THE CHOQUET THEORY AND REPRESENTATION OF ORDERED BANACH SPACES

BY

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1. Introduction

Choquet boundary theory has mainly been developed so far for ordered Banach spaces which have a strict order unit and the order unit norm, or equivalently for the space of continuous affine functionals on a compact convex set in a locally convex topological linear space. However Choquet, [4], showed that much of the theory can be extended to the case where there is no order unit, and in particular he showed how to define conical measures and their barycentres for such spaces. In [6] his methods were used to characterize intrinsically the ordered Banach spaces whose duals are Banach lattices; these spaces are called R-spaces.

In this paper we show how all these concepts are preserved under the continuous embedding of one ordered Banach space as a subspace of another. Under the weak filtering condition of §3, we find that there is a very close connection between the Choquet theories of the two spaces, and if the one space is also dense in the other the two theories coincide in a certain exact sense.

In §4 this is used to provide a representation of any ordered Banach space with a topological order unit as a space of extended-valued affine functionals on a compact convex set. The Choquet theory of such spaces reduces to the usual Choquet theory of a compact convex set. We then analyse a large class of R-spaces, including all separable ones.

For a Banach lattice with a topological order unit this provides a representation as a vector lattice of extended-valued continuous functions on a compact Hausdorff space, which is unique up to homeomorphism. We indicate how this representation is related to that of Bernau, [3], which exists under much weaker hypotheses.

2. The general theory

We recall some of the basic definitions and notation of the Choquet theory for ordered Banach spaces developed in [4], [6]. An ordered Banach space is said to be *regular* if it satisfies the conditions:

(i) if $x, y \in V$ and $-x \leq y \leq x$ then $\| y \| \leq \| x \|$;

(ii) if $x \in V$ and $\varepsilon > 0$ then there is some $y \in V$ with $y \geq x, -x$ and $\| y \| < \| x \| + \varepsilon$.

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The *stump* of the positive cone V^+ of V is defined as the set

$$\{x \in V : 0 \leq x \quad \text{and} \quad \| x \| \leq 1\}.$$

If V is a regular ordered Banach space then V^* is regular, and the stump X of the positive cone of V^* is a compact convex set in the weak* topology. V is canonically order-isomorphic and homeomorphic with $A_0(X)$, the space of continuous linear functionals on X. We let S be the cone of functions on X which are the pointwise suprema of a finite number of functions of $A_0(X)$. If $L = S - S$ then L is a vector lattice of continuous functions on X, and in [6] we showed how to give L a norm so that it is a normal lattice and the natural injection $\alpha : V \to L$ is an isometric order injection. The positive elements of L^* are called *conical measures* and the stump of the positive cone of L^* is denoted P. The injection $\alpha : V \to L$ has a dual $\beta : L^* \to V^*$ such that $\beta(P) \subseteq X$; this is called the *barycentre map*. We denote the set of conical measures μ such that $\beta\mu = x \in V^*$ by $R(x, V)$ and observe that it follows quickly from the definition of the norm in L, [6], that if $x \in X$ then $R(x, V) \subseteq P$.

If $\lambda, \mu \in P$ we write $\lambda \leq \mu$ if $(\lambda, f) \leq (\mu, f)$ for all $f \in S$. This makes P into a partially ordered set and it is shown in [4, 6] that every element of P is dominated by a maximal element. If $x_i \in X$ for $i = 1, \cdots, n$ and $\sum x_i = x \in X$ then the functional

$$f \to f(x_i) + \cdots + f(x_n)$$

defined for all $f \in L$ is an element of $R(x, V) \subseteq P$ and is called a *discrete conical representing measure* for x. In [6] we showed that the discrete conical measures are dense in $R(x, V)$ for all $x \in X$.

Now suppose that V_1, V_2 are two regular ordered Banach spaces. We call a one-one continuous map $i : V_1 \to V_2$ with $\| i \| = 1$ an *embedding* if for any $x \in V_1$ we have $0 \leq x$ if and only if $0 \leq ix$.

