambda, \tau\lambda, \pi\mu\tau\lambda\}$. Since $\mathbf{C}_X(\lambda) \nsubseteq S\langle\mu', \tau'\rangle$, we have either $\mathbf{C}_X(\lambda) \subseteq (S\langle\mu', \tau'\rangle)^u$ or $\mathbf{C}_X(\lambda) \subseteq (S\langle\mu', \tau'\rangle)^{u^{-1}}$. For the action of u on S one gets $\pi^u = \tau$, $\tau^u = \pi\tau$, $\mu^u = \lambda$, $\lambda^u = \mu\lambda$.

We know that $(\mu\tau')^u$ is equal to one of the four elements in $S\langle\tau'\rangle$ the squares of which are equal to τ. These elements are $\lambda\tau'$, $\tau\lambda\tau'$, $\pi\lambda\tau'$, $\pi\tau\lambda\tau'$. We know that $\mu^u = \lambda$. It follows that $(\tau')^u$ is equal to τ', $\tau\tau'$, $\pi\tau'$, or $\pi\tau\tau'$. The set $\mathfrak{S} = \{\tau', \tau\tau', \pi\tau', \pi\tau\tau'\}$ is u-invariant. Hence u centralizes an element in $\mathfrak{S}$. The group $\langle\mu, \lambda\rangle$ operates transitively on $\mathfrak{S}$, and so, transforming u by an element in $\langle\mu, \lambda\rangle$, we may and shall assume that $u\tau' = \tau' u$.

We consider now $u\mu'$. We have $u\mu' \in \mathbf{C}_X(\lambda) \cap \mathbf{C}(\tau)$, and so

$$(u\mu')^{u^{-1}} \in \mathbf{C}_X(\mu) \cap C(\pi) = S\langle\mu'\rangle.$$

Further,

$$(u\mu')^{u^{-1}} \in S\langle\mu'\rangle \cap \mathbf{C}_X(\tau') = \langle\pi, \tau\rangle\langle\mu'\rangle.$$

Clearly, $(u\mu')^{u^{-1}} \notin \langle\pi, \tau\rangle$ since otherwise $u \in \langle\pi, \tau\rangle\langle\mu'\rangle$ against $u^3 = 1$. Considering the possibilities for $u\mu'$, we get that $(u\mu')^{u^{-1}} = \mu'$ or $(u\mu')^{u^{-1}} = \pi\tau\mu'$. If the last possibility holds then $u\mu' = \mu'\pi\tau u^{-1}$. Put $\bar{u} = \pi u$ and note that the order of πu is 3 and that $\bar{u}$ has all the properties of u required so far. Compute $(\bar{u}\mu')^2 = \pi u\mu'\pi u\mu' = u\tau\pi\pi\tau u^{-1} = 1$. It follows that $\mu'\bar{u}\mu' = \bar{u}^{-1}$ or equivalently $(\bar{u}\mu')^{\bar{u}^{-1}} = \mu'$. Hence we may and shall assume that $\mu' u\mu' = u^{-1}$.

We turn now to the determination of $\mathbf{C}_G(\mu)$. Put $\bar{\mathfrak{G}} = \mathbf{C}_G(\mu)$ and $\bar{\mathfrak{G}}/\langle\mu\rangle = \mathfrak{G}$. In the epimorphism $\bar{\mathfrak{G}} \to \mathfrak{G}$ put $\pi \to p$, $\tau \to t$, $\lambda \to l$, $\mu' \to m$, $\xi \to z$, $\nu \to n$ and $\alpha^{\tau'} \to a$.

It is $\mathbf{C}_{\mathfrak{G}}(p) = \langle l, z\rangle \times \langle p, t\rangle\langle m\rangle = \mathfrak{T}$, where $\langle p, t\rangle\langle m\rangle$ is dihedral of order 8, $\mathbf{Z}(\mathfrak{T}) = \langle l, z, p\rangle$ and $\mathfrak{T}' = \langle p\rangle$. $\mathfrak{T}$ is a Sylow 2-subgroup of $\mathfrak{G}$ and $\mathbf{N}_{\mathfrak{G}}(\mathfrak{T}) = \mathfrak{T}$. Application of [17; Lemma, p. 169] yields that no two different elements of $\mathbf{Z}(\mathfrak{T})$ are conjugate in $\mathfrak{G}$.

