ON THE LATTICE $D(X)$

BY
PHILIP NANZETTA[1]

The lattice, $D(X)$, of continuous extended (real-valued) functions on a compact space X is used in virtually every representation theorem for "nice" ordered algebraic systems (see [P_1] and [P_2] for groups, [JK] for linear spaces, and [HJ] for algebras). In this note we ignore the (partially defined) algebraic operations and concentrate on the lattice structure of $D(X)$. Specifically, we answer the question "when does $D(X)$ characterize X?" and give a (partially satisfying) answer to the question "when is the Dedekind completion of $D(X)$ isomorphic to $D(Y)$ for some space Y?"

For the first question, we show that if X is compact, then X may be constructed as the Isbell structure space of $D(X)$ (see [IM]). It is evident that, for noncompact (completely regular) X, $D(X)$ is isomorphic to $D(\beta X)$—where βX is the Stone-Čech compactification of X (see [GJ, Chapt. 6]).

For the second question, we show that the Dedekind completion of $D(X)$ is isomorphic to $D(Y)$ for some Y iff it is isomorphic to $D(X_\infty)$—where X_∞ is the minimal projective extension of X [G]—and that the Dedekind completion of $D(X)$ is isomorphic in a canonical fashion to $D(X_\infty)$ iff X is "z-thin".

The author wishes to thank T. Memp for a conscientious reading of this note.

Throughout this note, *all given spaces are assumed to be completely regular Hausdorff spaces.*

1. $D(X)$ and its prime ideals

Let X be a compact space; then $D(X)$ is the set of all continuous functions on X to the two-point compactification, $\gamma\mathbf{R}$ of $\mathbf{R}$ which are real-valued on a dense subset of X (the dense set depending on the function). For $f \in D(X)$, $\mathcal{R}(f)$ denotes the subset of X on which f is real-valued, and $\mathfrak{N}(f)$ is its complement. By defining order pointwise, $D(X)$ becomes a distributive lattice.

For $r \in \mathbf{R}$, we will denote by $\mathbf{r}$ the constant function whose value is r. For $f : A \to B$ and $C \subseteq A$, $f[C]$ denotes the set $\{f(c) : c \in C\}$. For $f \in D(X)$, $Z(f)$ denotes the set of zeros of f, and $Z(f)$ is referred to as a *zero-set* of $D(X)$.

A *prime ideal* of $D(X)$ is a nonempty proper sublattice of $D(X)$ which contains an infimum $f \wedge g$ iff it contains either f or g.

The theorem of this section shows that the set of prime ideals of $D(X)$ is composed of fibers, each fiber lying above a unique point of the space X.

For $x \in X$, $a \in \gamma\mathbf{R}$, let

$$J(x, a) = \{f \in D(X) : f(x) < a\} \quad \text{and} \quad I(x, a) = \{f \in D(X) : f(x) \leq a\}.$$

Received June 5, 1967.

[1] Supported by a National Science Foundation grant.

It is clear that $J(x, a)$ and $I(x, a)$ are prime ideals whenever they are nonempty proper subsets of $D(X)$.

THEOREM. *Let X be a compact space and P a prime ideal of $D(X)$. Then for unique $x \in X$, $a \in \gamma\mathbf{R}$,*

$$J(x, a) \subseteq P \subseteq I(x, a).$$

Proof. Suppose P contains no $J(x, a)$ for $a \neq -\infty$. Then, for $n \in N$ and each $x \in X$, there is $f \in D(X) \sim P$ with $f(x) < -n$. By compactness of X, there are $f_1, \cdots, f_m$ not in P such that $\inf \{f_i : i \leq m\} < -\mathbf{n}$. Hence no element of P is bounded below by $-\mathbf{n}$. Since this can be done for each $n \in N$, every function in P takes the value $-\infty$. For $g \in D(X)$, let

$$\mathfrak{N}^-(g) = \{x \in X : g(x) = -\infty\}.$$

Since $\bigcap \{\mathfrak{N}^-(g_i) : i \leq n\} = \mathfrak{N}^-(\sup \{g_i : i \leq n\}), \{\mathfrak{N}^-(g) : g \in P\}$ is a family of closed subsets of a compact space with the finite intersection property. Hence there is an x in the intersection. In this case,