THEOREM 1. *Let $i : V_1 \to V_2$ be an embedding between the regular ordered Banach spaces V_1, V_2. Then i induces a one-one lattice homomorphism $i : L_1 \to L_2$ with $\| i \| = 1$ such that if $\alpha_r : V_r \to L_r$, $r = 1, 2$, are the natural embeddings then $i\alpha_1 = \alpha_2 i$. The maps induce dual maps*

$$j : X_2 \to X_1 \quad \text{and} \quad j : P_2 \to P_1$$

such that if $\beta_r : P_r \to X_r$, $r = 1, 2$, are the barycentre maps then $j\beta_2 = \beta_1 j$. The map $j : P_2 \to P_1$ is order-preserving.

Proof. The dual $j : V_2^* \to V_1^*$ of $i : V_1 \to V_2$ is positive and of norm $= 1$ so we have $j(X_2) \subseteq X_1$. If V_r^{*+} is the positive cone of V_r^* for $r = 1, 2$, then $j(V_2^{*+})$ is weak* dense in V_1^{*+}. For otherwise by the Hahn-Banach theorem we could find some $x \in V_1$ with $x \ngeq 0$ but $x \mid j(V_2^{*+}) \geq 0$. But then we would have $ix \mid (V_2^{*+}) \geq 0$ so that $ix \geq 0$, and this is impossible as i is an embedding.

The map $j : X_2 \to X_1$ induces a dual map $i : L_1 \to L_2$ which is an extension of $i : V_1 \to V_2$. It is clear that i is a lattice homomorphism. Suppose $f \in L_1$ and $if = 0$. Then $f \mid (jX_2) = 0$ and as f is linear on the rays of V_1^{*+} so $f \mid (jV_2^{*+}) = 0$ Now f is the restriction to X_1 of a continuous function defined on the cone V_1^{*+} with the weak* topology and $j(V_2^{*+})$ is dense in V_1^{*+}. Therefore $f = 0$ and we conclude that $i : L_1 \to L_2$ is one-one. Now suppose that $f \in L_1$ and $\| f \| < 1$. Then by the definition of the norm of L_1 [6] there is some $g \in V_1$ with $g \geq f$, $-f$ and $\| g \| < 1$. As $i : L_1 \to L_2$ is a +ve map and an extension of $i : V_1 \to V_2$ so $ig \geq (if), (-if)$, and so as L_2 is a normed lattice we have $\| if \| \leq \| ig \| \leq \| g \| < 1$. Therefore $i : L_1 \to L_2$ has norm $= 1$.

The dual $j : L_2^* \to L_1^*$ of $i : L_1 \to L_2$ is positive and of norm $= 1$ and so $j(P_2) \subseteq P_1$. The equation $j\beta_2 = \beta_1 j$ is the dual of the equation $i\alpha_1 = \alpha_2 i$. We now show that $j : P_2 \to P_1$ preserves order. Let $\lambda, \mu \in P_2$ and $\lambda \leq \mu$. For any $f \in S_1$ we have $if \in S_2$ and so $(f, j\lambda) = (if, \lambda) \leq (if, \mu) = (f, j\mu)$. Therefore $j\lambda \leq j\mu$.

COROLLARY 2. *If $i : V_1 \to V_2$ is an embedding of the regular ordered Banach space V_1 onto a dense subspace of the regular ordered Banach space V_2, then the maps*

$$j : X_2 \to X_1 \quad \textit{and} \quad j : P_2 \to P_1$$

are one-one. For $\lambda, \mu \in P_2$ we have $\lambda \leq \mu$ if and only if $j\lambda \leq j\mu$.

For if iV_1 is dense in V_2 we see that, as $i : L_1 \to L_2$ is an extension of $i : V_1 \to V_2$ and the lattice operations in a normed lattice are continuous [7] so iL_1 is dense in L_2 and iS_1 is dense in S_2.

Without stronger conditions on the embedding $i : V_1 \to V_2$ little more can be said about the above situation. We now call a subspace L of an ordered Banach space V a *weakly filtering subspace* of V when

If $x \in L$, $y \in V$, $y \geq x, 0$ and $\varepsilon > 0$ then there exist $x_1 \in L$ and $y_1 \in V$ such that $y_1 \geq x_1 \geq x, 0$ and $\| y - y_1 \| < \varepsilon$.