Assume $p \sim t$ in $\mathfrak{G}$. Then there exists and $x \in \bar{\mathfrak{G}}$ such that $x^{-1}\pi x = \tau$ or $\mu\tau$. We have $|\mathbf{C}(\tau) \cap \mathbf{C}_G(\mu)| = |\mathbf{C}(\pi) \cap \mathbf{C}_G(\mu\lambda)| = 32$ against $|\mathbf{C}(\pi) \cap \mathbf{C}_G(\mu)| = 64$. Hence $p \nsim t$ in $\mathfrak{G}$. Further, $p \nsim m$, $p \nsim lm$, $p \nsim zt$, $p \nsim zlt$ because $(\pi\xi)^\nu = \pi\tau\lambda\xi$ and therefore $(pz)^n = ptlz$ and $(zlt)^m = ptlz$. Certainly, one has $p^n = ptl$ and $p^a = pmz$. Whether $p \sim zlm$ in $\mathfrak{G}$ or not has not been decided so far.

Application of [17; Theorem 5, p. 170] yields that the transfer of $\mathfrak{G}$ into $\mathfrak{T}$

is isomorphic to $\mathfrak{T}/\langle p, lt, zm\rangle$ if $p \not\sim zlm$ in $\mathfrak{G}$, or to $\mathfrak{T}/\langle p, t, l, zm\rangle$ if $p \sim zlm$ in $\mathfrak{G}$.

Assume by way of contradiction that $\mathfrak{G}$ has no normal subgroup of index 4. Then $\mathfrak{G}$ has a normal subgroup $\mathfrak{M}$ with $[\mathfrak{G}:\mathfrak{M}] = 2$. Since $\mathfrak{G}' \subseteq \mathfrak{M}$ we get $\mathfrak{T}' \subseteq \mathfrak{M}$ and so $\langle p, t, l, zm\rangle \subseteq \mathfrak{M}$. Since $p \sim zm \sim zmp \sim zlm \sim zlmp$ in $\mathfrak{G}$ and $z \notin \mathfrak{M}$ we get that these five elements are conjugate in $\mathfrak{M}$. We have

$$\mathbf{C}_{\mathfrak{M}}(p) = \langle l\rangle \times \langle p, t\rangle\langle zm\rangle = \mathfrak{F}.$$

Because of $\mathfrak{F}' = \langle p\rangle$ we get $\mathbf{N}_{\mathfrak{M}}(\mathfrak{F}) = \mathfrak{F}$ and so l, p and lp lie in three different conjugate classes of $\mathfrak{M}$. Consider

$$\mathbf{C}_{\mathfrak{M}}(p) \cap \mathbf{C}(zm) = \mathbf{C}_{\mathfrak{M}}(p) \cap \mathbf{C}(zpm) = \mathbf{C}_{\mathfrak{M}}(p) \cap \mathbf{C}(zlm)$$
$$= \mathbf{C}_{\mathfrak{M}}(p) \cap \mathbf{C}(zplm) = \langle l\rangle \times \langle p, zm\rangle = \mathfrak{F}_1 .$$

$\mathfrak{F}_1$ is an elementary abelian group of order 8 and is normalized by Sylow 2-subgroups of $\mathfrak{M}$ the commutator groups of which are $\langle p\rangle$, $\langle zm\rangle$, $\langle zpm\rangle$, $\langle zlm\rangle$, $\langle zlpm\rangle$. It follows $[\mathbf{N}_{\mathfrak{M}}(\mathfrak{F}_1):\mathfrak{F}] \geq 5$ and so 7 must divide $|\mathbf{N}_{\mathfrak{M}}(\mathfrak{F}_1)/\mathfrak{F}_1|$ from which would follow that all involutions of $\mathfrak{F}_1$ are conjugate against $p \not\sim l$ in $\mathfrak{M}$. We have shown that $\mathfrak{G}$ has a normal subgroup $\mathfrak{M}$ of index 4 and that $p \not\sim zlm$ in $\mathfrak{G}$.

We prove next that $\bar{\mathfrak{G}}$ has no non-trival normal subgroup of odd order. We have

$$|\mathbf{C}(\pi) \cap \bar{\mathfrak{G}}| = 64, \ |\mathbf{C}(\tau) \cap \bar{\mathfrak{G}}| = 32$$

and

$$|\mathbf{C}(\pi\tau) \cap \bar{\mathfrak{G}}| = |\mathbf{C}(\pi) \cap \mathbf{C}(\lambda)| = 32.$$

Using [15; p. 146], we get from the action of $\langle \pi, \tau\rangle$ on $\mathbf{O}(\bar{\mathfrak{G}})$ that $\mathbf{O}(\bar{\mathfrak{G}})$ is trivial. It follows from [17; Theorem 4, p. 169] that $\mathbf{O}(\mathfrak{G}) = 1$.