$$(\emptyset =) J(x, -\infty) \subseteq P \subseteq I(x, -\infty).$$

Suppose P contains some $J(x, a)$, $a \neq -\infty$. Let

$$b = \sup \{a : J(x, a) \subseteq P\}.$$

Clearly $J(x, b) \subseteq P$. If $b = +\infty$, then $P \subseteq I(x, b) (= D(X))$, so suppose $b \in \mathbf{R}$. We will assume $P \nsubseteq I(x, b)$ and deduce a contradiction. Let $f \in P$ with $f(x) = c > b$; we can assume $c < +\infty$ since $f \in P$ implies $f \wedge (\mathbf{b} + \mathbf{1}) \in P$. Let $g \in D(X)$ with $g(x) = \frac{1}{2}(b + c)$. Let U be an open neighborhood of x whose closure is contained in

$$\{y : g(y) < f(y)\} \cap \mathfrak{R}(g).$$

Let $h \in D(X)$ with $h(x) = b - 1$ and $h|_{X \sim U} = g|_{X \sim U}$. Then $h \in J(x, b) \subseteq P$ and $h \vee f \geq g$. Hence $P \supseteq J(x, \frac{1}{2}(b + c))$, contradicting the definition of b. We have shown that, in any case, $J(x, b) \subseteq P \subseteq I(x, b)$ for some x and some b.

Let $x \neq y$. If $J(x, a) \subseteq I(y, b)$, then either $a = -\infty$ or $b = +\infty$; for if $a \neq -\infty$, $b \neq +\infty$, there is $f \in C(X)$ with $f(x) = a - 1$ and $f(y) = b + 1$. Hence, if

$$J(x, a) \subseteq P \subseteq I(x, a) \quad \text{and} \quad J(y, b) \subseteq P \subseteq I(y, b),$$

then either

$$a = b = -\infty \qquad (\text{and} \quad P \subseteq I(x, -\infty) \cap I(y, -\infty))$$

or

$$a = b = +\infty \qquad (\text{and} \quad J(x, +\infty) \cup J(y, +\infty) \subseteq P).$$

Now, suppose $P \subseteq I(x, -\infty) \cap I(y, -\infty)$. Let $f \in P$ (since $f \wedge \mathbf{0} \in P$, we can suppose $f \leq \mathbf{0}$). Let U, V be open neighborhoods of x, y, resp. with disjoint closures. Let $g \in D(X)$ be defined as follows: $g'(x) = 0$, $g'(z) = f(z)$

for $z \in$ bdry U—extend g' over cl U ($\{x\} \cup$ bdry U is compact, hence C^*-embedded [GJ, 1.17]) so that $g' \leq 0$, and let $g' |_{X \sim U} = f |_{X \sim U}$; g' is continuous; let $g = g' \vee f$. Then $\mathfrak{N}(g) = \mathfrak{N}^-(g) \subseteq \mathfrak{N}^-(f)$, so $g \in D(X)$, $g(x) = 0$, and $g |_{X \sim U} = f |_{X \sim U}$. Let $h \in D(X)$ with $h(y) = 0$ and $h |_{X \sim V} = f |_{X \sim V}$. Then $g \wedge h \leq f$, so $g \wedge h \in P$—but $g \notin I(x, -\infty)$, hence $g \notin P$; and $h \notin I(y, -\infty)$, hence $h \notin P$. This contradicts primeness of P. Thus, in this case, $x = y$. For the last case, suppose

$$J(x, +\infty) \cup J(y, +\infty) \subseteq P.$$

Let $f \notin P$. As above, let $g, h \in D(X)$ with $g \vee h \geq f$, $g(x) = 0$, $h(y) = 0$; then $g \in J(x, +\infty)$, $h \in J(y, +\infty)$, so $g \vee h \in P$—a contradiction. This concludes the proof of uniqueness.

It should be remarked here that there usually are prime ideals strictly between $J(x, a)$ and $I(x, a)$, and that these need not even form a chain.

Example. (This is an easy modification of an example of Kaplansky [K, 3].) Let M (resp. L) be the set of functions f in $D([-1, 1])$ for which $f(x) \leq -|x|$ for all $x \geq 0$ (resp., for all $x \leq 0$) in some neighborhood of 0. Then M and L are ideals. Let M' (resp. L') be the set of all f in $D([-1, 1])$ for which $f(x) \geq |x|$ for all $x \geq 0$ (resp., for all $x \leq 0$) in some neighborhood of 0. Then M' and L' are dual ideals of $D([-1, 1])$. By [S_1, Theorem 6], M and L are contained in prime ideals P, Q, disjoint from M' and L', resp. Since M meets L' and M' meets L, P and Q are not comparable; clearly both lie between $J(0, 0)$ and $I(0, 0)$.

Theorem 1 leaves completely untouched the problem of describing the sets of prime ideals between $J(x, a)$ and $I(x, a)$. This question will have to be examined before there is much hope of solving the problem of recognizing $D(X)$: given a lattice L, when is L isomorphic to $D(X)$ for some space X?

