SOME HOMOTOPY OF STUNTED COMPLEX PROJECTIVE SPACE

BY

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1. Introduction

The 2-components of the stable homotopy groups $\pi^s_{2n+i}(CP/CP^{n-1})$ of stunted complex projective space are here tabulated, up to group extension, for $8 \le i \le 13$. For earlier work, including computation of these groups for $i \le 7$, see [13], [4], [11], [6], [7], and [8] as corrected by [9]. See [3] for odd components.

A result of Toda [13] relates these stable groups to the metastable homotopy groups of unitary groups as follows: Let $0 \le t < n$. Then

$$\pi^s_{2n+2t+1}(CP/CP^{n-1}) = \pi_{2n+2t+1}U(n),$$

while there exists a commutative diagram with an exact row

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & \pi^s_{2n+2t}(CP/CP^{n-1}) & \longrightarrow & \pi_{2n+2t}U(n) & \longrightarrow & 0 \\ & & {\scriptstyle (n+t)!}\downarrow & & {\scriptstyle h}\downarrow & & & & \\ & & Z & = & H_{2n+2t}(CP/CP^{n-1}) & & & & \end{array}$$

in which h is the Hurewicz homomorphism.

In view of Toda's formula the value of h is needed to deduce $\pi_{2n+2t}\, U(n)$. We include this datum as (2.3) and give in (2.5) the order of the image of each element of the 2-component of $\pi^s_{2n+i}\, S^{2n}$ in $\pi^s_{2n+i}(CP/CP^{n-1})$.

Our basic method is the stable homotopy exact couple resulting from the standard cell filtration of CP/CP^{n-1}. By naturality, differentials in the resulting spectral sequence for CP/CP^{n-1} may be computed in the analogous spectral sequence for CP. The study in [9] of this sequence for CP is the basis of the calculation here; we keep its notations and have used its techniques to determine most of the requisite new differentials.

We tabulate our results in Section 2; Section 3 describes the computation and lists the values of the needed differentials. In two cases the methods of [9] were not adequate to evaluate these differentials. Proofs for these cases are indicated in Section 4.

The results of Section 2 were announced in [10].

2. Results on homotopy groups

THEOREM 2.1. *The 2-component of the torsion of the stable homotopy group* $\pi^s_{2n+i}(CP/CP^{n-1})$, $8 \le i \le 13$, *is given by Table* 2.2.

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TABLE 2.2

i	$n(8)$: 0	1	2	3
8	$3Z_2$	0	$2Z_2$	Z_4
9	$3Z_2 + Z_8$ 8 (16) $3Z_2 + Z_{16}$	Z_4 9 (32) Z_8 25 (32) Z_{16}	$Z_2 + Z_2 ? 2Z_2$ 10 (32) $Z_2 + Z_2 ? (Z_2 + Z_4)$ 26 (64) $Z_2 + Z_2 ? (Z_2 + Z_8)$ 58 (64) $Z_2 + Z_{16} + Z_2 ? Z_2$	Z_8
10	Z_2	0	$2Z_2$	Z_2
11	$Z_8 ? Z_4$ 0 (128) $Z_8 + Z_4$ 8 (32) $Z_8 ? Z_8$ 24 (32) $Z_4 + Z_{16}$	$Z_4 ? 2Z_2$ 9 (32) $Z_4 ? (Z_2 + Z_4)$ 25 (64) $Z_4 ? (Z_2 + Z_8)$ 57 (128) $Z_4 ? Z_2 + Z_{16}$ 121 (128) $Z_4 ? Z_2 + Z_{32}$	$Z_2 ? Z_8$	$Z_2 + Z_8$
12	0	Z_2	Z_2	0
13	$Z_4 ? 2Z_2$ 8 (32) $Z_4 ? (Z_4 + Z_2)$ 24 (64) $Z_4 ? (Z_8 + Z_2)$ 56 (128) $Z_4 ? Z_2 + Z_{16}$ 120 (128) $Z_4 ? Z_2 + Z_{32}$	$Z_2 ? Z_8$	$Z_2 + Z_8$	Z_8

i	$n(8)$: 4	5	6	7
8	$3Z_2$	0	$2Z_2$	Z_2
9	$2Z_2 + Z_2 ? Z_4$	Z_2	$Z_2 + Z_2 ? Z_2$	Z_8 15 (16) Z_{16}
10	Z_2	0	Z_2	Z_2
11	$Z_4 ? Z_2$	$Z_8 ? Z_2$	Z_{16} 22 (32) Z_{32} 6 (32) Z_{64}	$2Z_2 + Z_8$ 23 (32) $2Z_2 + Z_{16}$ 39 (64) $2Z_2 + Z_{32}$ 7 (64) $2Z_2 + Z_{64}$
12	0	0	Z_2	0
13	$Z_8 ? Z_2$	Z_{16} 21 (32) Z_{32} 5 (32) Z_{64}	$2Z_2 + Z_8$ 22 (32) $2Z_2 + Z_{16}$ 38 (64) $2Z_2 + Z_{32}$ 6 (64) $2Z_2 + Z_{64}$	$Z_8 ? Z_4$ 23 (32) $Z_4 + Z_{16}$ 7 (32) $Z_8 ? Z_8$

In (2.2) the symbol $A ? B$ denotes a group satisfying an exact sequence $0 \to A \to A ? B \to B \to 0$.