This condition was first used for the special case of a subspace of the space of all continuous affine functionals on a Choquet simplex by Jellett, [13]. See also [10].

THEOREM 3. *Let L be a weakly filtering subspace of an ordered positively generated Banach space V. Let ϕ be a positive functional on L and ψ a positive functional on V such that $\phi \leq \psi \mid L$. Then there exists a linear extension $\bar{\phi}$ of ϕ to V such that $0 \leq \bar{\phi} \leq \psi$.*

Proof. First recall [15] that a positive functional on an ordered positively generated Banach space is continuous, and suppose for definiteness that $\| \psi \| \leq 1$. We define a sublinear functional p on V by

$$p(x) = \inf \{(\psi, y) : 0, x \leq y \in V\}$$

and observe that for $0 \leq x \in V$ we have $p(x) = (\psi, x)$ and for $0 \geq x \in V$ we have $p(x) = 0$. We now assert that for all $x \in L$ we have $(\phi, x) \leq p(x)$. For let $\varepsilon > 0$ and let $0, x \leq y \in V$ satisfy $(\psi, y) < p(x) + \varepsilon/2$. Using the fact that L is a weakly filtering subspace of V let $x_1 \in L$, $y_1 \in V$, $y_1 \geq x_1 \geq x, 0$ and $\| y - y_1 \| < \varepsilon/2$. Then we have the chain of inequalities

$$(\phi, x) \leq (\phi, x_1) \leq (\psi, x_1) \leq (\psi, y_1) \leq \psi(y) + \varepsilon/2 < p(x) + \varepsilon$$

and as $\varepsilon > 0$ is arbitrary so $(\phi, x) \leq p(x)$. We use the Hahn-Banach theorem to obtain an extension $\bar{\phi}$ of ϕ to V such that $(\bar{\phi}, x) \leq p(x)$ for all $x \in V$. If $0 \geq x \in V$ then $(\bar{\phi}, x) \leq p(x) = 0$, so $\bar{\phi}$ is a positive functional. If $0 \leq x \in V$ then $(\bar{\phi}, x) \leq p(x) = (\psi, x)$ so $\bar{\phi} \leq \psi$.

We now define an *ideal* I in an ordered vector space V as a positively generated subspace such that if $0 \leq x \leq y \in I$ then $x \in I$.

COROLLARY 4. *Let $i : V_1 \to V_2$ be an embedding of the regular ordered Banach space as a dense weakly filtering subspace of the regular ordered Banach space V_2. Then the dual map $j : V_2^* \to V_1^*$ is an embedding of V_2^* as a weak* dense ideal in V_1^*.*

THEOREM 5. *Let $i : V_1 \to V_2$ be an embedding of the regular ordered Banach space V_1 as a weakly filtering subspace of the regular ordered Banach space V_2. Then the induced map $j : P_2 \to P_1$ between the sets of conical measures has range equal to all the conical measures in P_2 whose barycentre is in jX_2. Specifically we have the formula*

$$j\{R(x, V_2)\} = R(jx, V_1)$$

for all $x \in X_2$. $j : P_2 \to P_1$ preserves the partial ordering and maps maximal conical measures to maximal conical measures.

Note. A related theorem for the space of continuous affine functionals on a Choquet simplex has been proved in [13].

Proof. It follows inductively from Theorem 3 that if $x_r \in X_1$ for $r = 1, \cdots, n$ and $\sum x_r = jy \in X_1$ where $y \in X_2$ then there are $y_r \in X_2$ for $r = 1, \cdots, n$ with $\sum y_r = y$ and $jy_r = x_r$ for $r = 1, \cdots, n$. Now the map $j : P_2 \to P_1$ is linear so

$$j\left(\sum_{r=1}^{n} \varepsilon_{y_r}\right) = \sum_{r=1}^{n} j\varepsilon_{y_r} = \sum_{r=1}^{n} \varepsilon_{jy_r} = \sum_{r=1}^{n} \varepsilon_{x_r}.$$

Therefore $j\{R(x, V_2)\}$ is compact and contains all discrete conical measures in $R(jx, V_1)$. As is shown in [4, 6], the set of all discrete conical measures is dense in $R(jx, V_1)$ so we see that the formula of the theorem holds.