2. Recovery of X from $D(X)$

Let L be an arbitrary distributive lattice. We repeat Isbell's definition of the structure space $\kappa(L)$ of L [IM]. Define the relation k on the set of prime ideals of L to be the smallest equivalence relation containing "$\subseteq$". In the case of $D(X)$, the k-classes are just the fibers over points. The k-class of a prime ideal P will be denoted $[P]$. Topologize the set $\kappa(L)$ of k-classes of prime ideals of L as follows: a class c is an *immediate limit point* of a set $H \subseteq \kappa(L)$ if the members h_α of H have representatives $P_{\alpha\beta}$ whose kernel $\bigcap P_{\alpha\beta}$ is nonempty and contained in some representative P of c. Then c is a *limit point* of H if it is an immediate limit point of some subset of H. For a proof that this defines a topology, see [IM].

Note that if f belongs to a prime ideal P of $D(X)$ and $g \leq f$ on a neighborhood U of x, where $J(x, a) \subseteq P \subseteq I(x, a)$, then $g \in P$. For, if $a \geq 0$, let $h \in D(X)$ with $h(x) < a$ and $h |_{X \sim U} = g |_{X \sim U}$; then $h \in J(x, a) \subseteq P$ and

$h \vee f \geq g$; if $a \leq 0$, let $h \in D(X)$ with $h(x) > a$, $h|_{X\sim U} = f|_{X\sim U}$; then $h \notin I(x, a) \supseteq P$ and $g \wedge h \leq f$, so $g \in P$.

THEOREM. *The structure space $\kappa(D(X))$ is homeomorphic to X if X is compact.*

Proof. Define $h: X \to \kappa\ (D(X))$ by $h\ (x) = [I(x, 0)]$. From Theorem 1, it is clear that h is a bijection. We prove that

$$\text{for } x \in X \text{ and } S \subseteq X, \quad x \in \text{cl}\, S \text{ iff } h(x) \in \text{cl}\, h[S].$$

If $x \in \text{cl}\, S$, then $[I(x, 0)] \in \text{cl}\, \{[I(y, 0)] : y \in S\}$ follows from the statement $f(y) \leq 0$ for all $y \in S$ implies $f(x) \leq 0$.

Suppose $x \notin \text{cl}\, S$. Let V be a closed neighborhood of x disjoint from cl S and let W be a closed neighborhood of x contained in int V. Let $\{P_{\alpha\beta}\}$ be a set of representatives of elements of $h[S]$ and let $f \in \bigcap P_{\alpha\beta}$. Let Q be a representative of $[I(x, 0)]$, and let $g \notin Q$. Finally, let k' be a continuous function on X to $\gamma\mathbf{R}$ such that $k'|_{X\sim V} = f|_{X\sim V}$ and $k'|_W = g|_W$. Let $k = (k' \vee f) \wedge g$. Then $k \in D(X)$ and $k \leq f$ on a neighborhood of each point of S, so $k \in \bigcap P_{\alpha\beta}$; also, $k \geq g$ on a neighborhood of x, so $k \notin Q$. Hence $\bigcap P_{\alpha\beta} \not\subseteq Q$. Thus $h(x) \notin \text{cl}\, h[S]$.

As remarked above, if X is not compact, then $D(X)$ and $D(\beta X)$ are isomorphic, so in general, $\kappa(D(X))$ is homeomorphic to βX.

In view of this theorem it is (in some sense) possible to recover the lattice-ordered algebra $C(X)$ from the lattice structure of $D(X)$ alone.

3. The Dedekind completion of $D(X)$

Let L be a lattice and M be a sublattice. Then M is *order dense* in L iff for each $e \in L$,

$$e = \sup\{m \in M : m \leq e\} = \inf\{m \in M : m \geq e\}.$$

The lattice L is *Dedekind complete* iff every bounded subset has a sepremum and an infimum. Finally, an order isomorphism $\varphi : L \to P$ between lattices is *complete* iff whenever a supremum or infinum exists in L it is preserved by φ.

If L is a lattice, a pair (P, φ) is a *Dedekind completion* of L iff φ is a complete isomorphism of L onto an order dense subset of the Dedekind complete lattice P. If a lattice L has a Dedekind completion, it is determined up to an isomorphism "leaving L pointwise fixed".

In investigating the above properties in the lattice $D(X)$, it is frequently enough to check only half of the condition, since $f \to -f$ is an order automorphism of $D(X)$.