Modulo torsion, $\pi^s_{2n+i}(CP/CP^{n-1})$ vanishes for i odd, but is infinite cyclic for i even.

THEOREM 2.3. *Let x_{n+k} generate $H_{2n+2k}(CP/CP^{n-1})$. Let $h_{n+k,k}\, x_{n+k}$ generate*

TABLE 2.4

$n(8)$	k = 1	2	3	4		5		6	
0	1	2	2		8		8		16
						8 (16)	4	8 (16)	8
1	2	2	8		8		16		16
				9 (16)	4	9 (32)	8	9 (32)	8
						25 (32)	4	25 (64)	4
								57 (128)	2
								121 (128)	1
2	1	8	4		8		8		128
				10 (16)	4	10 (32)	4		
						26 (64)	2		
						58 (64)	1		
3	2	4	4		8		32		64
				11 (32)	4				
				27 (64)	2				
				59 (64)	1				
4	1	4	1		16		16		64
5	2	1	8		16		32		16
6	1	8	4		16		16		64
								22 (32)	32
								6 (32)	16
7	2	4	2		16		16		32
						15 (16)	8	23 (32)	16
								39 (64)	8
								7 (64)	4

the image of the Hurewicz homomorphism

$$h : \pi^s_{2n+2k}(CP/CP^{n-1}) \to H_{2n+2k}(CP/CP^{n-1}).$$

Then, up to multiplication by an odd integer, for $1 \leq k \leq 6$ $h_{n+k,k}$ *is given by Table* 2.4, *while* $h_{n+7,7} = h_{n+7,6}$.

THEOREM 2.5. *Let* β *be in the* 2-*component of* G_i, $0 < i \leq 13$. *Let*

$$j : S^{2n} \to CP/CP^{n-1}$$

be the inclusion. Then the order of $j_* \beta \in \pi^s_{2n+i}(CP/CP^{n-1})$ *is given by Table* 2.6.

Nomenclature for elements of G, the stable homotopy of spheres, is as in [12].

3. Description of the calculation

According to [9, Section 3], the E^1-term of the spectral sequence $\{E^r CP; d^r CP\}$ has the following description. For each p, column p contains

TABLE 2.6

	β																
$n\,(8)$	η	η^2	ν	ν^2		σ	ε	$\bar{\nu}$	$\eta\sigma$	$\eta\varepsilon$	$\eta\bar{\nu}$	$\eta^2\sigma$		μ	$\eta\mu$		ζ
0	2	2	8	2		16	2	2	2	2	2	2		2	2		8
																24 (32)	4
1	0	0	2	2		8	0	0	0	0	0	0		0	0		2
					9 (16)	16							25 (32)	2		121 (128)	4
2	2	2	2	0		8	2	2	2	2	0	2		2	2		2
					10 (16)	16											
3	0	0	0	0		2	2	2	0	0	0	0		0	0		0
					11 (32)	4											
					27 (64)	8											
					59 (64)	16											
4	2	2	4	2		8	2	2	2	2	2	2		2	2		4
5	0	0	4	2		4	0	0	0	0	0	0		0	0		4
6	2	2	2	0		2	2	2	2	2	0	2		2	2		0
																22 (32)	4
																6 (32)	8
7	0	0	0	0		2	0	0	0	0	0	0		0	0		0
													15 (16)	2		39 (64)	2
																7 (64)	4

a copy of G, the stable homotopy of spheres, beginning with $\iota_p \in E^1_{p,p}$. For $\beta \in G_q$, we denote by β_p the corresponding element in $E^1_{p,p+q}$.

For CP/CP^{n-1}, E^1 has the same description in column p for $p \geq n$, while column p is zero for $p < n$. The differentials follow by naturality from those for CP. To deduce the results of Section 2, it is necessary to compute, in the 2-component, the differentials d^r on each element of total degree no more than $2n + 14$. We now restrict attention to such elements and to the 2-component.

For $r \leq 3$, these differentials are given by (5.1), (5.2), and (5.4) of [9]. Inspection reveals that we need evaluate d^4 only on the diagonal terms $E^4_{n+i,n+i}$ for $4 \leq i \leq 7$. These differentials are given by (5.6) of [9].

