ON POINCARE SERIES WITH APPLICATION TO H^p SPACES ON BORDERED RIEMANN SURFACES[1]

BY

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Introduction

In this paper, as in [5], we use Poincaré Θ-series to study the Hardy spaces of a compact bordered Riemann surface. Our fundamental tool for projecting theorems from the disk D to the surface R is the conditional expectation operator E of Forelli [6], which we define in §2 by means of Θ-series. Our definition allows us in §3 to interpret E as a map from $C(\partial D)$, the space of continuous functions on ∂D, to $C(\partial R)$. The adjoint map E^* enables us to lift measures from ∂R to ∂D. Using E and E^*, we give easy proofs of the Cauchy-Read theorem and the decomposition of $L^p(\partial R)$ in §2, and of the F. and M. Riesz theorem for R in §3. In addition, we obtain a pair of theorems about Θ-series. The more surprising one, Theorem 4, states that every differential which is analytic in R and continuous in $\bar{R}$ is the Θ-series of a function analytic in D and continuous in $\bar{D}$.

If $R \neq D$, the real parts of functions continuous in $\bar{R}$ and analytic in R do not generate $C(\partial R)$. There is a complementary subspace of finite but positive dimension (see [1], [3], [6], [7]). Forelli [6] described such a subspace N, the image under E of a certain subspace of $H^\infty(D)$. Our definition of E shows that N coincides with the complementary subspace obtained by Heins [7] (see §2.3).

Interference from N makes it hard to obtain satisfactory forms of the invariant subspace theorem or Szegö's theorem on R. We illustrate the difficulties in §3.6 by giving a form of Szegö's theorem. One way around them may be found in [1].

In the final §4 we examine some of our formulas more deeply to find their relation to two classical reproducing formulas on R: the Poisson and Cauchy formulas. Indeed we give explicit representations of the Poisson and Cauchy kernels in terms of Θ-series.

Except in §4.2, all our Θ-series have dimension -2. Since series of that dimension are a bit unfamiliar, we devote §1 to an exposition, based on Tsuji's book [12], of their elementary properties.

1. Poincare series

1.1. We shall consider a compact bordered Riemann surface $\bar{R} = R \cup \partial R$ whose boundary ∂R consists of $n \geqq 1$ analytic curves. The universal covering surface of R can be identified with the unit disk $D = \{z \in \mathbf{C}: |z| < 1\}$. Then

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the group G of cover transformations is a free group of Möbius transformations, and R can be identified with the orbit space D/G so that the natural map $\pi : D \rightarrow D/G$ is holomorphic. G acts in the extended plane; the set of limit points $L(G)$ is a closed subset of ∂D. If we set $\hat{D} = \mathbf{C} \cup \{\infty\} - L(G)$, $\hat{D}/G$ can be identified with the double $\hat{R}$ of R, and the extended map $\pi : \hat{D} \rightarrow \hat{D}/G = \hat{R}$ is holomorphic. Note that $\pi^{-1}(\partial R) = \partial D - L(G)$. We can choose (in many ways) a relatively compact set $\mathcal{I}$ in $\partial D - L(G)$, consisting of n half-open intervals, so that π maps each interval 1-1 onto a component of ∂R, different intervals corresponding to different components. Then

$$(1.1) \qquad A(\mathcal{I}) \cap B(\mathcal{I}) = \emptyset, \quad \text{if} \quad A \neq B, A, B \in G,$$

$$(1.2) \qquad G(\mathcal{I}) = \partial D - L(G).$$

Using π, we will identify functions f and differentials of the form $g(z)\, dz$ on R or $\hat{R}$ with functions in D or $\hat{D}$ which satisfy, respectively,

$$(1.3) \qquad f(Az) = f(z) \quad \text{for all} \quad A \in G$$

$$(1.4) \qquad g(Az)A'(z) = g(z) \quad \text{for all} \quad A \in G.$$

A function satisfying (1.3) is said to be *automorphic*.

1.2. We will call $g(z)\, dz$ a *meromorphic differential* on R or $\hat{R}$ if $g(z)$ is a meromorphic function in D or $\hat{D}$ satisfying (1.4). If $g(z)$ has no poles and has at least a double zero at ∞, we call $g(z)\, dz$ an analytic differential. The condition at ∞ expresses the regularity of $g(z)\, dz$ in terms of the local parameter $\zeta = 1/z$. It is fulfilled automatically if $g(z)$ satisfies (1.4) and is regular at $A(\infty)$ for some $A \in G$.

The anti-conformal involution $j(z) = 1/\bar{z}$ induces an involution of $\hat{R}$ and an involution $f \rightarrow \bar{f} \circ j$ of meromorphic functions on $\hat{R}$. A meromorphic function $f(z)$ on $\hat{R}$ is *symmetric* if $f(z) = \bar{f}(1/\bar{z})$ for all $z \in \hat{D}$, or equivalently, if $f(z)$ is real on $\mathcal{I}$. j also induces an involution j^* of meromorphic differentials $\beta = g(z)\, dz$ on $\hat{R}$ by

$$(1.5) \qquad j^*(g(z)\, dz) = -z^{-2}\bar{g}(1/\bar{z})\, dz = \bar{g}(jz)\, d(1/z).$$

β is *symmetric* if $j^*(\beta) = \beta$. If $g_1(z) = izg(z)$, that is described by the condition $g_1(z) = \bar{g}_1(1/\bar{z})$ for all $z \in \hat{D}$; equivalently

$$(1.6) \qquad g(z)\, dz = izg(z) \mid dz \mid, \; z \in \mathcal{I}, \quad \text{is real.}$$

Every differential β can be written in the form $\beta = \beta_1 + i\beta_2$, where β_1 and β_2 are symmetric. Simply put

$$\beta_1 = \tfrac{1}{2}(\beta + j^*(\beta)), \qquad \beta_2 = (1/2i)(\beta - j^*(\beta)).$$

1.3. Let m be the linear measure on $\partial D : m(S) = \int_S \mid dz \mid$, $S \subset \partial D$ a Baire set.

$$m(L(G)) = 0. \tag{1.7}$$

In fact, let $\varphi(z)$ be the characteristic function of $L(G)$. The Poisson integral of φ is a harmonic function $u(z)$ on D which vanishes on $\partial D - L(G)$. Since $L(G)$ is a G-invariant set, $\varphi(z)$ and hence $u(z)$ satisfy (1.3). In other words $u(z)$ is a harmonic function on R which vanishes on ∂R. By the maximum principle $u \equiv 0$ and hence $\varphi = 0$ a.e., proving (1.7).