Choquet, [4], has shown that a conical measure μ on a regular ordered Banach space V is maximal if and only if for all $f \in S$ we have

$$(\mu, f) = \inf \{(\mu, g) : f \leq g \in -S\}.$$

Now let $\mu \in P_2$ be maximal and let $f \in S_1$, $g \in -S_2$, $if \leq g$ and $\varepsilon > 0$. It follows inductively from the fact that iV_1 is a weakly filtering subspace of V_2 and from the continuity of the lattice operations in a normed lattice that there exist $f_1 \in -S_1$, $g_1 \in -S_2$ with $if \leq if_1 \leq g_1$ and $\| g - g_1 \| < \varepsilon$. Therefore

$$\begin{aligned}(j\mu, f) &= (\mu, if)\\ &= \inf\{(\mu, g) : if \leq g \in -S_2\}\\ &= \inf\{(\mu, g) : if \leq g \in -iS_1\}\\ &= \inf\{(j\mu, h) : f \leq h \in -S_1\}\end{aligned}$$

and we see that $j\mu$ is a maximal conical measure.

COROLLARY 6. *If in the situation of Theorem 5, iV_1 is a dense weakly filtering subspace of V_2 then $j : P_2 \to P_1$ maps P_2 homeomorphically and order isomorphically onto the set of conical measures in P_1 whose barycentre is in jX_2.*

3. Topological order units

An interesting case of the theory of the least section occurs when V_1 is an order unit norm space. The Choquet theory of these spaces is well understood, see for example [2, 16], and it is indicated in [6] how our theory reduces to the usual one for order unit norm spaces. Thus under the conditions of Corollary 6 the Choquet theory of V_2 can be reduced to the Choquet theory of a compact convex set, and in particular the maximal conical measures on V_2 can be regarded as those which are concentrated on the extreme rays of the locally compact positive cone of V_1^*; these extreme rays can be regarded as "virtual" extreme rays of the positive cone of V_2^*. We now show how this situation arises in the general case.

If V_2 is a regular ordered Banach space then any element $0 \leq e \in V_2$ such that $\| e \| = 1$ defines an ideal

$$V_1 = \{x \in V_2 : -ne \leq x \leq ne \quad \text{for some} \quad n\}$$

and if we give V_1 the order unit norm

$$\| x \| = \inf\{\alpha : -\alpha e \leq x \leq \alpha e\}$$

then V_1 becomes a regular ordered Banach space and the injection $i : V_1 \to V_2$ is an embedding of V_1 into V_2. We now define a *topological order unit* e in V_2 as a non-negative element generating the ideal V_1 with $\| e \| = 1$ and such that

(i) *if $x \in V_1$, $y \in V_2$, $\varepsilon > 0$ and $y - \varepsilon e \geq x, 0$ then there exists some $z \in V_1$ with $y \geq z \geq x, 0$ and $\| y - z \| < \varepsilon$;*

(ii) *if $0 \leq x \in V_1$, $0 \leq y_1, y_2 \in V_2$, $\varepsilon > 0$ and $y_1 + y_2 \geq x$ then there exist $x_1, x_2 \in V_1$ with $y_i \geq x_i$ for $i = 1, 2$ and $x_1 + x_2 + \varepsilon e \geq x$.*

Example. Let V be the subspace of $L^1[0, 1] \oplus L^1[2, 3]$ of measurable functions f such that

$$\int_0^1 f\,dx = \int_2^3 f\,dx$$

with the norm

$$\|f\| = \max\left\{\int_0^1 |f|\,dx, \int_2^3 |f|\,dx\right\}$$

and the ordering given by saying that $0 \leq f \in V$ if and only if $f \geq 0$ almost everywhere. Then V is a regular ordered Banach space and the element e which is constantly one is a topological order unit. We note from this example that it is not generally possible to eliminate the $\varepsilon > 0$ from (i) and (ii) in the above definition.