The 2-group $\langle p, lt, zm\rangle$ is dihedral of order 8 and is a Sylow 2-subgroup of $\mathfrak{M}$. Further, $\mathbf{C}_{\mathfrak{M}}(p) = \langle p, lt, zm\rangle$, $\mathbf{O}(\mathfrak{M}) = 1$ and $\langle n, a\rangle \subseteq \mathfrak{M}$. Assume that $\mathfrak{M}$ has a subgroup of index 2. If $\mathfrak{N}$ is the intersection of all subgroups of index 2 of $\mathfrak{M}$, then $2 \leq [\mathfrak{M}:\mathfrak{N}] \leq 4$, and so $\langle p\rangle$ and $\langle p, lt, zm\rangle \subseteq \mathfrak{N}$ which is not possible. Hence $\mathfrak{M}$ does not possess subgroups of index 2. We are in the situation to apply [6; Theorem 1, p. 553] and get that $\mathfrak{M} \cong A_6$ or $\mathfrak{M} \cong PSL(2, 7)$.

Denote by $\bar{\mathfrak{M}}$ the counter image of $\mathfrak{M}$ in $\bar{\mathfrak{G}}$. A Sylow 2-subgroup of $\bar{\mathfrak{M}}$ is $\langle \mu\rangle \times \langle \pi\mu, \tau\lambda\rangle\langle \mu'\xi\rangle$. From a result in [3] we get $\bar{\mathfrak{M}} = \langle \mu\rangle \times A$ where A is isomorphic to A_6 or $PSL(2, 7)$. Since A char $\bar{\mathfrak{M}}$ we get $A \lhd \bar{\mathfrak{G}}$. Clearly, $\langle \nu, \alpha^{\tau'}\rangle \subseteq A$, and since $\langle \pi\mu, \tau\lambda\rangle\langle \nu\rangle$ is isomorphic to A_4, also $\langle \pi\mu, \tau\lambda\rangle\langle \nu\rangle \subseteq A$. Because of $(\pi\mu)^{\tau'\alpha\tau'} = \pi\mu\mu'\xi$, it follows $\mu'\xi \in A$. Hence $\langle \pi\mu, \tau\lambda\rangle\langle \mu'\xi\rangle$ is a Sylow 2-subgroup of A.

We shall consider now $A\langle \mu'\rangle = X$. Assume that $\mathbf{C}_X(A) = \langle y\mu'\rangle$ is of order 2 for some $y \in A$. Then $[y, \mu'] = [y, \pi\mu] = 1$ and $\nu^{-1} = y^{-1}\nu y$. Since $(y\mu')^2 = 1$ we have $y^2 = 1$. Since

$$\mathbf{C}_A(\pi\mu) = \langle \pi\mu, \tau\lambda\rangle\langle \mu'\xi\rangle \quad \text{and} \quad \langle \pi\mu, \tau\lambda\rangle\langle \nu\rangle\langle \mu'\xi\rangle \cong S_4 ,$$

we obtain $y = \mu'\xi$. We must have $[y\mu', \tau'\alpha\tau'] = [\xi, \tau'\alpha\tau'] = 1$. Consequently,

$$1 = \xi\tau'\alpha^{-1}\tau'\xi\tau'\alpha\tau' = \xi\tau'\alpha^{-1}\mu'\xi\alpha\tau' = \xi\tau'(\alpha^{-1}\mu'\alpha)\mu\tau',$$

and so

$$\alpha^{-1}\mu'\alpha = \tau'\xi\tau'\mu = \mu'\xi\mu \sim \pi$$

which is not possible. It follows that $\mathbf{C}_X(A) = 1$ and $A\langle\mu'\rangle$ is isomorphic to an automorphism group of A. Since a Sylow 2-subgroup of $A\langle\mu'\rangle$ has no elements of order 8, we get $A \cong A_6$ and $A\langle\mu'\rangle \cong S_6$.

We have $|\mathfrak{S}| = 8\cdot|A|$, and $\mathfrak{S}/\mathbf{C}_{\mathfrak{S}}(A) \cong S_6$ since $\mathfrak{S}$ has no elements of order 8. It follows that $|\mathbf{C}_{\mathfrak{S}}(A)| = 4$. Obviously, $A \cap \mathbf{C}_{\mathfrak{S}}(A) = 1$. Since $\mathfrak{S}/A$ is dihedral of order 8, we have to discuss the following three cases:

(1) $A\mathbf{C}_{\mathfrak{S}}(A) = A\langle\mu, \mu'\rangle$,
(2) $A\mathbf{C}_{\mathfrak{S}}(A) = A\langle\mu'\lambda\rangle$,
(3) $A\mathbf{C}_{\mathfrak{S}}(A) = A\langle\mu, \lambda\rangle$.