If $f(z)$ is integrable on ∂D we obtain from (1.1), (1.2) and (1.7) that

$$\begin{aligned} \int_{\partial D} f(z) \mid dz \mid &= \sum_{A \epsilon G} \int_{A(\mathcal{J})} f(z) \mid dz \mid \\ &= \int_{\mathcal{J}} (\sum_{A \epsilon G} f(A\zeta) \mid A'(\zeta) \mid) \mid d\zeta \mid \end{aligned} \tag{1.8}$$

If in addition f satisfies (1.3) on ∂D, so that f is a function on ∂R, (1.8) simplifies to

$$\int_{\partial D} f(z) \mid dz \mid = \int_{\mathcal{J}} f(z)(\sum_{A \epsilon G} \mid A'(z) \mid) \mid dz \mid .$$

We introduce the function

$$\rho(z) = \sum_{A \epsilon G} \mid A'(z) \mid \tag{1.9}$$

so that our formula becomes

PROPOSITION 1. *For every integrable function $f(z)$ on ∂R,*

$$\int_{\partial D} f(z) \mid dz \mid = \int_{\mathcal{J}} f(z)\rho(z) \mid dz \mid . \tag{1.10}$$

1.4. Applying (1.10) with $f(z) = 1$ we obtain

$$\int_{\mathcal{J}} \rho(z) \mid dz \mid = 2\pi,$$

from which we conclude $\rho(z) < \infty$ a.e. in $\mathcal{J}$. Much more is true:

PROPOSITION 2. *The series* (1.9) *converges uniformly on every compact subset of $\hat{D}$ which does not intersect $G(\infty) = \{A(\infty) : A \epsilon G\}$.*

Proof. Let $\{A_n\}$ be an enumeration of G with $A_1 = I$. Each A_n is of the form

$$A_n(z) = (a_n z + b_n)/(\bar{b}_n z + \bar{a}_n), \quad | a_n |^2 - | b_n |^2 = 1.$$

Since no element of G has a fixed point in $\hat{D}$, $b_n \neq 0$ for $n \neq 1$. For $z \epsilon \mathcal{J}$,

$$| A'_n(z)| = | \bar{b}_n z + \bar{a}_n |^{-2} \geqq (| b_n | + | a_n |)^{-2} \geqq (2| a_n |)^{-2}.$$

Since $\rho(z)$ is finite for some $z \in \mathcal{I}$, we have

$$\sum |a_n|^{-2} < \infty,$$

and since $|a_n|^2 - |b_n|^2 = 1$,

$$\sum_2^\infty |b_n|^{-2} < \infty. \tag{1.11}$$

Now let K be a compact set in $\hat{D}$ disjoint from $G(\infty)$, and let $\delta > 0$ be the distance of the closure $G(\infty) \cup L(G)$ of $G(\infty)$ from K. For $z \in K$ and $n > 1$,

$$\begin{aligned} |A_n'(z)| &= |b_n|^{-2} |z + \bar{a}_n \bar{b}_n^{-1}|^{-2} \\ &= |b_n|^{-2} |z - A_n^{-1}(\infty)|^{-2} \leqq \delta^{-2} |b_n|^{-2}, \end{aligned} \tag{1.12}$$

and Proposition 2 immediately follows. Note that each point of $G(\infty)$ interferes with only one term of (1.9).

Corollary. $\rho(z)$ is a bounded continuous function on $\mathcal{I}$.

1.5. One way to obtain a meromorphic differential on R or $\hat{R}$ is to start with an arbitrary meromorphic function $F(z)$ in D or $\hat{D}$ and form the *Poincaré series*

$$(\Theta F)(z) = \sum_{A \in G} F(Az)A'(z). \tag{1.13}$$

If the series (1.13) converges uniformly on compact subsets of D or $\hat{D}$, $(\Theta F)(z)\,dz$ will be a meromorphic differential on R or $\hat{R}$. Proposition 2 implies the convergence of (1.13) for many functions $F(z)$. For instance

Proposition 3. *Let $r(z)$ be a rational function with no poles in $L(G)$. Then $(\Theta r)(z)\,dz$ is a meromorphic differential on $\hat{R}$.*

Proof. If K is a relatively compact subregion of $\hat{D}$ then $A(K) \cap K \neq \emptyset$ for only a finite number of $A \in G$. Hence if K contains the poles of $r(z)$ and $M = \sup_{z \notin K} |r(z)|$, then $|r(Az)| \leqq M$ for all $A \in G$, with a finite number of exceptions.

1.6. As an example consider our basic meromorphic differential

$$\alpha = \Theta(1/z)\,dz = \sum_{A \in G} (A'(z)/A(z))\,dz, \qquad z \in \hat{D}. \tag{1.14}$$

α is analytic in $\hat{R}$ except for simple poles at $\pi(0)$, $\pi(\infty)$ (of residue $+1$, -1 respectively). Hence by the Riemann-Roch theorem, α has $2\,\mathfrak{g}$ zeros in $\hat{R}$, where $\mathfrak{g}$ is the genus of $\hat{R}$.

The formula $|A'(z)| = zA'(z)/A(z)$ for $z \in \partial D$ and $A \in G$, with (1.9) and (1.14), yields

$$\alpha = z^{-1}\rho(z)\,dz = i\rho(z)\,|dz|, \qquad z \in \mathcal{I}. \tag{1.15}$$

Comparing (1.15) with (1.6), we find that $i\alpha$ is symmetric on $\hat{R}$ and, by (1.5), has symmetric zeros. Since $\rho(z) \geqq 1$ for all z, no zero appears on ∂R. Thus, α has exactly $\mathfrak{g}$ zeros in R.

It will turn out (§§2, 3) that α has fundamental importance on R. But this is hardly surprising, because α is closely related to Green's function $g(z)$ on R with pole at $\pi(0)$. Indeed, since

$$g(z) = \textstyle\sum_{A \in G} \log | A(z) |, \qquad \alpha = dg + i *dg.$$

Because $dg = 0$ along ∂R, (1.15) gives

$$i\rho(z) \mid dz \mid = \alpha = i(\partial g/\partial n) \mid dz \mid \quad \text{on } \partial R,$$

so that we can write (1.10) in the form

$$\int_{\partial D} f(z) \mid dz \mid = \int_{\mathfrak{g}} f(z) \frac{\partial g}{\partial n} \mid dz \mid .$$

2. The conditional expectation

2.1. For $f(z)$ defined in D, $\hat{D}$, or ∂D, set

$$(2.1) \qquad \begin{aligned} (Ef)(z) &= \sum_{A \in G} \frac{f(Az)A'(z)}{A(z)} \Big/ \sum_{A \in G} \frac{A'(z)}{A(z)} \\ &= \Theta(f/z)/\Theta(1/z). \end{aligned}$$

Obviously, Ef is an automorphic function whenever it exists. Its existence for suitable functions f is guaranteed by Propositions 2 and 3. For example, Ef is a meromorphic function on $\hat{R}$ whenever f is rational with no poles in $L(G)$. If f is a bounded analytic function in D, then Ef is meromorphic in R with poles only at the zeros of the differential α defined in §1.6. If f itself is automorphic, then $Ef = f$.