If e is a topolocial order unit in the regular ordered Banach space V_2 then $e : V_1 \rightarrow V_2$ embeds V_1 as a dense weakly filtering subspace of V_2 and so all our previous theory applies. Define $B \subseteq V_1^*$ as

$$B = \{\phi \in V_1^* : 0 \leq \phi \quad \text{and} \quad \phi(e) = 1\}$$

so that B is a compact convex base for the locally compact positive cone in V_1^*, [8]. It is well known that V_1 is canonically isometrically and order isomorphic with $A(B)$, the space of continuous affine functionals on B, in such a way that $e \in V_1$ corresponds to the function one. $j : V_2^* \rightarrow V_1^*$ identifies V_2^* with a dense ideal in V_1^*.

THEOREM 7. *Let e be a topological order unit in the regular ordered Banach space V_2 and let B be the natural base of the dual cone of the ideal V_1 generated by e. Then there is a natural one-one, linear, order-preserving map j from the positive cone V_2^+ to the cone of lower semi-continuous affine functionals on B which extends the identification $j : V_1 \rightarrow A(B)$. There is also a natural, one-one, linear, order preserving map j' from the positive cone L_2^{*+} to the cone of regular Borel measures on B such that if $0 \leq f \in V_2$ and μ is a conical measure for V_2 then*

$$(f, \mu) = \int_B (jf)\, d(j'\mu).$$

Proof. If $f \in V_2^+$ we define the function jf on B by

$$\begin{aligned} jf &= \sup\{jg : f \geq g \in V_1\} \\ &= \sup\{jg : f - \varepsilon e \geq g \in V_1 \quad \text{for some} \quad \varepsilon > 0\}. \end{aligned}$$

By condition (i) on e we can show that the second family of $g \in V_1$ filters upwards and converges in norm to f. Therefore jf is lower semi-continuous affine and

$$(f, \mu) = \sup\{(g, \mu) : f - \varepsilon e \geq g \in V_1 \quad \text{for some} \quad \varepsilon > 0\}.$$

Also because of the filtering condition we see that for $g \in V_1$ we have $f - \varepsilon e \geq g$ for some $\varepsilon > 0$ if and only if $jf > jg$ on B, and it follows that j is one-one. It clearly preserves order and has the properties that $j(\alpha f) = \alpha(jf)$ for all $\alpha \geq 0$ and $f \in V_2^+$, and that $j(f + g) \geq jf + jg$ for all $f, g \in V_2^+$.

We now show that j is subadditive. Let $f, g \in V_2^+$ and let $f + h \in V_1$. For some positive integer n we have

$$(f + ne) + (g + ne) \geq h + 2ne \geq 0.$$

By condition (ii) on e, for any $\varepsilon > 0$ we can find $h_1, h_2 \in V$ with $f + ne \geq h_1$, $g + ne \geq h_2$ and

$$h_1 + h_2 + \varepsilon e \geq h + 2ne.$$

Therefore $f \geq (h_1 - ne)$, $g \geq (h_2 - ne)$ and

$$(h_1 - ne) + (h_2 - ne) \geq h - \varepsilon e.$$

Therefore

$$jf + jg \geq j(h_1 - ne) + j(h_2 - ne) \geq jh - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary and $h \in V_1$ is arbitrary subject to $h \leq f + g$ so by the definition of $j(f + g)$ we have

$$jf + jg \geq j(f + g).$$

The other map j' of the theorem is the restriction of the map from L_2^* into L_1^* defined earlier. Our general theory tells us that the formula of this theorem holds for all $f \in V_1$. It therefore holds for all $f \in V_2^+$ in consequence of the formula

$$\int (jf)\, d(j'\mu) = \sup\left\{ \int (jg)\, d(j'\mu) : g \in V_1 \text{ and } jg < jf \right\}$$

and the remarks at the beginning of the proof.

In [6] we defined an *R-space* as a regular ordered Banach space with the Riesz decomposition property, and proved that an ordered Banach space is an R-space if and only if its dual is a Banach lattice. We also investigated the ideal structure of these spaces. Further light on their structure is thrown by the following theorem.

THEOREM 8. *An element $0 \leq e \in V_2$ in an R-space V_2 is a topological order unit if and only if $\| e \| = 1$ and the ideal V_1 generated by e is dense in V_2; every separable R-space has a topological order unit. If V_2 is an R-space with a topological order unit e then there is a natural one-one map j'' from the set of closed ideals of V_2 to a sublattice of the set of closed faces of the Choquet simplex B associated with V_1, such that if I is a closed ideal in V_2 then*

$$I^+ = \{f \in V_2^+ : (jf)|(j''I) = 0\}.$$

where j on V_2^+ is the map of the last theorem.