For every compact space X there exists [G] a compact *extremally disconnected* space X_∞ (i.e., every open subset of X_∞ has open closure) and a continuous map τ of X_∞ onto X which is *tight* (i.e., τ maps no proper closed subset onto— equivalently, every nonempty open subset of X_∞ contains the preimage of a nonempty open subset of X). The pair (X_∞, τ) is called the *minimal pro-*

jective extension of X, and it is characterized up to a homeomorphism "respecting τ" by the above properties.

For any X, the map $\tau : X_\infty \to X$ induces a lattice isomorphism τ^* of $D(X)$ into $D(X_\infty)$ by sending $f \in D(X)$ to $f \circ \tau \in D(X_\infty)$ (since τ is tight, $f \circ \tau$ is real-valued on a dense subset of X_∞).

THEOREM. *For compact* X, $(D(X_\infty), \tau^*)$ *is the Dedekind completion of* $D(X)$ *iff* $\tau^*[D(X)]$ *is order dense in* $D(X_\infty)$.

Proof. Let f'' be an upper bound in $D(X_\infty)$ of $F \subseteq D(X_\infty)$, and let $f' \in F$. Let $Y = \mathcal{R}(f'') \cap \mathcal{R}(f')$. Since Y is dense in the extremally disconnected space X_∞, Y is extremally disconnected [GJ, 1H]. By [S_2, 12], $C(Y)$ is a Dedekind complete lattice. Let $g \in C(Y)$ be the supremum of $\{f|_Y : f' \leq f \in F\}$. Since Y is C^*-embedded in X_∞[GJ, 1H], g has a continuous extension, h, over X_∞; h is the supremum of F. Hence $D(X_\infty)$ is Dedekind complete.

It remains only to show that τ^* is a complete isomorphism (this argument is patterned after a proof of E. C. Weinberg for $C(X)$). Let g be an upper bound in $D(X)$ of $F \subseteq D(X)$ and suppose $\tau^*(g)$ is not the supremum of $\tau^*[F]$. Then there exists $r > 0$ such that

$$\{x \in X_\infty : \tau^*(f)(x) + r < \tau^*(g)(x) \quad \text{for all} \quad f \in F\}$$

has nonempty interior. Since τ is tight, there is a nonempty open subset U of X such that $\tau^*(f)(x) + r < \tau^*(g)(x)$ for all $f \in F$ whenever $\tau(x) \in U$. Hence $f(y) + r < g(y)$ for all $f \in F$, $y \in U$, so g is not the supremum of F. Hence τ^* is complete.

4. Necessary and sufficient conditions

A function $f : X \to \gamma\mathbf{R}$ is *lower semi-continuous* (lcs) iff for each $\lambda \in \gamma\mathbf{R}$, the set, $\{x \in X : f(x) > \lambda\}$ is open.

Following Dilworth [*D*], if $f : X \to \gamma\mathbf{R}$ is any function, we define

$$f^*(x) = \inf\{\sup\{f(y) : y \in U\} : U \text{ a neighborhood of } x\}$$

and

$$f_*(x) = \sup\{\inf\{f(y) : y \in U\} : U \text{ a neighborhood of } x\}.$$

Dilworth proves the following statements for bounded functions f—the proofs can easily be modified to apply to unbounded functions.

(1) f is lsc iff $f = f_*$.
(2) $f^* \geq f \geq f_*$, and $f^*{}_*{}^*{}_* = f^*{}_*$

A lsc function f is *normal* lsc iff for each $\lambda \in \gamma\mathbf{R}$, $\{x \in X : f(x) < \lambda\}$ is a union of regular closed sets.

(3) If $f = f^*{}_*$, then f is normal lsc.
(4) If f is normal lsc, then for each $x \in X$, $d > f(x)$, and neighborhood U of x, there exists a nonempty open set $V \subseteq U$ such that $f(y) < d$ for all $y \in V$.

(5) Since for extremally disconnected X, the closure of an open set is open, a normal lsc function on an extremally disconnected space is continuous.

A space X is *z-thin* iff whenever S is a nowhere dense subset of X which can be written $S = \bigcap_{n \in N} U_n$ with $(U_n)_{n \in N}$ a decreasing sequence of closed sets, each of which is a union of regular closed sets, then S lies in a nowhere dense zero-set of X.[2] A slightly less obscure definition is the following: X is z-thin iff every nowhere dense "minus-infinity-set" of a normal lsc function on X lies in a nowhere dense zero-set of X.