2.2. G is a free group of rank $\mathfrak{g}$, where $\mathfrak{g}$ is the genus of $\hat{R}$. Choose a set of generaters $\{A_j\}$, $1 \leqq j \leqq \mathfrak{g}$, and define

$$(2.2) \qquad h_j(z) = z\bar{\zeta}_j/(1 - \bar{\zeta}_j z), \qquad \zeta_j = A_j(0), \qquad 1 \leqq j \leqq \mathfrak{g}$$

LEMMA 1. *$(Eh_j)\alpha$ is an analytic differential on $\hat{R}$.*

Proof. From the definitions we have

$$(2.3) \qquad (Eh_j)\alpha = \Theta\left(\frac{\bar{\zeta}_j}{1 - \bar{\zeta}_j z}\right) dz.$$

Thus $(Eh_j)\alpha$ is a meromorphic differential on $\hat{R}$. Since $A_j(\infty) = 1/\bar{\zeta}_j$, $(Eh_j)\alpha$ can have a pole only at $\pi(\infty)$ ($= \pi(1/\bar{\zeta}_j)$). But

$$\Theta\left(\frac{\bar{\zeta}_j}{1 - \bar{\zeta}_j z}\right) = \bar{\zeta}_j\left\{\frac{1}{1 - \bar{\zeta}_j z} + \frac{(A_j^{-1})'(z)}{1 - \bar{\zeta}_j(A_j^{-1})(z)}\right\} + f(z)$$

where $f(z)$ is analytic in a neighborhood of $1/\bar{\zeta}_j$. Elementary calculation shows that the bracketed expression is regular at $1/\bar{\zeta}_j$. That proves the lemma.

2.3. Let $\mathfrak{A}(R)$ be the (complex) vector space of analytic differentials on R which are continuous in $\bar{R}$. The Dirichlet integral [2] defines an inner product

$$(\beta_1, \beta_2) = \iint_R \beta_1 \wedge \overline{*\beta_2} = i \iint_R \beta_1 \wedge \bar{\beta}_2$$

on $\mathfrak{A}(R)$. Let Γ_j be the closed curve in R covered by the line segment in D joining 0 to $\zeta_j = A_j(0)$. It is well known [2] that there is an analytic differential $\psi(\Gamma_j)$ on R such that

$$2\pi \int_{\Gamma_j} \beta = (\beta, \psi(\Gamma_j)) \quad \text{for all } \beta \in \mathfrak{A}(R).$$

LEMMA 2. $\psi(\Gamma_j) = (Eh_j)\alpha$.

Proof. Set $\beta = f(z)\,dz$. Then f is integrable in D, for if $\mathfrak{R}$ is a fundamental polygon for G in D we compute

$$\begin{aligned}\iint_D |f(\zeta)|\,d\xi\,d\eta &= \sum_{A\in G} \iint_{A(\mathfrak{R})} |f(\zeta)|\,d\xi\,d\eta \\ &= \sum_{A\in G} \iint_{\mathfrak{R}} |f(Az)|\,|A'(z)|^2\,dx\,dy \\ &= \iint_{\mathfrak{R}} |f(z)|\,\rho(z)\,dx\,dy,\end{aligned}$$

where of course $\rho(z)$ is defined by (1.9). But $\rho(z)\,|f(z)|$ is continuous, hence bounded, in the closure of $\mathfrak{R}$.

Since f is integrable in D, it satisfies

$$\pi f(z) = \iint_D f(\zeta)(1 - \bar{\zeta}z)^{-2}\,d\xi\,d\eta, \qquad z \in D.$$

Integrating from 0 to ζ_j we obtain

$$\begin{aligned}\pi \int_0^{\zeta_j} f(z)\,dz &= \iint_D f(\zeta)\zeta_j(1 - \bar{\zeta}\zeta_j)^{-1}\,d\xi\,d\eta \\ &= \sum_{A\in G} \iint_{A(\mathfrak{R})} f(\zeta)\zeta_j(1 - \bar{\zeta}\zeta_j)^{-1}\,d\xi\,d\eta \\ &= \sum_{A\in G} \iint_{\mathfrak{R}} f(Az)\zeta_j(1 - \zeta_j\overline{A(z)})^{-1}A'(z)\overline{A'(z)}\,dx\,dy \\ &= \iint_{\mathfrak{R}} f(z)\bar{\Theta}(\bar{\zeta}_j(1 - \bar{\zeta}_j z)^{-1})(z)\,dx\,dy.\end{aligned}$$

In view of (2.3), that proves Lemma 2.

DEFINITION. N is the vector space spanned by the functions $\{Eh_j\}$, $1 \leqq j \leqq \mathfrak{g}$.

COROLLARY 1. (i) *N has dimension $\mathfrak{g}$.*

(ii) *N consists of the meromorphic functions $f(z)$ on $\hat{R}$ such that $f\alpha$ is an analytic differential on $\hat{R}$.*

(iii) *N has a basis consisting of functions real on ∂R.*

Proof. (i) The vector space of analytic differentials on $\hat{R}$ has dimension $\mathfrak{g}$. If the differentials $\psi(\Gamma_j) = (Eh_j)\alpha$ were not independent, there would be a non-zero analytic differential on $\hat{R}$ which was exact in R. That is impossible [2, p. 296].

(ii) Lemma 1 asserts that N is a linear subspace of the vector space $M(-(\alpha))$ of functions f such that $f\alpha$ is analytic in $\hat{R}$. But $M(-(\alpha))$ has dimension $\mathfrak{g}$, for every analytic differential β on $\hat{R}$ can be written $\beta = (\beta/\alpha)\alpha$, and $\beta/\alpha \in M(-(\alpha))$. By (i), $N = M(-(\alpha))$.

(iii) Choose a basis $\{\beta_j\}$, $1 \leqq j \leqq \mathfrak{g}$, for the analytic differentials on $\hat{R}$ such that each β_j is symmetric (see §1.2). Since $i\alpha$ is a symmetric differential, the functions $i\beta_j/\alpha$ form a symmetric basis for N. In particular, they are real on ∂R. (A closer examination of the differentials $\psi(\Gamma_j)$ would reveal them to be symmetric.)

COROLLARY 2. (Heins [7]). *If $f \in N$ and is analytic in R, then $f \equiv 0$.*

Proof. Let $f = \sum C_j(Eh_j)$. If f is analytic in R, then $df \in \mathfrak{A}(R)$, and Lemma 2 gives

$$0 = 2\pi \sum \bar{C}_j \int_{\Gamma_j} df = (df, f\alpha) = i \iint_R df \wedge \overline{f\alpha} = i \int_{\partial R} |f|^2 \bar{\alpha},$$

where the last equality is Stokes' theorem. Equation (1.15) shows that the differential $i\bar{\alpha}$ is positive along ∂R. Therefore f vanishes on ∂R, hence everywhere.