Before we prove this theorem we shall need a lemma on ideals in R-spaces which is of the same type as the results in [6].

LEMMA 9. *If I is an ideal in an R-space V then its closure $\bar{I}$ is also an ideal* *and if* $0 \leq f \in \bar{I}$ *we can write*

$$f = \sum_{n=1}^{\infty} f_n$$

where $0 \leq f_n \in I$ *and* $\sum_{n=1}^{\infty} \| f_n \| < \infty$.

Proof. We first show that $\bar{I}$ is positively generated.

If $f \in \bar{I}$ we can certainly find $f_n \in I$ such that $\sum_{n=1}^{\infty} \| f_n \| < \infty$ and $\sum_{n=1}^{\infty} f_n = f$. Now for each n let $\pm f_n \leq g_n, h_n$ where $g_n \in I$ and $h_n \in V$ with $\| h_n \| < 2 \| f_n \|$. By the Riesz decomposition property we obtain $k_n \in V$ with

$$\pm f_n \leq k_n \leq g_n, h_n$$

and then see that $k_n \in I$ and $\| k_n \| < 2 \| f_n \|$. Now as $\sum_{n=1}^{\infty} \| k_n \| < \infty$ so $\sum_{n=1}^{\infty} k_n = k \in \bar{I}$ converges and as $\pm \sum_{n=1}^{m} f_n \leq \sum_{n=1}^{m} k_n$ for all m so by the closedness of V^+ we have $\pm f \leq k$.

Now let $0 \leq 1 \leq f \in \bar{I}$. Then we have

$$0 \leq 1 \leq \sum_{n=1}^{\infty} k_n = k_1 + \sum_{n=2}^{\infty} k_n .$$

Using the Riesz decomposition property we can write $l = l_1 + m_1$ where $0 \leq l_1 \leq l, k_1$ and

$$0 \leq m_1 \leq l, \sum_{n=2}^{\infty} k_n .$$

Proceeding inductively we see that we can write

$$l = \sum_{n=1}^{N} l_n + m_N$$

where $0 \leq l_n \leq l, k_n$ and

$$0 \leq m_N \leq l, \sum_{n=N+1}^{\infty} k_n .$$

Then $\sum_{n=1}^{\infty} \| l_n \| \leq \sum_{n=1}^{\infty} k_n < \infty$
and

$$\| l - \sum_{n=1}^{N} l_n \| = \| m_N \| \leq \| \sum_{n=N+1}^{\infty} k_n \| \to 0$$

so $l = \sum_{n=1}^{\infty} l_n$.

Finally as $0 \leq l_n \leq k_n \in I$ so $l_n \in I$. This both proves that $\bar{I}$ is an ideal and on putting $l = f$ gives us the formula of the lemma.

Proof of theorem. For $0 \leq e \in V_2$ to be a topological order unit it is clearly necessary for V_1 to be dense in V_2. Conversely suppose this is the case. We prove a strengthened form of condition (i) on e. Let $y \in V_2$, $x \in V_1$, $y \geq x, 0$ and $\varepsilon > 0$. By the lemma there exists $\omega \in V_1$ with $0 \leq \omega \leq y$ and $\| \omega - y \| < \varepsilon$. By a simple use of the Riesz decomposition property we can now find $z \in V_1$ with $x, \omega \leq z \leq y$. Then $0, x \leq z \leq y$ and $\| z - y \| < \varepsilon$. A strengthened form of condition (ii) on e is immediate from the Riesz decomposition property.

Let V be a separable R-space. Then if e_n is a countable dense set in V^+ then

$$e = \alpha \sum_{n=1}^{\infty} e_n / 2^n \| e_n \|$$

is a topological order unit for some $\alpha > 0$, using the previous criterion.