THEOREM. *Let X be a compact space and let (X_∞, τ) be the minimal projective extension of X. The following are equivalent:*

(a) *$(D(X_\infty), \tau^*)$ is the Dedekind completion of $D(X)$.*

(b) *For $f \in D(X_\infty)$ there exists $g \in D(X)$ with $\tau^*(g) \leq f$.*

(c) *Every normal lsc function f on X with $\mathfrak{N}^-(f)$ nowhere dense is bounded below by an element g of $D(X)$.*

(d) *X is z-thin.*

Proof. (a) implies (b) is clear.

(b) implies (c). Let f be normal lsc on X with $\mathfrak{N}^-(f)$ nowhere dense. We suppose $f \leq \mathbf{0}$. Then $h = (f \circ \tau)^*{}_*$ is normal lsc on an extremally disconnected space, hence (by (5)) h is continuous; since $h \geq (f \circ \tau)_* = f \circ \tau$, $h \in D(X_\infty)$. Let $g \in D(X)$ with $g \circ \tau \leq h$. We will show $g \leq f$. Let $x \in X$ and suppose $h(y) > f(x)$ for all $y \in \tau^{\leftarrow}(x)$. Since $\tau^{\leftarrow}(x)$ is compact, there is a neighborhood U of $\tau^{\leftarrow}(x)$ on which $h(y) > d > f(x)$ for some $d \in \mathbf{R}$. By [Ke, 3.12], τ induces an upper semi-continuous decomposition, so we can assume $U = \tau^{\leftarrow}\tau[U]$; i.e., $\tau[U]$ is a neighborhood of x. Since f is normal lsc, there is a nonempty open subset V of $\tau[U]$ such that $f[V] \subseteq [-\infty, d)$. But then $f \circ \tau[\tau^{\leftarrow}(V)] \subseteq [-\infty, d)$, so $h[\tau^{\leftarrow}(V)] \subseteq [-\infty, d)$, a contradiction. Hence $h(y) \leq f(x)$ for some $y \in \tau^{\leftarrow}(x)$; thus $g \circ \tau(y) \leq h(y) \leq f(x)$, so $g(x) \leq f(x)$.

(c) implies (a). Let $f \in D(X_\infty)$. Define $f^\# : X \to \gamma\mathbf{R}$ by

$$f^\#(x) = \inf\{f(y) : \tau(y) = x\}.$$

Clearly $\mathfrak{N}^-(f^\#)$ is nowhere dense. For $\lambda \in \gamma\mathbf{R}$,

$$\{x \in X : f^\#(x) \leq \lambda\} = \{x \in X : f(y) \leq \lambda \text{ for some } y \in \tau^{\leftarrow}(x)\}$$
$$= \tau[\{x \in X_\infty : f(x) \leq \lambda\}],$$

which is closed in X; hence $f^\#$ is lsc. Let $\lambda \in \mathbf{R} \cup \{+\infty\}$, and let $x \in X$ with $f^\#(x) < \lambda$. Let $f^\#(x) < d < \lambda$, $W = \{y \in X_\infty : f(y) < d\}$. Let U be any neighborhood of x. Then, since $f^\#(x) < d$, $\tau^{\leftarrow}[U] \cap W \neq \emptyset$. Since τ is tight, there is a nonempty open set V_U for which

$$\tau^{\leftarrow}(V_U) \subseteq W \cap \tau^{\leftarrow}[U].$$

[2] The condition of z-thinness will be explored fully in a forthcoming paper of J. E. Mack entitled *The Dedekind completion of* $D(X)$.

If $z \in V_U$, then $\tau^{\leftarrow}(z) \subseteq W$, so $f^{\#}(z) < d$. Hence

$$V_U \subseteq \{z \in X : f^{\#}(z) \leq d\},$$

and the latter is closed. Then clearly

$$x \in \text{cl} \bigcup \{V_U : U \text{ a neighborhood of } x\} \subseteq \{z \in X : f^{\#}(z) < \lambda\}.$$

This shows that $f^{\#}$ is normal lsc.

Let $h \in D(X)$ such that $h \leq f^{\#}$. Then, clearly, $\tau^*(h) \leq f$. Let

$$f' = \sup \{\tau^*(g) : \tau^*(g) \leq f, g \in D(X)\},$$

and suppose $f' < f$. Then there is $r \in \mathbf{R}$ such that

$$U = \{x \in X_\infty : f'(x) < r < f(x)\}$$

is nonempty. Let V be an open subset of X such that

$$\text{cl}\, V \subseteq \mathcal{R}(h) \quad \text{and} \quad \emptyset \neq \tau^{\leftarrow}(V) \subseteq U.$$