THEOREM 1. *If f is meromorphic in R and $f\alpha$ is regular in R, there is a unique $h \in N$ such that $f - h$ is analytic in R.*

Proof. The space P of principal parts of such functions f is a vector space of dimension $\mathfrak{g}$, for α has $\mathfrak{g}$ zeros in R. Corollaries 1 and 2 imply that the map from N to P which sends each function to its principal parts is a vector space isomorphism.

2.4. Since $|A'(z)| = zA'(z)/A(z)$ for $z \in \partial D$, we can write (2.1) in the form

$$(Ef)(z) = \sum\nolimits_{A \in G} f(Az) \,|A'(z)|/\rho(z), \qquad z \in \partial D, \tag{2.4}$$

where $\rho(z)$ is given by (1.9). Set $L^p = L^p(dm)$, $1 \leqq p \leqq \infty$, where m is the linear measure on ∂D, and let $L^p \mid G$ be the subspace of automorphic functions. We claim that $E : L^p \to L^p \mid G$ is a projection of norm one; in other words

$$(2.5) \qquad \| Ef \|_p \leqq \| f \|_p , \qquad 1 \leqq p \leqq \infty.$$

That is clear if $p = \infty$ because the series (1.9) converges almost everywhere on ∂D. For $p < \infty$ Hölder's inequality and (1.9) give

$$\begin{aligned} \rho(z)^p \mid Ef(z) \mid^p &\leqq \left(\textstyle\sum_{A \in G} \mid f(Az) \mid \mid A'(z) \mid \right) \\ &\leqq \left(\textstyle\sum_{A \in G} \mid f(Az) \mid^p \mid A'(z) \mid \right) \rho(z)^{p-1}, \end{aligned}$$

or

$$(2.6) \qquad \mid Ef \mid^p \leqq E(\mid f \mid^p), \qquad 1 \leqq p < \infty.$$

For any $g \in L^1$, (1.8), (1.10), and (2.4) yield

$$(2.7) \qquad \int_{\partial D} g \mid dz \mid = \int_{\mathcal{S}} (Eg)\rho \mid dz \mid = \int_{\partial D} (Eg) \mid dz \mid .$$

From (2.6) and (2.7) we obtain

$$\| Ef \|_p^p = \int_{\partial D} \mid Ef \mid^p \mid dz \mid \leqq \int_{\partial D} E(\mid f \mid^p) \mid dz \mid = \int_{\partial D} \mid f \mid^p \mid dz \mid = \| f \|_p^p ,$$

proving (2.5).

We should also note the obvious facts that $Ef = f$ for all $f \in L^p \mid G$ and that $E\bar{f} = \overline{Ef}$ for all $f \in L^p$.

Remark. The identity

$$E(fg) = fEg, \qquad f \in L^p \mid G, \ g \in L^q$$

is immediate from (2.4). With (2.7) it implies that

$$(2.8) \qquad \int_{\partial D} fg \mid dz \mid = \int_{\partial D} f(Eg) \mid dz \mid , \qquad f \in L^p \mid G, g \in L^q,$$

whence

$$\int_{\partial D} f(Eg) \mid dz \mid = \int_{\partial D} (Ef)g \mid dz \mid, \qquad f \in L^p, g \in L^q.$$

Thus E is the *conditional expectation* operator considered by Forelli [6]. (Of course the numbers p and q above satisfy $p^{-1} + q^{-1} = 1$.)

2.5. The Hardy space $H^p(D)$, $1 \leqq p \leqq \infty$, is the Banach space of analytic functions in D which satisfy the equivalent conditions

(i) $$\| f \|_p^p = \lim_{r \to 1} \int_{|z| = r} \mid f \mid^p \mid dz \mid / \mid z \mid < \infty \qquad (p < \infty)$$

$$\| f \|_\infty = \lim_{r \to 1} \max \{ \mid f(z) \mid : \mid z \mid = r \} < \infty,$$

(ii) $\mid f \mid^p$ has a harmonic majorant in D $(p < \infty)$.

For each $f \in H^p(D)$, $f(e^{i\theta}) = \lim_{r\to1} f(re^{i\theta})$ exists a.e. on ∂D and is in L^p. Furthermore, its L^p norm equals the norm given by (i), and f is equal to the Poisson integral of its boundary values [8]. We may therefore identify $H^p(D)$ with a subspace of L^p.

The Hardy space $H^p(R)$, $1 \leqq p \leqq \infty$ is the Banach space of analytic functions in R satisfying the equivalent conditions (see [11]):

(i) $$\| f \|_p'^p = \lim_{r\to1} \int_{l_r} | f |^p(\partial g/\partial n)\, ds < \infty \qquad (p < \infty)$$

$$\| f \|_\infty' = \lim_{r\to1} \max \{| f(z) | : z \in l_r\} < \infty,$$

(ii) $| f |^p$ has a harmonic majorant in R $(p < \infty)$,
(iii) $f \in H^p(D)$ and f is automorphic.

Here g is Green's function on R with pole at $\pi(0)$, and

$$l_r = \{z \in R : g(z) = 1 - r\}.$$

Furthermore $\| f \|_p' = \| f \|_p$. Using (iii) we shall identify $H^p(R)$ with a subspace of L^p; in fact $H^p(R) = L^p \mid G \cap H^p(D)$.

Finally, $H_0^p(D)$ is the set of $f \in H^p(D)$ satisfying the equivalent conditions $f(0) = 0$ and

$$\int_{\partial D} f| dz | = 0;$$

set $H_0^p(R) = H_0^p(D) \cap H^p(R)$.

2.6. The operator E is a powerful tool for the study of $H^p(R)$, as Forelli has shown in [6]. The basic fact is

PROPOSITION 4 ([6]). $EH^p(D) = H^p(R) \oplus N$, $1 \leqq p \leqq \infty$.

Proof. The inclusion $H^p(R) \subset EH^p(D)$ is obvious because E leaves $H^p(R)$ fixed. Since the functions h_j belong to $H^p(D)$ for all $p \geqq 1$, we also have $N \subset EH^p(D)$. Corollary 2, §2.3, implies that $H^p(R) \cap N = \{0\}$. Moreover, $H^p(R) \oplus N$ is closed in L^p/G, and the natural projection from $H^p(R) \oplus N$ to $H^p(R)$ is continuous, because N is finite dimensional. (That justifies the direct sum notation.) We have proved that

$$H^p(R) \oplus N \subset EH^p(D).$$

Suppose now that $f \in H^\infty(D)$. As we observed in §2.1, Ef is meromorphic in R with poles only at the zeros of α. By Theorem 1, there exists $h \in N$ such that $Ef - h \in H^\infty(R)$. Thus, $EH^\infty(D) \subset H^\infty(R) \oplus N$.