The case (1) cannot happen, since then $A\mathbf{C}_{\mathfrak{S}}(A) = \langle\mu\rangle \times A\langle\mu'\rangle$ against $|\mathbf{C}_{\mathfrak{S}}(A)| = 4$. Assume that we are in the case (2). Then $\mathbf{C}_{\mathfrak{S}}(A) = \langle y\mu'\lambda\rangle$ would be of order 4 for some $y \in A$. We have

$$[y, \mu'\lambda] = [y, \pi\mu] = [y\mu', \nu] = 1 \quad\text{and}\quad (y\mu'\lambda)^2 = y^2\mu \in \mathbf{C}(A),$$

and so $y^2 \in \mathbf{C}(A) \cap A = 1$. It follows that $y = \mu'\xi$. Hence $\mathbf{C}_{\mathfrak{S}}(A) = \langle\xi\lambda\rangle$. Therefore $[\xi\lambda, \tau'\alpha\tau'] = 1$ which means

$$\tau'\alpha^{-1}\tau'(\xi\lambda)\tau'\alpha\tau' = \tau'\alpha^{-1}(\mu'\xi\tau\lambda)\alpha\tau' = \tau'(\alpha^{-1}\mu'\alpha)\mu\tau\lambda\tau' = \xi\lambda,$$

and therefore

$$\alpha^{-1}\mu'\alpha = \tau'\xi\lambda\tau'\lambda\tau\mu = \mu'\xi\tau\lambda\lambda\tau\mu = \mu\mu'\xi \sim \pi$$

yields a contradiction.

We are necessarily in case (3). Since $\mu \in \mathbf{C}_{\mathfrak{S}}(A)$ we get $A\mu \cap \mathbf{C}(A) = \mu$ and hence $A\lambda \cap \mathbf{C}_{\mathfrak{S}}(A) \neq \emptyset$ since $|\mathbf{C}_{\mathfrak{S}}(A)| = 4$. There exists $y \in A$ such that $y\lambda \in \mathbf{C}(A)$. It follows that $[y, \lambda] = [y, \nu] = [y, \pi\mu] = 1$. Because of

$$\mathbf{C}_A(\pi\mu) = \langle\pi\mu, \tau\lambda, \mu'\xi\rangle \quad\text{and}\quad \langle\pi\mu, \tau\lambda\rangle\langle\nu\rangle\langle\mu'\xi\rangle \cong S_4,$$

it follows that $y = 1$. Hence $\mathbf{C}_{\mathfrak{S}}(A) = \langle\mu, \lambda\rangle$. The lemma is proved.

5. The identification of G with A_{10}

(5.1) LEMMA. *$[u, \nu] = 1$ and $u\nu$ is of order 3. $\langle\mu', \tau'\rangle$ normalizes $\langle u, \nu\rangle$.*

Proof. Denote by R a Sylow 3-subgroup of $\mathbf{N}_G(S)$ which contains u. We know that R is elementary abelian of order 9, and that $SR \lhd \mathbf{N}_G(S)$. Consider $SR\langle\tau', \mu'\rangle = X$ and compute $\mathbf{C}_X(u)$. It is $\mathbf{C}_X(u) = R(S\langle\tau', \mu'\rangle \cap \mathbf{C}(u)) = R\langle\tau'\rangle$. Further, $R \lhd R\langle\mu', \tau'\rangle$. The element ν possesses precisely four conjugates in RS under RS. These are $\nu, \nu^\pi, \nu^\tau, \nu^{\pi\tau}$. Hence $\nu^x \in R$, for some x in $\{1, \pi, \tau, \pi\tau\}$. If $x = \tau$, then ν^τ and $\mu'\nu^\tau\mu'$ lie in R and hence $[\nu^\tau, \mu'\nu^\tau\mu'] = 1$ which is not possible. Therefore $x \neq \tau$. Similarly, one proves that $x \neq \pi\tau$. It follows that $x = 1$ or $x = \pi$. Interchanging ν and ν^π if necessary, we may and shall assume $[u, \nu] = 1$.

(5.2) LEMMA. *The element* $u\nu$ *of order* 3 *centralizes* A. *Further,*

$$\mathbf{N}_G(\langle\mu, \lambda\rangle) = (\langle\mu, \lambda\rangle \times A)\langle u, \mu'\rangle.$$

Proof. Clearly,

$$u\nu \in \mathbf{N}_G(\langle\mu, \lambda\rangle), \qquad \mathbf{C}_G(\langle\mu, \lambda\rangle) = \langle\mu, \lambda\rangle \times A.$$

It follows that $u\nu$ normalizes A. The automorphism group of A is an extension of A by a four-group. Hence $u\nu$ induces an inner automorphism on A. We have $[\pi\mu, u\nu] = 1$ and since $\mathbf{C}_A(\pi\mu) = \langle\pi\mu, \tau\lambda, \mu'\xi\rangle$, it follows that $(u\nu)^4$ induces the identity automorphism on A. Because $u\nu$ is of order 3, we obtain $[u\nu, A] = 1$.