Let $a \in V$; let $s \in C(X)$ with $s(a) = 1$, $s[X \sim V] = \{0\}$, and $\mathbf{0} \leq s \leq \mathbf{1}$. Since s vanishes on the neighborhood $X \sim \text{cl}\, V$ of $\mathfrak{N}(h)$, $h' = rs + (1 - s)h$ can be defined as an element of $D(X)$. Now, $h'(a) = r$, $h'(x) \leq r$ for all $x \in V$, and $h' \leq h$ on $X \sim V$. Hence $\tau^*(h') \leq f$ and $\tau^*(h')(x) = r > f'(x)$ for $x \in \tau^{\leftarrow}(a)$, so $\tau^*(h') \not\leq f'$, contradicting the assumption. Hence $f = f'$ and $\tau^*[D(X)]$ is order dense in $D(X_\infty)$.

(c) implies (d). Let S be a nowhere dense subset of X with $S = \bigcap_{n \in N} U_n$ where $(U_n)_{n \in N}$ is a decreasing sequence of closed sets each of which is a union of regular closed sets. Let $U_0 = X$. Define $f : X \to \gamma\mathbf{R}$ by

$$f[U_n \sim U_{n+1}] = \{-n\}, \qquad f[S] = \{-\infty\}.$$

It is easy to check that f is normal lsc; using (c), $\mathfrak{N}^-(g)$ is the desired nowhere dense zero-set containing S.

(d) implies (c). Let f be a normal lsc function on X with $\mathfrak{N}^-(f)$ nowhere dense.

Let Z be a nowhere dense zero-set containing $\mathfrak{N}^-(f)$. By [MJ, 3.1, 3.2], there is an $h' \in C(X \sim Z)$ with $h' \leq (f|_{X \sim Z}) \wedge \mathbf{0}$. Let $k \in C(X)$ and $Z(k) = Z$ and $k \leq \mathbf{0}$. Define $h : X \to \gamma\mathbf{R}$ by $h(x) = -\infty$ for $x \in Z$, $h(x) = (1/k(x)) + h'(x)$ for $x \notin Z$. Since $h' \leq f|_{X \sim Z}$, we have $h \leq f$; h is real-valued on the dense set $X \sim Z$. For $x \in Z$ and $n \in N$, $\{y \in X : k(y) > -1/n\}$ is a neighborhood of x on which $h(y) < -n$. Hence $h \in D(X)$.

In view of the proof of this theorem,

$$A = \{f \in D(X_\infty) : \tau^*(g') \leq f \leq \tau^*(g''), \text{ for some } g', g'' \in D(X)\}$$

contains $\tau^*[D(X)]$ as an order dense sublattice; A is clearly Dedekind complete. Hence every $D(X)$ has a Dedekind completion contained in $D(X_\infty)$.

5. A characterization

If $(D(Y), \varphi)$ is a Dedekind completion of $D(X)$, then $(D(Y), \varphi)$ is said to be *regular* if there exists an automorphism θ of $D(Y)$ such that $\theta\varphi$ takes bounded functions to bounded functions.

THEOREM. *Let X, Y be compact spaces, and suppose $(D(Y), \varphi)$ is the Dedekind completion of $D(X)$. Then*

(a) *there exists $\tau : Y \to X$ such that (Y, τ) is the minimal projective extension of X, and*

(b) *if $(D(Y), \varphi)$ is regular, then τ can be chosen so that $\tau^* = \theta\varphi$ for some automorphism θ of $D(Y)$.*

PROOF. If $(D(Y), \varphi)$ is regular, replace φ by $\theta'\varphi$ where $\theta'\varphi$ takes bounded functions to bounded functions.

We will define $\tau' : \kappa(D(Y)) \to \kappa(D(X))$. Let

$$\tau'([I(y, 0)]) = [\varphi^{\leftarrow}(I(y, 0))];$$

it is clear that τ' is well defined; we will show that it is onto.

Let P be a prime ideal of $D(X)$. Let

$$Q' = \{f \in D(Y) : f \leq \varphi(g) \text{ for some } g \in P\}$$

and

$$Q'' = \{f \in D(Y) : f \geq \varphi(g) \text{ for some } g \notin P\}.$$

Then Q' is a nonempty proper ideal of $D(Y)$ and Q'' is a nonempty proper dual ideal of $D(Y)$. Hence, by [S_1, Theorem 6] Q' lies in a prime ideal Q of $D(Y)$ missing Q''. It is clear that $P = \varphi^{\leftarrow}(Q')$ and $D(X) \sim P = \varphi^{\leftarrow}(Q'')$; hence $P = \varphi^{\leftarrow}(Q)$; i.e., $\tau'([Q]) = [P]$, so τ' is onto.