If $f \in H^p(D)$, $p < \infty$, and $f_r(z) = f(rz)$, $r < 1$, then $f_r \to f$ in L^p as $r \to 1$ (see [8]). From (2.5) it follows that $Ef_r \to Ef$ in $L^p \mid G$. But

$$Ef_r \in H^\infty(R) \oplus N \subset H^p(R) \oplus N.$$

Since $H^p(R) \oplus N$ is closed, we conclude that $EH^p(D) \subset H^p(R) \oplus N$ for all $p \geqq 1$.

PROPOSITION 5 ([6], [7]). *For* $1 < p < \infty$,

$$L^p \mid G = H_0^p(R) \oplus H^p(R) \oplus N.$$

Proof. It is classical (see [8]) that $L^p = H_0^p(D) \oplus \overline{H^p(D)}$ if $1 < p < \infty$. Writing $f \in L^p \mid G$ in the form $f = g + \bar{h}$, with g and $h \in H^p(D)$, and applying E, we obtain

$$f = Ef = Eg + \overline{Eh}.$$

To complete the proof we apply Proposition 4 and observe that $N = \bar{N}$ because of Corollary 1, §2.3.

PROPOSITION 6. [3], [7], [9], [10]. $f \in L^1 \mid G$ *is in* $H^1(R)$ *if and only if*

$$\int_{\partial R} f\beta = 0 \quad \textit{for all } \beta \in \mathfrak{A}(R). \tag{2.9}$$

Proof. If $f \in H^1(R)$ is continuous in $\bar{R}$, (2.9) follows immediately from Stokes' theorem. For any $f \in H^1(R)$, Ef_r is continuous on ∂R, $r < 1$. If Q is the (continuous) projection from $H^1(R) \oplus N$ to $H^1(R)$, then QEf_r belongs to $H^1(R)$ and is continuous in $\bar{R}$. Since $QEf_r \to QEf = f$ as $r \to 1$, (2.9) holds for all $f \in H^1(R)$.

Conversely, let $f \in L^1 \mid G$ satisfy (2.9). Then, for all $n \geqq 0$,

$$0 = \int_{\partial R} f(z)\Theta(z^n)\, dz = \int_{\partial R} f(z)E(z^{n+1})\alpha$$

$$= i\int_{\mathcal{G}} f(z)E(z^{n+1})\rho(z) \mid dz \mid = i\int_{\partial D} f(z)z^{n+1} \mid dz \mid,$$

by (2.1), (1.15) and (2.7). A classical theorem implies that $f \in H^1(D)$. Thus, $f \in H^1(D) \cap L^1 \mid G = H^1(R)$.

Remark. Proposition 6 is a weak form of the Cauchy-Read theorem [9], [10]. We shall obtain the strong form in §3.2 as a consequence of the F. and M. Riesz theorem.

2.7. *Remark.* Let g be any meromorphic function on $\hat{R}$ having the same zeros as α, with no other zeros or poles in $\bar{R}$. Then, it is clear that

$$E(gH^\infty(D)) = g(H^\infty(R) \oplus N) = H^\infty(R).$$

For on the one hand $g(H^\infty(R) \oplus N)$ is obviously contained in $H^\infty(R)$, and on the other hand Theorem 1 implies that $f/g \in H^\infty(R) \oplus N$ whenever $f \in H^\infty(R)$.

As Forelli showed in [6], the corona conjecture for $H^\infty(R)$ can be proved in a few lines as soon as $g \in H^\infty(D)$ with $E(gH^\infty(D)) = H^\infty(R)$ is found. He found such a g by methods quite different from ours.

3. Functions with continuous boundary values

3.1. Let $C(\partial D)$ and $C(\partial R)$ be the Banach spaces of continuous complex-valued functions on ∂D and ∂R, respectively. Proposition 2 and the formula (2.4) show that E maps $C(\partial D)$ into $C(\partial R)$. Formula (2.5) shows that $E : C(\partial D) \to C(\partial R)$ has norm one. We shall calculate the adjoint map $E^* : C(\partial R)^* \to C(\partial D)^*$. In addition, we shall use a map

$$\pi_* : C(\partial D)^* \to C(\partial R)^*$$

induced by the natural map $\pi : \hat{D} \to \hat{R}$.

By the Riesz representation theorem, $C(\partial D)^*$ is the space of finite complex Baire measures on ∂D, and $C(\partial R)^*$ is the space of finite complex Baire measures on ∂R, or equivalently, on $\mathcal{J} \subset \partial D$.

LEMMA 3. *For each $\mu \in C(\partial R)^*$ and each Baire set $S \subset \partial D$,*

$$(E^*\mu)(S) = \sum_{A \in G} \int_{A^{-1}(S) \cap \mathcal{J}} | A'(z) | \rho(z)^{-1} \, d\mu(z). \tag{3.1}$$

Proof. Let $\mu^*(S)$ denote the right side of (3.1). It is clear that μ^* is a finite complex Baire measure on ∂D. We will show that it has the properties

$$\mu^*(L(G)) = 0 \tag{3.2}$$

$$\mu^*(B(S)) = \int_S | B'(z) | \, d\mu^*(z), \qquad B \in G \tag{3.3}$$

$$\int_{\partial D} f(z) \, d\mu^*(z) = \int_{\mathcal{J}} (Ef)(z) \, d\mu(z), \qquad f \in C(\partial D). \tag{3.4}$$

The truth of (3.2) is clear. (3.4) implies that $\mu^* = E^*\mu$. By a change of variable $w = B(z)$, $B \in G$, in (3.1) we find that $\mu^*(S)$ is equal to the series in (3.1) with $\mathcal{J}$ replaced by $B(\mathcal{J})$. Hence $d\mu^*(B(z)) = \rho(w)^{-1} \, d\mu(w) = | B'(z) | \rho(z)^{-1} \, d\mu(z) = | B'(z) | \, d\mu^*(z)$, first for $z \in \mathcal{J}$ and then for arbitrary $z \in \partial D - L(G)$. This is the differentiated form of (3.3). To prove (3.4):

$$\begin{aligned} \int_{\mathcal{J}} Ef(z) \, d\mu(z) &= \sum_{A \in G} \int_{\mathcal{J}} f(Az) \, | A'(z) | \, \rho(z)^{-1} \, d\mu(z) \\ &= \sum_{A \in G} \int_{\mathcal{J}} f(Az) \, | A'(z) | \, d\mu^*(z) \\ &= \sum_{A \in G} \int_{\mathcal{J}} f(Az) \, d\mu^*(Az) = \int_{\partial D} f \, d\mu^*. \end{aligned}$$

LEMMA 4. *Define $\pi_* : C(\partial D)^* \to C(\partial R)^*$ by*

$$(\pi_* \mu)(S) = \mu(G(S)) = \textstyle\sum_{A \in G} \mu(A(S)) \tag{3.5}$$

for each $\mu \in C(\partial D)^*$ *and each Baire set* $S \subset \partial R$. π_* *is linear of norm one. Moreover,* $\pi_* \circ E^*$ *is the identity on* $C(\partial R)^*$, *and* $P = E^* \circ \pi_*$ *is a projection of norm one from* $C(\partial D)^*$ *onto the closed subspace of measures which satisfy* (3.2) *and* (3.3).