(5.3) LEMMA. *Denote by* ω *an element of order* 5 *in* $A\langle\mu'\rangle$. $\mathbf{C}_G(\omega)$ *is equal to* $(\langle\mu, \lambda\rangle\langle u\nu\rangle) \times \langle\omega\rangle$ *or* $L \times \langle\omega\rangle$ *where* $L \cong A_5$.

Proof. There is only one conjugate class of elements of order 5 in $\mathbf{C}_G(\mu)$. We have $\mathbf{C}_G(\omega) \cap \mathbf{C}_G(\mu) = \langle\mu, \lambda\rangle \times \langle\omega\rangle$. Let U be a Sylow 2-subgroup of $\mathbf{C}_G(\omega)$ containgin $\langle\mu, \lambda\rangle$. Assume $\langle\mu, \lambda\rangle \subset U$. If $\mathbf{Z}(U) \not\subseteq \langle\mu, \lambda\rangle$, then 2^3 divides $|\mathbf{C}_G(\omega) \cap \mathbf{C}_G(\mu)|$ which is not the case. Hence $\mathbf{Z}(U) \subseteq \langle\mu, \lambda\rangle$ and μ, λ or $\mu\lambda$ is contained in $\mathbf{Z}(U)$. But then $|\mathbf{C}_G(\omega) \cap \mathbf{C}_G(x)|$ is divisible by 2^3 where $x \in \{\mu, \mu\lambda, \lambda\}$. However, in G we have $\mu \sim \lambda \sim \mu\lambda$, and so, $\mathbf{C}_G(x) \cap \mathbf{C}_G(\omega)$ is conjugate to $\mathbf{C}_G(\mu) \cap \mathbf{C}_G(\omega)$ in G against $2^3 \nmid |\mathbf{C}_G(\omega) \cap \mathbf{C}_G(\mu)|$. We have proved that $U = \langle\mu, \lambda\rangle$. Put $K = \mathbf{O}(\mathbf{C}_G(\omega))$. It follows from [15; p. 146] that

$$|K|\cdot|\mathbf{C}_K(\langle\mu, \lambda\rangle)|^2 = |\mathbf{C}_K(\mu)|\cdot|\mathbf{C}_K(\lambda)|\cdot|\mathbf{C}_K(\mu\lambda)| = 5^3.$$

Therefore $|K| = 5$ and $K = \langle\omega\rangle$. It follows from (5.2) that $u\nu \in \mathbf{C}_G(\omega)$. Hence all involutions of $\mathbf{C}_G(\omega)$ are conjugate under $\mathbf{C}_G(\omega)$. Application of [12; Main Theorem, p. 191] yields the lemma.

(5.4) LEMMA. $\mathbf{C}_G(u\nu) = \langle u\nu\rangle \times W$ *where* $W \cong A_7$ *and* $A \subset W$.

Proof. It is $\mu^{\tau'} = \pi\mu$. Hence

$$\mathbf{C}_G(\pi\mu) = (\langle\pi\mu, \tau\lambda\rangle \times \tilde{A})\langle\mu'\rangle$$

and

$$\langle u\nu, \alpha, \mu, \lambda, \xi\rangle \subseteq \tilde{A}.$$

We know that $\tilde{A} \cong A_6$. There exists an element β in $\tilde{A}$ such that $(\beta\mu')^2 = 1$ and $[\beta\mu', u\nu] = 1$. Put

$$Y = \mathbf{C}_G(\pi\mu) \cap \mathbf{C}_G(u\nu).$$

The group $T = \langle\pi\mu, \tau\lambda\rangle\langle\beta\mu'\rangle$ is dihedral of order 8 and a Sylow 2-subgroup of Y. The structure of $\tilde{A}\langle\mu'\rangle$ yields $|Y| = 2^3 3^2$. Let U be a Sylow 2-subgroup of $\mathbf{C}_G(u\nu)$ which contains T. Suppose $T \subset U$. If $\mathbf{Z}(U) \not\subseteq T$, then 2^4

divides $|Y|$ which cannot happen. If $\mathbf{Z}(U) \subseteq T$, then $\mathbf{Z}(U) = \langle \pi\mu \rangle$ and again we get a contradiction to $|Y|$. Hence $T = U$.

Put $K = \mathbf{O}(\mathbf{C}_G(u\nu))$. We have

$$|K| \cdot |\mathbf{C}_K(\langle \pi\mu, \tau\lambda \rangle)|^2 = |\mathbf{C}_K(\pi\mu)| \cdot |\mathbf{C}_K(\tau\lambda)| \cdot |\mathbf{C}_K(\pi\mu\tau\lambda)|.$$

Since $\mathbf{C}_G(\mu)$ does not contain subgroups of order divisible by $3 \cdot 5$, we obtain' that K is a 3-group with $3 \leq |K| \leq 81$. We know that $A \subseteq \mathbf{C}_G(u\nu)$. Hence ω induces an automorphism on $K/\langle u\nu \rangle$. Since a 3-group of order at most 27 does not have an automorphism of order 5 which follows from [7; Theorem 12.2.2, p. 178], we know that ω stabilizes the chain $K \supseteq \langle u\nu \rangle \supset \langle 1 \rangle$. It is a consequence of [9; Lemma 7, p. 6] that ω centralizes K. Application of (5.3) yields $K = \langle u\nu \rangle$ is of order 3.