Further inspection reveals that the only additional possible non-zero differentials on terms of total degree no more than $2n + 14$ are d^5 and d^6 on the diagonal, namely d^5 on $E^5_{n+i,n+i}$ for $5 \leq i \leq 7$ and d^6 on $E^6_{n+i,n+i}$ for $6 \leq i \leq 7$. These differentials are deduced from their following values in CP.

LEMMA 3.1. *$d^5_{n+5,n+5}\,CP$ is zero, except for $n \equiv 3$ or 5 (8), 1 or 7 (16), or 9 (32), in which cases its image is generated by $\mu_n \in E^5_{n,n+9}$.*

LEMMA 3.2. *The image of $d^6_{n+6,n+6}\,CP$ is generated by $\lambda\zeta_n \in E^6_{n,n+11}$, where $\lambda = 1$ if $n \equiv 3$ (8), 14 or 15 (16), or 23 (32); $\lambda = 2$ if $n \equiv 2$ (8), 1 (16),* 9 (32), 25 or 39 (64), *or* 57 (128); $\lambda = 4$ *if* $n \equiv 4$ (8) *or* 22 *or* 24 (32); *and* $\lambda = 0$ *otherwise.*

Except for group extensions, (2.1), (2.3), and (2.5) follow by direct calculation from (3.1), (3.2), and the preceding remarks. As an adjunct to the calculation, as a check, and as a tool for evaluating the group extensions we have, as in [9], computed the Adams spectral sequence for the 2-component of $\pi^s_* CP/CP^{n-1}$ by the method of [5]. We have used these computations and some *ad hoc* methods to determine some of the group extensions, but we omit to reproduce this work.

4. Proofs for two cases

(3.1) and most of (3.2) are proved, case by case, using only techniques of [9], in particular (3.1), (3.2), (4.7), (4.9), (4.11), inspection of cell structures and comparison with the Adams spectral sequence. However, refinements were needed to compute $d^6_{n+6,n+6}$ for $n \equiv 24$ (32) and $n \equiv 6$ (16).

We first consider $n \equiv 6$ (16). For such n, $E^6_{n+6,n+6} = Z$ generated by $16\iota_{n+6}$ and $E^6_{n,n+11} = Z_8$ generated by ζ_n. The best result obtained by methods of [9] is that $d^6 32\iota_{n+6} = 0$ and thus $d^6 16\iota_{n+6} = 4\zeta_n$ or 0. The difficulty in applying (4.11) of [9] is that e_C does not detect 4ζ.

However, since n is even, CP^{n+1}/CP^{n-1} has the homotopy type of $S^{2n} \vee S^{2n+2}$. Thus we may view S^{2n+2} as a subcomplex of CP^{n+6}/CP^{n-1}. Let L be the complex obtained from CP^{n+6}/CP^{n-1} by collapsing S^{2n+2}. A calculation in the homotopy exact couple spectral sequence for L reveals that in this spectral sequence d^6 is defined on $8\iota_{n+6}$.

We then use the idea of (4.11) of [9] to evaluate $d^6 8\iota_{n+6}$ in L, considering an element $\gamma \in \tilde{K}_C L$ corresponding to

$$\mu^n - \frac{1}{2} n\mu^{n+1} \in \tilde{K}_C(CP^{n+6}/CP^{n-1}),$$

the latter of which restricts to zero on S^{2n+2}. Calculating $ch_{n+6}\gamma$, we obtain in L that $d^6 8\iota_{n+6}$ is zero mod $4\zeta_n$ if $n \equiv 6$ (32), but congruent to $2\zeta_n$ mod $4\zeta_n$ if $n \equiv 22$ (32). The value of $d^6 16\iota_{n+6}$ in CP follows by naturality.

We now consider the case $n \equiv 24$ (32). Once again (4.11) of [9] fails by a factor of 2 since e_C does not detect 4ζ. In this case we pass to the real K-theory and apply the sharper Adams invariant $e_{R'}$ [1], which does detect 4ζ. The proof of (4.11) of [9] may be readily adapted to yield the following, where $y \in H^2 CP$ is the generator.

LEMMA 4.1. *Suppose* $n \equiv 0$ (8) *and* $k \equiv 2$ (4). *Let* $\beta_n \in d^k s\iota_{n+k}$. *Let* $\alpha \in \tilde{K}_R CP^{n+k}/CP^{n-1}$ *be such that* $ch_R\, \alpha = y^n + \cdots + qy^{n+k}$. *Then* $e_{R'}\, \beta \equiv \frac{1}{2} sq$ mod 1.

For $n \equiv 24$ (32) the value of $d^6 4\iota_{n+6}$ follows from (4.1), where α and $ch_R\, \alpha$ are found by a long calculation based on (2.2 v) of [2].

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