Proof. Let $\mu_* = \pi_* \mu$, $\mu \in C(\partial D)^*$. Then $d\mu_*(z) = \sum d\mu(Az)$, and

$$\int_{\partial D - L(G)} f\, d\mu = \sum \int_{\mathcal{g}} f(w)\, d\mu(Aw) = \int_{\mathcal{g}} f\, d\mu_*$$

for all $f \in C(\partial R)$. Thus π_* has norm one. Let $\mu \in C(\partial R)^*$ and suppose $S \subset \mathcal{g}$ is a Baire set. Setting $\mu^* = E^*\mu \in C(\partial D)^*$ we obtain

$$\begin{aligned}(\pi_*\mu^*)(S) &= \sum_{A \in G} \mu^*(A(S)) = \sum_{A \in G} \int_S | A'(z) |\, d\mu^*(z) \\ &= \int_S \rho(z)\, d\mu^*(z) = \int_S d\mu(z) = \mu(S),\end{aligned}$$

proving that $\pi_* \circ E^*$ is the identity.

Finally, each $\mu^* \in C(\partial D)^*$ which satisfies (3.2) and (3.3) is in the range of E^*; in fact, $\mu^* = P\mu^* = E^*\mu$, where $\mu = \pi_* \mu^*$. For by (3.3),

$$\mu(S) = \int_S \rho(z)\, d\mu^*(z) \quad \text{if } S \subset \mathcal{g}.$$

Hence for any Baire set $T \subset \partial D$

$$\begin{aligned}(E^*\mu)(T) &= \sum_{A \in G} \int_{A^{-1}(T) \cap \mathcal{g}} | A'(z) |\, d\mu^*(z) \\ &= \sum_{A \in G} \int_{T \cap A(\mathcal{g})} d\mu^*(Az) = \mu^*(T),\end{aligned}$$

by (3.2) and (3.3).

Remark. We map L^1 into $C(\partial D)^*$ by identifying each $f \in L^1$ with the measure $d\mu = f(z) \mid dz \mid$ on ∂D. Each subspace of L^1 will be identified with its image in $C(\partial D)^*$. The restriction of P to L^1 is simply E. In particular, $P(H^1(D)) = H^1(R) \oplus N$.

3.2. Our work in §3.1 has two immediate applications.

THEOREM 2. *E maps* $C(\partial D)$ *onto* $C(\partial R)$.

Proof. A standard result in functional analysis [4, p. 488] says that E has dense range if and only if E^* is one-to-one and E has closed range if and only if E^* does. Therefore Theorem 2 is equivalent to the assertion that E^* is one-to-one and has closed range. These properties of E^* are immediate consequences of Lemma 4, specifically of the fact that E^* has a left inverse.

We will now introduce the two Banach spaces

$$A_0(D) = \{f \in H_0^\infty(D) : f \text{ is continuous in } \bar{D}\}$$
$$A_0(R) = \{f \in H_0^\infty(R) : f \text{ is continuous in } \bar{R}\}.$$

The functions z^n, $n \geq 1$, are dense in $A_0(D)$. We have by uniform convergence that Ef is meromorphic in R, continuous on ∂R, and vanishes at $\pi(0)$. Hence as in §2.6,

$$E(A_0(D)) \subset A_0(R) \oplus N \tag{3.6}$$

but the opposite inclusion is not obvious.

Lemma 5 (F. and M. Riesz) [3], [7], [10]. *Let μ be a finite complex Baire measure on ∂R such that*

$$\int_{\partial R} f \, d\mu = 0, \quad \text{all } f \in E(A_0(D)).$$

Then $d\mu = h(z)\rho(z) \mid dz \mid$ for some $h \in H^1(R)$.

Proof. Set $\mu^* = E^*\mu$. (3.4) implies that $\int_{\partial D} z^n \, d\mu^* = 0$ for all $n \geq 1$, and hence the classical result in D implies that $d\mu^* = h(z) \mid dz \mid$ for some $h \in H^1(D)$. But (3.3) implies that

$$h(B(z)) \mid B'(z) \mid \mid dz \mid = \mid B'(z) \mid h(z) \mid dz \mid$$

so that $h(B(z)) = h(z)$ for all $z \in \partial D$ and $B \in G$. Hence $h \in H^1(R)$.

Corollary 1 ([9], [10]). $[A_0(R) \oplus N]^\perp = \pi_*(H^1(R))$.

Proof. Since $\pi_*(H^1(R))$ consists of the measures on ∂R of the form $d\mu = h(z)\rho(z) \mid dz \mid$, $h \in H^1(R)$, (3.6) and Lemma 5 imply that

$$[A_0(R) \oplus N]^\perp \subset [E(A_0(D))]^\perp \subset \pi_*(H^1(R)). \tag{3.7}$$

Conversely, if $f \in A_0(R) \oplus N$ and $\mu \in \pi_*(H^1(R))$, then

$$i\int_{\mathcal{g}} f \, d\mu = i\int_{\mathcal{g}} f(z)h(z)\rho(z) \mid dz \mid = \int_{\partial R} hf\alpha = 0,$$

by (1.15) and Proposition 6, since $f\alpha \in \mathfrak{A}(R)$ when $f \in A_0(R) \oplus N$.

Corollary 2. *$E(A_0(D))$ is dense in $A_0(R) \oplus N$.*

In fact, Corollary 1 and (3.7) imply that every linear functional which vanishes on $E(A_0(D))$ vanishes on $A_0(R) \oplus N$.

Remark. Corollary 1 is the strong form of the Cauchy-Read theorem which we promised in §2.6. It corresponds to the classical theorem that

$$A_0(D)^\perp = H^1(D).$$

3.3. We are now ready to prove the main result of this chapter.

THEOREM 3. $E(A_0(D)) = A_0(R) \oplus N$.