If e is a topological order unit in the R-space V_2 then V_1 is a simplex space and the base B of V_1^{*+} is a Choquet simplex. See [5], [9], [11]. The facial structure of Choquet simplexes is described in [11]. If I is a closed ideal in V_2 then $I \cap V_1$ is a closed ideal in V_1 and so corresponds to a face $j''I$ of B. In fact $I = \overline{I \cap V_1}$ by the lemma so we see that j'' is a one-one map. To prove that the set of closed faces $j''I$ which arise in this way is a sublattice of the lattice of all faces in B it is sufficient to prove that if I, J are closed ideals in V_2 then

$$(I \cap J) \cap V_1 = (I \cap V_1) \cap (J \cap V_1)$$

and

$$(I + J) \cap V_1 = (I \cap V_1) + (J \cap V_1)$$

since it is shown in [6] that the sum and intersection of two closed ideals in an R-space are closed ideals. The first equation is trivial and it is also obvious that the right-hand side of the second equation is contained in the left-hand side, both sides representing ideals in V_2. Now let $0 \leq f \in (I + J) \cap V_1$. As in Theorem 5.3 of [6] we see that we can write $f = g + h$ where $0 \leq g \in I$ and $0 \leq h \in J$. As V_1 is an ideal so $g \in I \cap V_1$ and $h \in J \cap V_1$. As $(I + J) \cap V_1$ is an ideal it is positively generated and this concludes the proof of the second equation.

For $0 \leq f \in V_1$ we have by the definition of $j''I$ that $f \in I$ if and only if $(jf)|(j''I) = 0$. Now using the lemma and the definition of jf for $0 \leq f \in V_2$ it is clear that the formula of the theorem holds.

4. Banach lattices

For a Banach lattice V_2 it is more natural to present this theory in a rather different form, although the situation is essentially the same as that of Theorem 8. An element $0 \leq e \in V_2$ with $\| e \| = 1$ is a topological order unit if and only if for all $0 \leq f \in V_2$, $\lim_{n\to\infty} f \wedge ne = f$. The ideal V_1 generated by e is a Kakutani M-space [13] under the order unit norm and the Choquet simplex B has a closed boundary Ω and we can identify

$$V_1 \cong A(B) \cong C(\Omega)$$

by [1]. If $j : V_1 \to C(\Omega)$ is this identification then the representation j of V_2^+ of Theorems 7, 8 is essentially the same as the map j from V_2^+ to the cone of lower semi-continuous functions on Ω given by

$$j(f) = \sup_{n\to\infty} \{j(f \wedge ne)\}$$

and this map j is also one-one, linear, and preserves the lattice operations of V_2^+. Now for any $0 \leq f \in V_2$ we have

$$(jf) \wedge n = jf \wedge j(ne) = j(f \wedge ne) \in C(\Omega).$$

We can conclude that each function $jf : \Omega \to [0, \infty]$ is actually continuous.

Moreover as (jf, μ) is finite for a weak*-dense family of measures on Ω so jf is finite on an open dense subset of Ω.

If Ω is any compact Hausdorff space then the set $\tilde{C}(\Omega)$ of all continuous functions $f : \Omega \to [-\infty, \infty]$ which are finite on an open dense set is a lattice but not generally a vector space. $\tilde{C}(\Omega)$ is a sublattice of the vector lattice $D(\Omega)$ of all finite continuous functions defined on open dense sets with the obvious operations and identification of two functions which are almost everywhere equal. By a *vector sublattice of* $\tilde{C}(\Omega)$ we shall mean a vector sublattice of $D(\Omega)$ each element of which is in $\tilde{C}(\Omega)$. By an *ideal* in $\tilde{C}(\Omega)$ we shall mean a vector sublattice L of $\tilde{C}(\Omega)$ such that if $0 \leq f \leq g \in L$ and $f \in \tilde{C}(\Omega)$ then $f \in L$.

THEOREM 10. *If V is a Banach lattice with a topological order unit, for example a separable Banach lattice, then each topological order unit e defines a compact Hausdorff space Ω and a faithful representation j of V as an ideal in $\tilde{C}(\Omega)$. The space Ω is independent of the unit e up to homeomorphism and there is a one-one correspondence between the closed ideals of V and a sublattice of the set of closed subsets of Ω. V^* may be identified with an ideal in $M(\Omega)$, the dual of $C(\Omega)$.*

Proof. We shall not prove those parts of this theorem which are obvious corollaries of previous theorems though in fact simple direct proofs for this special case can often be produced.