Since $D(Y)$ is Dedekind complete, the sublattice $C(Y)$ is Dedekind complete; by [S_2, 12], Y is extremally disconnected.

Let $C \subseteq \kappa(D(X))$ be closed and suppose $[Q] \in \text{cl } \tau'^{\leftarrow}(C)$. Then there is a family $\{R_{\alpha\beta}\}$ of representatives of some of the elements of $\tau'^{\leftarrow}(C)$ such that $\bigcap R_{\alpha\beta}$ is nonempty and is contained in some representative Q' of $[Q]$. Now, $\bigcap \varphi^{\leftarrow}(R_{\alpha\beta}) = \varphi^{\leftarrow}(\bigcap R_{\alpha\beta})$ is nonempty and is contained in $\varphi^{\leftarrow}(Q')$, so

$$\tau'([Q]) = [\varphi^{\leftarrow}(Q)] = [\varphi^{\leftarrow}(Q')] \in \text{cl } C;$$

hence $[Q] \in \tau'^{\leftarrow}(C)$. Therefore τ' is continuous. Define $\tau : Y \to X$ via the homeomorphisms of Theorem 2.

Suppose C is a proper closed subset of Y for which $\tau[C] = X$. Let $f \in D(Y)$ such that $\varphi(\mathbf{0}) < f$ and $\varphi(\mathbf{0})|_U = f|_U$ for some proper open subset U of Y containing C. For each $x \in X$, let $y(x) \in \tau^{\leftarrow}(x) \cap C$. Let K_x be a prime ideal of $D(Y)$ for which

$$J(y(x), a) \subseteq K_x \subseteq I(y(x), a)$$

for some $a \in \gamma\mathbf{R}$, and $\varphi^{\leftarrow}(K_x) = I(x, 0)$. ($K_x$ can be generated as above from $K'_x = \{g \in D(Y) : g \leq \varphi(h)$ on some neighborhood of $y(x)$ for some

$h \in I(x, 0)\}$.) Suppose $h \in D(X)$ and $\varphi(h) \leq f$. Then $\varphi(h) \leq \varphi(\mathbf{0})$ on a neighborhood of each $y(x)$, $x \in X$, so $\varphi(h) \in K_x$ for all $x \in X$. This implies $h \in I(x, 0)$, so $h \leq \mathbf{0}$. Hence

$$f > \varphi(\mathbf{0}) = \sup\{\varphi(h) : h \in D(X), \varphi(h) \leq f\},$$

a contradiction. Thus τ is tight and (Y, τ) is the minimal projective extension of X. This completes the proof of (a).

Let $k \in D(Y)$ and let $\mathbf{0} \geq k' \in D(X)$ such that $\varphi(k') \leq k$. Suppose $x \in Y$ is such that $k(x) = -\infty$. Let $b \in \mathbf{R}$, and let $K(x, b)$ be a prime ideal of $D(Y)$ for which $\varphi^{\leftarrow}(K(x, b)) = I(\tau(x), b)$ and $J(x, c) \subseteq K(x, b) \subseteq I(x, c)$ for some $c \in \gamma\mathbf{R}$. Since $\varphi(\mathbf{b})$ is bounded, $c > -\infty$. Hence $k \in J(x, c)$, so $\varphi(k') \in J(x, c)$, so

$$k' \in \varphi^{\leftarrow}(J(x, c)) \subseteq I\ (\tau\ (x)\ , b).$$

Since this is true for all $b \in \mathbf{R}$, $k'(\tau(x)) = -\infty$.

Let $Z = \mathfrak{N}^{-}(k')$; by [MJ, 3], $X \sim Z$ is a weak cb space; as in the proof of Theorem 4, $k^{\#}$ is normal lsc; hence [MJ, 3], there is $\mathbf{0} \geq k'' \in C(X \sim Z)$ such that $k'' \leq k^{\#}|_{X \sim Z}$. Define $h : X \rightarrow \gamma\mathbf{R}$ by $h(x) = k''(x) + k'(x)$ for $x \notin Z$, $h[Z] = \{-\infty\}$. Clearly $h \in D(X)$ and $h \circ \tau \leq k$. Hence $\tau^{*}[D(X)]$ is order dense in $D(Y)$. By uniqueness of Dedekind completions, there exists an automorphism θ'' of $D(Y)$ such that $\tau^{*} = \theta''\varphi$.