We shall now apply [6; Theorem 1, p. 553]. If $\mathbf{C}_G(u\nu) = B$ has a normal subgroup of index 4, then B would have a normal 2-complement against $\omega \in B$ and $\mathbf{O}(B) = \langle u\nu \rangle$. Put $B/\langle u\nu \rangle = \mathfrak{B}$ and $\langle u\nu \rangle A/\langle u\nu \rangle = \mathfrak{A}$. Assume that $\mathfrak{B}$ has a subgroup $\mathfrak{U}$ of index 2. Clearly, $\mathfrak{A} \not\subseteq \mathfrak{U}$ since 8 does not divide $|\mathfrak{U}|$. Hence $\mathfrak{U}\mathfrak{A} = \mathfrak{B}$ and $\mathfrak{U} \cap \mathfrak{A} \lhd \mathfrak{A}$. If $\mathfrak{U} \cap \mathfrak{A} = 1$, then $\mathfrak{B}/\mathfrak{U} \cong \mathfrak{A}\mathfrak{U}/\mathfrak{U} \cong \mathfrak{A}/\mathfrak{U} \cap \mathfrak{A} = \mathfrak{A}$ yields a contradiction. If $\mathfrak{U} \cap \mathfrak{A} = \mathfrak{A}$, then $\mathfrak{A} \subseteq \mathfrak{U}$ which we had ruled out. Hence $\mathfrak{B}$ does not have subgroups of index 2. It follows that $\mathfrak{B}$ is isomorphic to $PSL(2, q)$, q odd, or $\mathfrak{B}$ is isomorphic to A_7. In any case, $\mathfrak{B}$ is a simple group. In the epimorphism $B \to \mathfrak{B}$ put $b \to \bar{b}$ for an element $b \in B$. We have

$$|\mathbf{C}_{\mathfrak{B}}(\bar{\pi}\bar{\mu})| = 2^3 3 \text{ and } \mathbf{C}_{\mathfrak{B}}(\bar{\pi}\bar{\mu}) = (\langle \bar{\pi}\bar{\mu}, \bar{\tau}\bar{\lambda} \rangle \times \langle \bar{x} \rangle)\langle \bar{\beta}\bar{\mu}' \rangle$$

where $\bar{x}^3 = 1$ for an $x \in A$ and $\langle \bar{x}, \bar{\beta}\bar{\mu}' \rangle \cong S_3$ since in $\bar{A}\langle \mu' \rangle$ a group of order 9 is not centralized by an involution. It follows that $\mathbf{C}_{\mathfrak{B}}(\bar{\pi}\bar{\mu}) = \mathbf{C}_{A_7}((12)(34))$ and so by the result of [13] we must have $\mathfrak{B} \cong A_7$. Since $\langle u\nu \rangle \times A \subseteq \mathbf{C}(u\nu)$ we get from a result in [3] that $\mathbf{C}_G(u\nu) = \langle u\nu \rangle \times W$, where $W \cong A_7$. Since A has no subgroup of index 3, it follows $A \subset W$. The proof is complete.

(5.5) LEMMA. *$\mathbf{N}_G(\langle u\nu \rangle) = (\langle u\nu \rangle \times W)\langle \mu' \rangle$ and $W\langle \mu' \rangle \cong S_7$.*

Proof. Put $W\langle \mu' \rangle = X$. Suppose $\mathbf{C}_X(W) = \langle w\mu' \rangle$ is of order 2 for some $w \in W$. Then $[w\mu', W] = 1$ but no involution of G centralizes a group isomorphic to A_7. Hence $W\langle \mu' \rangle$ is an automorphism group of W and so

$$W\langle \mu' \rangle \cong S_7.$$

(5.6) LEMMA. *$\mathbf{N}_G(\langle u\nu \rangle) \cap \mathbf{C}_G(\mu) = A\langle \mu' \rangle$.*

Proof. We have
$$\mathbf{N}_G(\langle u\nu \rangle) \cap \mathbf{C}_G(\mu) = \langle \mu' \rangle((\langle u\nu \rangle \times W) \cap \mathbf{C}_G(\mu)) = \langle \mu' \rangle(W \cap \mathbf{C}(\mu)) = \langle \mu' \rangle A.$$