Proof. By Corollary 2 of Lemma 5, we need to prove only that

$$E : A_0(D) \to A_0(R) \oplus N$$

has closed range. As in Theorem 2, we shall prove instead that E^* has closed range. Corollary 1 of Lemma 5 allows us to interpret E^* as a map from the coset space $C(\partial R)^*/\pi_*(H^1(R))$ into $C(\partial D)^*/H^1(D)$. The image of E^* is therefore

$$E^*(C(\partial R)^*)/H^1(D) = P(C(\partial D)^*)/H^1(D),$$

where $P : C(\partial D)^* \to C(\partial D)^*$ is the projection defined in Lemma 4. It is not obvious that $P(C(\partial D)^*)/H^1(D)$ is closed. The difficulty is that P does not preserve $H^1(D)$. To compensate for that we use the projection

$$Q : H^1(D) \oplus N \to H^1(D)$$

with kernel N. Here we interpret $H^1(D)$ and N as closed subspaces of $C(\partial D)^*$. The subspace $H^1(D) \oplus N$ is closed, and Q is continuous, because N has finite dimension.

Let $\{\mu_n\} \subset P(C(\partial D)^*)$ and $\{\nu_n\} \subset H^1(D)$ be sequences such that

$$\mu_n + \nu_n \to \lambda \in C(\partial D)^*.$$

We must find $\sigma \in H^1(D)$ such that $\sigma + \lambda = P(\sigma + \lambda)$. We assert that

$$\sigma = Q(P\lambda - \lambda) = \lim\ (QP\nu_n - \nu_n), \qquad n \to \infty,$$

suffices. First we verify that σ exists. Since $\mu_n + \nu_n \to \lambda$,

$$P(\mu_n + \nu_n) = \mu_n + P\nu_n \to P\lambda.$$

Therefore $P\lambda - \lambda = \lim\ (P\nu_n - \nu_n) \in H^1(D) \oplus N$, and σ exists, because

$$P\nu_n \in PH^1(D) = H^1(R) \oplus N \subset H^1(D) \oplus N,$$

a closed subspace. Since $QP\nu_n \in H^1(R)$, it is fixed by P, and we find that

$$\begin{aligned} P(\sigma + \lambda) &= \lim\ (PQP\nu_n - P\nu_n + P\nu_n + \mu_n) \\ &= \lim\ (QP\nu_n - \nu_n + \nu_n + \mu_n) = \sigma + \lambda, \end{aligned}$$

completing the proof.

3.4. Theorem 3 has an interesting application to Poincaré series Set $A(D) = A_0(D) \oplus C$; $A(D)$ is the closure in $C(\partial D)$ of the polynomials.

THEOREM 4. *The Poincaré series* (1.13), *maps* $A(D)$ *onto* $\mathfrak{A}(R)$.

Proof. The map $f(z) \to f_0(z) = zf(z)$ carries $A(D)$ onto $A_0(D)$. Comparing (1.13), (1.14) and (2.1) we find that

$$(\Theta f)(z)\,dz = (Ef_0)(z)\alpha.$$

By Theorem 3, the range of Θ is the set of all differentials $f\alpha$, $f \in A_0(R) \oplus N$. But the mapping $\beta \to \beta/\alpha$ is a one-to-one correspondence between $\mathcal{A}(R)$ and $A_0(R) \oplus N$, by Theorem 1.

Remark. Since polynomials are dense in $A(D)$, the Poincaré series of polynomials are dense in $\mathcal{A}(R)$. Thus each differential in $\mathcal{A}(R)$ can be uniformly approximated in $\bar{R}$ by meromorphic differentials in $\hat{R}$ which have poles only at $\pi(\infty)$.

3.5. The meromorphic differentials on $\hat{R}$ can also be described easily by Poincaré series. In fact, Proposition 3 has the following converse.

THEOREM 5. *Every meromorphic differential on $\hat{R}$ has the form $(\Theta r)(z)\,dz$, where $r(z)$ is rational with no poles in $L(G)$.*

Proof. Put $r_n(z) = (z - \zeta)^n$. If $\zeta \in \hat{D} - G(\infty)$, then $(\Theta r_n)(z)\,dz$ has a pole of order $-n$ at $\pi(\zeta)$ for $n < 0$, a pole of order $n + 2$ at $\pi(\infty)$ for $n > -2$, and no other poles in $\hat{R}$. Therefore, every meromorphic differential on $\hat{R}$ is the sum of an analytic differential and a linear combination of the differentials $(\Theta r_n)(z)\,dz$. From (2.3), Lemma 1, and Corollary 1 of Lemma 2, we conclude that every analytic differential on $\hat{R}$ is the Θ-series of a rational function with poles only in $G(\infty)$. That proves Theorem 5.

3.6. To illustrate some of the difficulties that can arise upon projecting a theorem on $H^p(D)$ we will present the theorem of Szegö and Kolmogoroff-Krein as presented in [8] (cf. [1, §5]).

Let μ be a finite positive Baire measure on ∂R with

$$d\mu = (1/2\pi)h(z)\rho(z)\,|\,dz\,| + d\mu_s,$$

μ_s singular. Then for

$$D(f) = \int_{\partial R} |\,1 - f\,|^2\,d\mu,$$

$$\inf_{f \in E(A_0(D))} D(f) \leqq \exp\,(1/2\pi) \int_{\partial R} (\log h)\rho(z)\,|\,dz\,| \leqq \inf_{f \in A_0(R)} D(f).$$

There is equality on both sides if $N \perp A(R)$ with respect to $d\mu$.

Proof. The corresponding theorem in D applied to $E^*\mu$ implies that

$$\inf_{g \in A_0(D)} \int_{\partial R} E(|\,1 - g\,|^2)\,d\mu = \exp\,(1/2\pi) \int_{\partial R} (\log h)\rho(z)\,|\,dz\,|.$$

On the one hand from (2.6) we have

$$E(|\,1 - g\,|^2) \geqq |\,E(1 - g)\,|^2 = |\,1 - E(g)\,|^2.$$

On the other hand if $f \in A_0(R)$ then $f_r(z) = f(rz) \in A_0(D)$ and it is not hard to show that

$$\lim E(|1 - f_r|^2) = E(|1 - f|^2) = |1 - f|^2$$

uniformly on $\mathcal{J}$. Finally if $N \perp A(R)$ with respect to $d\mu$ then writing $f \in E(A_0(D))$ as $f = f_1 + f_2$, $f_1 \in A_0(R)$, $f_2 \in N$, we have

$$\int_{\partial R} |1 - f|^2\, d\mu = \int_{\partial R} |1 - f_1|^2\, d\mu + \int_{\partial R} |f_2|^2\, d\mu.$$

4. Reproducing kernels on R

4.1. We will first construct the Poisson kernel for R. We recall that

$$P_\zeta(z) = (1 - |\zeta|^2)/|z - \zeta|^2, \qquad z \in \partial D, \quad \zeta \in D$$

is the Poisson kernel for D. Noting that

$$P_{A\zeta}(Az) = \frac{1 - |A\zeta|^2}{|Az - A\zeta|^2} = \frac{(1 - |\zeta|^2)\,|A'(\zeta)|}{|z - \zeta|^2\,|A'(z)|\,|A'(\zeta)|} = P_\zeta(z)\,|A'(z)|^{-1}$$

for all $A \in G$, we find that

$$E(P_{A\zeta})(z) = E(P_\zeta)(z), \qquad \text{all} \quad A \in G \quad \text{and} \quad z \in \partial D - L(G).$$

Thus $(EP_{A\zeta})(B(z)) = (EP_\zeta)(z)$ for all $z \in \partial D - L(G)$, $\zeta \in D$ and A, $B \in G$, so that $(EP_\zeta)(z)$ is a function on $\partial R \times R$. Furthermore if $f(\zeta)$ is any harmonic function in R, continuous on ∂R, we have, using (2.8),

$$2\pi f(\zeta) = \int_{\partial D} f(z)P_\zeta(z)\,|dz| = \int_{\partial R} f(z)(EP_\zeta)(z)\rho(z)\,|dz|.$$

Therefore $(EP_\zeta)(z)$ is the Poisson kernel for R.