If $j : V^+ \to \tilde{C}(\Omega)$ is as defined above then we extend j to V by defining

$$j(f) = j(f \vee 0) - j(-f \vee 0)$$

and see quickly that this is a faithful representation of V as a vector sublattice of $\tilde{C}(\Omega)$. Now let $0 \leq f \leq jg$ where $f \in \tilde{C}(\Omega)$ and $0 \leq g \in V$. We have $g = \lim_{n\to\infty} (g \wedge ne)$ and so there is a sequence m_n of integers such that $m_{n+1} > m_n$ and

$$\sum_{n=0}^{\infty} \| g \wedge m_{n+1} e - g \wedge m_n e \| < \infty .$$

Now

$$\begin{aligned}(jg) \wedge m_{n+1} - (jg) \wedge m_n &= \{(jg) \wedge m_{n+1}\} \vee m_n - m_n \\ &\geq (f \wedge m_{n+1}) \vee m_n - m_n \\ &= (f \wedge m_{n+1}) - (f \wedge m_n) \\ &\geq 0.\end{aligned}$$

As
$$(f \wedge m_{n+1}) - (f \wedge m_n) \in C(\Omega)$$

so we can find a unique $h_n \in V$ with

$$jh_n = (f \wedge m_{n+1}) - (f \wedge m_n)$$

and this h_n satisfies

$$0 \leq h_n \leq (g \wedge m_{n+1} e) - (g \wedge m_n e).$$

Therefore $\sum_{n=0}^{\infty} \| h_n \| < \infty$ and the sum

$$\sum_{n=0}^{\infty} h_n = h \in V$$

converges.

Now

$$\begin{aligned} jh \wedge m &= j(h \wedge me) \\ &= j \lim_{N\to\infty} \{me \wedge \textstyle\sum_{n=0}^{N} h_n\} \\ &= j \lim_{N\to\infty} j^{-1}\{m \wedge (f \wedge m_{N+1})\} \\ &= j \lim_{N\to\infty} j^{-1}(f \wedge m) \\ &= f \wedge m. \end{aligned}$$

Therefore $jh = f$ and we have shown that jV is an ideal in $\tilde{C}(\Omega)$.

As in Theorem 8 we see that there is a one-one correspondence between the set of closed ideals in V and a certain sublattice of the set of closed ideals in $C(\Omega)$. As the closed ideals in $C(\Omega)$ correspond exactly to the closed subsets of Ω so we get a natural one-one correspondence j'' between the set of closed ideals in V and a sublattice of the set of closed subsets in Ω, as in Theorem 8. Let $K \subseteq \Omega$ be a closed regular set, that is a closed subset of Ω with $K = \overline{\text{int } K}$. Let $I \subseteq V$ be the closed ideal given by

$$I = \{f \in V : |f| \wedge |g| = 0 \quad \text{for all} \quad g \in V \quad \text{such that} \quad \text{supp}(jg) \subseteq \text{int } K\}.$$

Then we can show that $j''I = K$, so that the family of sets $j''I$ where I are closed ideals in V, contains all regular closed sets.

We can now identify the points of Ω with the maximal increasing filtering families of proper closed ideals of V. We say a set $K \subseteq \Omega$ is in $\mathcal{K}$ if it consists of all the maximal filtering families containing a particular closed ideal of V. Then the family $\mathcal{K}$ forms a base for the closed sets of the topology of Ω, so that Ω is indeed independent of the unit $e \in V$.

Finally V^* can be identified with an ideal in $M(\Omega)$ as in Corollary 4 and Theorem 7. This concludes the proof.

We now indicate how this representation is related to that of Bernau, [3], obtained under more general conditions by purely algebraic methods. It is easy to show that his polar subspaces are precisely those closed ideals I such that $j''I$ are closed regular subsets of Ω. The space Bernau constructs is the Stone space $\tilde{\Omega}$ of the complete Boolean algebra of regular closed subsets of Ω, [12], and there is a natural map $\lambda : \tilde{\Omega} \to \Omega$. Bernau's representation is obtained by lifting our representation from Ω to $\tilde{\Omega}$. If V is order-complete then λ is a homeomorphism and the representations coincide.

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