(5.7) LEMMA. *In G we have $u\nu \sim \nu$, $u \sim \rho$ and $\nu \not\sim u$.*

Proof. Since $[u, \tau'] = 1$ and $\tau' \sim \pi$ in G and since all elements of order 3 in H are conjugate in H, we conclude that $\rho \sim u$ in G. We have $[\pi\mu\mu'\xi, \rho] = 1$ and $\pi\mu\mu'\xi \sim \mu$ in G. There is a Sylow 2-subgroup J of $\mathbf{C}_G(\pi\mu\mu'\xi) \cap$

$\mathbf{C}_G(\rho)$ which is dihedral of order 8 and contains $\langle\pi, \pi\mu\mu'\xi\rangle$. It follows that J is a Sylow 2-subgroup of $\mathbf{C}_G(\rho)$. If we had $\rho \sim uv$ in G, then J and

$$\langle\pi\mu, \tau\lambda, \mu'\xi\rangle$$

would be conjugate in G against $\langle\pi\mu, \tau\lambda, \mu'\xi\rangle \subseteq A$. Hence $\rho \not\sim uv$ in G. Since $\langle\mu, \lambda, \xi\rangle$ centralizes ν, we get $\nu \not\sim \rho$ in G. Since $\mathbf{C}_G(\mu)$ has precisely two classes of elements of order 3, it follows $u\nu \sim \nu$ in G.

(5.8) Lemma. *We have* $\xi u\nu\xi = u^{-1}\nu^{-1}$, $\xi u\xi = u^{-1}\nu$ *and* $\nu^{\tau'} = u^{-1}\nu^{-1}$.

Proof. The element $u\nu$ centralizes A and $\mu'\xi \in A$. We get $\mu'\xi u\nu\xi u' = u\nu$ and so $\xi u\nu\xi = u^{-1}\nu^{-1}$ and $\xi u\xi = u^{-1}\nu$. To complete the proof, one represents $\langle\mu', \tau'\rangle\langle\xi\rangle$ on $\langle u, \nu\rangle$ and uses (4.1) and (4.2).

(5.9) Lemma. *The elements α and ν of order 3 commute.*

Proof. From (5.8) we conclude that $\mathbf{C}_G(u\nu)$ is mapped onto $\mathbf{C}_G(\nu)$ under τ'. Since $\alpha^{\tau'} \in W$, we get $[\nu, \alpha] = 1$.

(5.10) Lemma. *The involutions μ', $\nu\mu'$, $\pi\mu\mu'$ and ξ are conjugate in $W\langle\mu'\rangle$ and are transpositions. The involution $\pi\mu\xi$ is a product of three transpositions.*

Proof. We have $(\pi\mu\mu')^{\tau\lambda} = \mu'$ and $\langle\pi\mu, \tau\lambda\rangle\langle\nu\rangle\langle\mu'\rangle \cong S_4$. Hence $\nu\mu' \sim \mu'$ in $W\langle\mu'\rangle$. The element α of order 3 normalizes L_2, $\langle\pi, \mu, \xi\rangle$ and $L_2\langle\mu'\rangle = \mathbf{C}_G(\langle\pi, \mu, \xi\rangle)$. Using the fact that $[\nu, \alpha] = 1$ one verifies that

$$(\mu')^{\alpha} \in \{\mu', \mu\mu', \xi\mu', \mu\mu'\xi\}.$$

Since $\pi \sim \mu\mu'\xi$, we get

$$(\mu')^{\alpha} \in \{\mu', \mu\mu', \xi\mu'\}.$$

If $(\mu')^{\alpha} = \mu'$, then $(\mu\mu')^{\alpha} = \mu\xi\mu' \sim \pi$ yields a contradiction. Also $(\mu')^{\alpha} = \xi\mu'$ is not possible since then $(\xi\mu')^{\alpha} = \mu\xi\mu' \sim \pi$ which is not possible. We must have $(\mu')^{\alpha} = \mu\mu'$ and so $(\mu\mu')^{\alpha} = \xi\mu'$. Hence $\mu' \sim \mu\mu' \sim \xi\mu'$ in $(W\langle\mu'\rangle)^{\tau'}$ since $\langle\alpha, \mu'\rangle \subseteq (W\langle\mu'\rangle)^{\tau'}$. Therefore $\mu' \sim \pi\mu\mu' \sim \xi$ in $W\langle\mu'\rangle$. Now, either μ' or $\pi\mu\xi$ is a transposition in $W\langle\mu'\rangle$. Since $\pi \sim \pi\mu\xi$ in G and 5 does not divide $|H|$ we get that μ' is a transposition and $\pi\mu\xi$ is a product of three transpositions.