4.2. We call the function $C(z, \zeta)$ a Cauchy kernel in R if for fixed $z \in \bar{D} - L(G)$, $C(z, \zeta)\,d\zeta$ is a meromorphic differential in $\bar{R}$ having one simple pole of residue one at $\pi(z)$, and for fixed ζ, $C(z, \zeta)$ is a meromorphic function in $\bar{R}$ having one simple pole of residue -1 at $\pi(\zeta)$. Thus $C(z, \zeta)$ must satisfy

$$C(Az, B\zeta)B'(\zeta) = C(z, \zeta), \qquad z, \ \zeta \in D; \ A, \ B \in G.$$

By analogy with §4.1 define

$$C_1(z, \zeta) = \sum_{A \in G} \frac{A'(\zeta)}{A(\zeta) - z} = E_\zeta\left(\frac{\zeta}{\zeta - z}\right)\alpha,$$

where the subscript ζ indicates that $\zeta(\zeta - z)^{-1}$ is interpreted as a function of ζ. For $f(z)$ analytic in R and continuous in $\bar{R}$ we find

$$\begin{aligned} 2\pi f(z) &= \int_{\partial D} \frac{\zeta f(\zeta)}{\zeta - z}\,|d\zeta| = \int_{\partial R} f(\zeta)E_\zeta(\zeta(\zeta - z)^{-1})(\zeta)\rho(\zeta)\,|d\zeta| \\ &= -i\int_{\partial R} f(\zeta)C_1(z, \zeta)\,d\zeta. \end{aligned} \tag{4.1}$$

Furthermore, $C_1(z, \zeta)\, d\zeta$ is a differential on R for each $z \in D$. However $C_1(z, \zeta)$, for fixed ζ, is not a function on R. To rectify this problem we will use a projection P that we constructed in [5]. Consider the Poincaré series

$$\Phi h = \textstyle\sum_{A \in G} h(A(z))A'(z)^2.$$

We choose a polynomial F so that ΦF is non-zero in $\bar{R}$ (see [5]), and we define

$$(Pf)(z) = (\Phi fF)(z)/(\Phi F)(z).$$

If f is analytic in $\bar{D}$, Pf is an analytic function in $\bar{R}$. If f is meromorphic in $\bar{D}$ with a simple pole of residue c at $z = \zeta$, then Pf is meromorphic in $\bar{R}$ with a simple pole of residue $cF(\zeta)/(\Phi F)(\zeta)$ at $\pi(\zeta)$.

Now we claim that

$$C(z, \zeta) = P_z\, C_1(z, \zeta) \tag{4.2}$$

where the subscript z indicates that $C_1(z, \zeta)$ is to be considered as a function of z, is the required Cauchy kernel. Explicitly

$$C(z, \zeta) = \sum_{A, B \in G} \frac{F(Bz)A'(\zeta)B'(z)^2}{(A\zeta - Bz)\varphi(z)} = \sum \frac{F(Bz)A'(\zeta)}{\varphi(Bz)(A\zeta - Bz)} \tag{4.2*}$$

where $\varphi(z) = (\Phi F)(z)$.

To prove that the double series involved in (4.2) converges, we need the identity

$$| B(A\zeta) - B(z) | = | A(\zeta) - z \,|\, | B'(A\zeta) |^{1/2} | B'(z) |^{1/2}$$

and the inequalities

$$| B'(z) | \leqq (| a | - | b |)^{-2} = (| a | + | b |)^2, \qquad z \in D,$$

$$| B'(z) | \leqq \sigma^{-2} | b |^{-2}, \qquad z \in \mathfrak{R}.$$

Here $B(z) = (az + b)/(\bar{b}z + \bar{a})$, $| a |^2 - | b |^2 = 1$, $\mathfrak{R}$ is a fundamental region for G in D, and σ is the distance from $\mathfrak{R}$ to the closed set $G(\infty) \cup L(G)$ (cf. (1.12)). Setting

$$M = \sup \{| F(z) | : z \in D\} \quad \text{and} \quad m = \inf \{| (\Phi F)(z) : z \in \mathfrak{R}\},$$

we obtain, for $z, \zeta \in \mathfrak{R}$,

$$\begin{aligned} | P_z C_1(z, \zeta) | &\leqq \frac{M}{m} \sum_B \sum_A \frac{| B'(z) |^2 \,| (BA)'(\zeta) |}{| (BA)(\zeta) - B(z) |} \\ &= \frac{M}{m} \sum_B \sum_A \frac{| B'(z) |^{3/2} \,| B'(A\zeta) |^{1/2} | A'(\zeta) |}{| A(\zeta) - z |} \\ &\leqq \frac{M}{m\delta^3} \left(\sum_A \frac{| A'(\zeta) |}{| A(\zeta) - z |} \right) \left(1 + {\sum_B}' \frac{| b | + | a |}{| b |^3} \right) \end{aligned}$$

where $\sum'$ denotes summation over all $B \neq I$. By (1.11), $\sum' | b |^{-2}$ converges. Since $| a/b | = | B^{-1}(\infty) |$, the terms $| a/b |$ are uniformly bounded, and the second series in parenthesis converges. The first converges uniformly for $z \in \mathfrak{R}$, provided the term $A = I$ is omitted.

Finally we note that the residue at $\pi(\zeta)$ for fixed ζ of $P_z C_1(z, \zeta)\, d\zeta$ is

$$-\sum F(A\zeta)/(\Theta F)(A\zeta) = -1$$

and similarly we see that $C(z, \zeta)\, d\zeta$ for fixed z is a meromorphic differential in ζ with simple pole at $\zeta = z$. Since $Pf = f$ for G-invariant functions f, the fact that C is a Cauchy kernel now follows from (4.1).

Remark. The essential part of our proof is the construction of C_1. At that point there is considerable freedom in choosing a projection P. Our construction of a Cauchy kernel appears to be simpler and, in a sense, more natural than the classical one.

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