6. Further remarks

The contents of Sections 4 and 5 can be summarized as follows: If $D(X)$ has any $D(Y)$ as a Dedekind completion, then it must be $D(X_{\infty})$; if $D(X)$ has any $(D(Y), \varphi)$ as a regular Dedekind completion, then (up to automorphism) it must be $(D(X_{\infty}), \tau^{*})$, and the latter can occur iff X is z-thin. I do not know whether every Dedekind completion which turns out to be a $D(Y)$ must be regular, but I strongly suspect the answer to be yes.

If X is a completely regular space for which the completion of $D(X)$ is $D(X_{\infty})$ (see [MJ] for the definition of X_{∞} for noncompact X), the same statement is true for βX: writing $D(X)^{\wedge}$ for the Dedekind completion of $D(X)$, we have $D(\beta X)^{\wedge} = D(X)^{\wedge} = D(X_{\infty}) = D(\beta(X_{\infty})) = D((\beta X)_{\infty})$—the last equality following from [HI, 1.5].

By arguments like those of Theorem 4, one can show that if X is, e.g., a metric space, then $D(X_{\infty})$ is the Dedekind completion of $D(X)$. Combining this with the previous paragraph, if Y is the Stone-Cech compactification of a metric space, then $D(Y_{\infty})$ is the Dedekind completion of $D(Y)$.

History

The question answered in Section 2 was motivated by [K], in which it was proved that, for compact X, the lattice $C(X)$ of all continuous real-valued functions on X characterizes X. Kaplansky's theorem has recently been extended by Subramanian [SU] and further improved by Isbell and Morse

[IM] in a direction slightly different from that taken here: for a rather general class of lattice-ordered rings, the maximal l-ideal space is characterized in terms of the lattice structure alone.

The question attacked in the remainder of this paper was motivaten by the following question for $C(X)$: is there a $C(Y)$ which is the Dedekind completion of $C(X)$ under a map which preserves the algebra structure of $C(X)$? An affirmative answer was given for compact X by Dilworth [D], for countable paracompact and normal spaces by Weinberg [W], and finally, necessary and sufficient conditions were given by J. E. Mack and D. G. Johnson [MJ] for realcompact X.

We have liberally used techniques of these authors.

It is worth mentioning here that the condition of regularity used in the study of the Dedekind completion of $D(X)$ is superfluous in the case of $C(X)$, since preservation of the algebra structure insures that the constant functions go to constant functions.

References

[D] R. P. Dilworth, *The normal completion of the lattice of continuous functions*, Trans. Amer. Math. Soc., vol. 68 (1950), pp. 427–438.

[G] A. M. Gleason, *Projective topological spaces*, Illinois J. Math., vol. 2 (1958), pp. 482–489.

[GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, D. van Nostrand, New York, 1960.

[HI] M. Henriksen and J. R. Isbell, *Some properties of compactifications*, Duke Math. J., vol. 25 (1958), pp. 83–105.

[HJ] M. Henriksen and D. G. Johnson, *On the structure of a class of archimedean lattice-ordered algebras*, Fund. Math., vol. 50 (1961), pp. 73–94.

[IM] J. R. Isbell and J. T. Morse, *Kaplansky spaces and l-groups*, to appear.

[JK] D. G. Johnson and J. E. Kist, *Prime ideals in vector lattices*, Canad. J. Math., vol. 14 (1962), pp. 517–528.

[K] I. Kaplansky, *Lattices of continuous functions*, Bull. Amer. Math. Soc., vol. 53 (1947), pp. 617–623.

[Ke] J. L. Kelley, *General topology*, D. van Nostrand, New York, 1955.

[MJ] J. E. Mack and D. G. Johnson, *The Dedekind completion of* $C(X)$, Pacific J. Math., vol. 20 (1967), pp. 231–243.

[P_1] D. Papert, *A representation theory for lattice-groups*, Proc. Lond. Math. Soc. (3), vol. 12 (1962), pp. 100–120.

[P_2] D. Papert, *A note on vector lattices and integration theory*, J. Lond. Math. Soc., vol. 38 (1963), pp. 477–485.

[S_1] M. H. Stone, *Topological representations of distributive lattices and Brouwerian logics*, Časopis Pěst. Mat., vol. 67 (1937), pp. 1–25.

[S_2] M. H. Stone, *Boundedness properties in function lattices*, Canad. J. Math., vol. 1 (1949), pp. 176–186.

[Su] H. Subramanian, *Kaplansky's theorem for f-rings*, to appear.

[W] E. C. Weinberg, *The alpha-completion of a ring of continuous real-valued functions*, Notices Amer. Math. Soc., vol. 7 (1960), pp. 533–534.

Case Western Reserve University
Cleveland, Ohio
University of Florida
Gainesville, Florida