(5.11) Lemma. *The group G contains a subgroup Q isomorphic to A_{10}.*

Proof. From [2; Section 161] follows that S_7 contains precisely one conjugate class of subgroups isomorphic to S_6. By S_6 we denote the symmetric group on the set $\{1, 2, 3, 4, 5, 6\}$. There exists an isomorphism φ of $W\langle\mu'\rangle$ onto S_7 which maps $A\langle\mu'\rangle$ onto S_6. $\{\mu', \nu\mu', \pi\mu\mu', \xi\}$ is a set of transpositions in $A\langle\mu'\rangle\backslash A$. Using φ, we can find a transposition $\sigma \in W\langle\mu'\rangle\backslash(W \cup A\langle\mu'\rangle)$ such that the order of $\sigma\mu'$ is 3 and $[\sigma, \nu\mu'] = [\sigma, \pi\mu\mu'] = [\sigma, \xi] = 1$. Also, we can find a transposition δ in $A\langle\mu'\rangle\backslash A$ such that $[\sigma, \delta] = [\mu', \delta] = [\nu\mu', \delta] = 1$, $(\pi\mu\mu'\delta)^3 = (\delta\xi)^3 = 1$. Clearly, both σ and δ invert $\mu\nu$ and $[\mu. \delta] = 1$.

We have $\langle\sigma, \mu\rangle \subseteq \mathbf{C}_G(\nu\mu') \cap \mathbf{C}(\pi\mu\mu') \cap \mathbf{C}(\xi) = X$. The group X is transformed by $\pi\mu\tau\lambda$ onto $\mathbf{C}_G(\nu) \cap \mathbf{C}(\mu') \cap \mathbf{C}(\xi) = \bar{X}$ since

$$\mathbf{C}(\nu\mu') \cap \mathbf{C}(\pi\mu\mu') = \mathbf{C}(\nu\pi\mu) \cap \mathbf{C}(\pi\mu\mu').$$

Obviously,

$$\mathbf{C}(\mu') \cap \mathbf{C}(\xi) = \mathbf{C}(\mu'\xi) \cap \mathbf{C}(\mu').$$

The elements μ' and $\mu'\xi$ are transpositions of $W^{\tau'}\langle\mu'\rangle$ and $[\mu', \mu'\xi] = 1$. It follows that 3 divides the order of X. Since $\mathbf{C}_G(\nu) \cap \mathbf{C}(\mu') \cong S_5$ by (5.7), (5.8) and (5.10), we get $\bar{X} = \langle\xi\rangle \times \langle k\rangle\langle z\rangle$, where $k^3 = z^2 = 1$ and $\langle k, z\rangle \cong S_3$ since $\xi \in \mathbf{Z}(\bar{X})$. Since $[\mu, \alpha] \neq 1$, we get that the order of $\mu\sigma$ is either 3 or 6. Denote by $\bar{\sigma}$ the element $\sigma^{\pi\mu\tau\lambda}$. Suppose that the order of $\mu\bar{\sigma}$ is 6. Then $\langle\mu\bar{\sigma}\rangle \lhd \bar{X}$ and $(\mu\bar{\sigma})^3 = \xi$. Since $\xi^{\pi\mu\tau\lambda} = \xi$ and $(\mu\alpha)^3 = \xi$, it follows from $[\mu\sigma, \pi\mu\mu'\delta] = 1$ that also $[\xi, \pi\mu\mu'\delta] = 1$ and so $[\xi, \delta] = 1$ against $1 \neq \delta\xi$ and $(\delta\xi)^3 = 1$. It follows that $\mu\sigma$ is of order 3.

Put $u\nu = M_1$, $\mu = M_2$, $\sigma = M_3$, $\mu' = M_4$, $\nu\mu' = M_5$, $\pi\mu\mu' = M_6$, $\delta = M_7$ and $\xi = M_8$. For the M_i we have obtained the following relations:

$$1 = M_1^3 = M_{i+1}^2 = (M_iM_{i+1})^3 = (M_iM_j)^2$$

where $i, j = 1, 2, \cdots, 8, j > i + 1$.

It follows from [4; chapter XIII] that $\langle M_1, M_2, \cdots, M_8\rangle = Q \cong A_{10}$.

(5.12) Lemma. $G = Q$.

Proof. From (4.2) and the fact that Q contains precisely two classes of involutions, and because $\mathbf{C}_G(\mu)$ is isomorphic to $\mathbf{C}_{A_{10}}((12)(34))$, we obtain that Q contains the centralizer in G of each of its involutions. Assume that Q is properly contained in G. Since by (3.1) the group G is simple, we get $\bigcap_{g\in G} Q^g = 1$. Application of a lemma in [14] yields that the number of conjugate classes of involutions of G is one against (2.11). We have proved that $Q = G$ and so $G \cong A_{10}$. The proof of Theorem B is complete.

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