# WALSH SERIES AND ADJUSTMENT OF FUNCTIONS ON SMALL SETS

BY

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## 1. Introduction

D. E. Menshov proved that a measurable function finite almost everywhere on $[0, 2\pi]$ can be changed on a set of measure less than $\varepsilon$ to a function whose Fourier series converges uniformly ([4]; see also [1, Chapter VI]). Recently, B. D. Kotlyar [3] proved an analogous theorem for Walsh series. Menshov proved also that for continuous functions the set where the adjustment is made can be chosen to depend only on $\varepsilon$ and the modulus of continuity.

In this paper, we present a different proof of Kotlyar's theorem that contains also an analogue of Menshov's theorem on continuous functions. Actually, our result contains somewhat more. Let $\{p_\nu\}$ and $\{q_\nu\}$ be increasing sequences of positive integers such that

$$(1) \qquad p_1 < q_1 < p_2 < q_2 < \cdots, \qquad \{q_\nu/p_\nu\} \quad \text{is unbounded.}$$

Define

$$(2) \qquad W = \bigcup_{\nu=1}^{\infty} [\psi_k : p_\nu \leqq k < q_\nu]$$

where $\psi_k$ is the $k$-th Walsh function. We shall prove that a measurable function can be changed on a small set to a function whose Walsh-Fourier series converges uniformly and contains only Walsh functions in $W$.

THEOREM. *Let $f$ be measurable and finite almost everywhere on* $[0, 1]$ *and let a positive $\varepsilon$ be given. Then there exists a function $g$ such that*

(a) *$g(x) = f(x)$ except on a set $E$ of measure less than $\varepsilon$,*

(b) *the Walsh-Fourier series of $g$ contains only Walsh functions in the set $W$ defined by* (1) *and* (2) *and converges uniformly.*

*Furthermore, suppose $\rho(\delta)$ is a nondecreasing function defined for $\delta > 0$ with*

$$\lim_{\delta\to 0} \rho(\delta) = 0.$$

*Then there is a set $E$ depending only on $\varepsilon$ and $\rho$ such that* (a) *and* (b) *hold for every continuous function $f$ whose modulus of continuity $\omega(\delta)$ satisfies*

$$\omega(\delta) \leqq \rho(\delta).$$

This theorem contains a previous result of the author [5].

COROLLARY. *The system of Walsh functions $W$ defined by* (1) *and* (2) *is total in measure on* $[0, 1]$.

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By the theorem, any bounded measurable function can be uniformly approximated by linear combinations of Walsh functions in $W$ except on sets of arbitrarily small measure. That means those linear combinations are dense in the sense of convergence in measure.

## 2. Definitions

Basic properties of Walsh functions. For a detailed treatment of Walsh functions, see the paper of N. J. Fine [2]. In this section, we review several of their properties.

For each $x$ in the interval $[0, 1)$, there is a dyadic expansion

$$x = \sum_{n=1}^{\infty} \frac{d_n(x)}{2^n}$$

where $d_n(x) = 0$ or $1$. The expansion is unique if the terminating form is chosen for dyadic rationals. Define

$$(3) \qquad \psi_0(x) \equiv 1, \qquad \psi_{2^n}(x) = \begin{cases} 1, & \text{if } d_{n+1}(x) = 0, \\ -1, & \text{if } d_{n+1}(x) = 1, \end{cases}$$

and if

$$N = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k} \quad \text{where} \quad 0 \leqq n_1 < n_2 < \cdots < n_k,$$

define

$$(4) \qquad \psi_N(x) = \psi_{2^{n_1}}(x)\psi_{2^{n_2}}(x) \cdots \psi_{2^{n_k}}(x).$$

These are the Walsh functions. They are a complete orthonormal set in $L^2[0, 1]$.

If $f$ is an integrable function, the $k$-th partial sum of its Walsh-Fourier expansion will be denoted by $s_k(x; f)$.

$V(n)$ will denote the set of linear combinations of the Walsh functions $\psi_i$ where $i < 2^n$. $V(n)$ consists of all step functions constant on each interval $I(n, j) = [j \cdot 2^{-n}, (j + 1)2^{-n})$. If $n < k$, $V(n, k)$ will denote the set of linear combinations of Walsh functions $\psi_i$ where $2^n \leqq i < 2^k$. $V(n, k)$ is the orthogonal complement of $V(n)$ in $V(k)$. If $f \in V(n, k)$, then

$$s_i(x; f) = \begin{cases} 0, & \text{if } i < 2^n, \\ f(x), & \text{if } i \geqq 2^k. \end{cases}$$

The dyadic interval $I(n, j)$ can be characterized as follows.

$$I(n, j) = [x : d_1(x) = d_1(\bar{x}), d_2(x) = d_2(\bar{x}), \cdots, d_n(x) = d_n(\bar{x})]$$

where $\bar{x}$ is any point of $I(n, j)$. From (3),

$$1 + \psi_{2^{i-1}}(\bar{x})\psi_{2^{i-1}}(x) = \begin{cases} 2, & \text{if } d_i(x) = d_i(\bar{x}), \\ 0, & \text{if } d_i(x) \neq d_i(\bar{x}). \end{cases}$$

Therefore,

$$\chi_{n,j}(x) = 2^{-n} \prod_{i=0}^{n-1} (1 + \psi_{2^i}(\bar{x})\psi_{2^i}(x))$$

where $\chi_{n,j}$ is the characteristic function of $I(n, j)$. Because of (4),

$$\chi_{n,j}(x) = 2^{-n} \sum_{i=0}^{2^n-1} \psi_i(\bar{x})\psi_i(x).$$

Since $| \psi_i(x) | = 1$ for every $x$ and all $i$.

$$\max_x | s_k(x; \chi_{n,j}) | \leqq 1, \qquad k \geqq 0.$$

## 3. Adjustment of characteristic functions

Lemma. *Let $\chi_{n,j}$ be the characteristic function of the interval*

$$I(n, j) = [j \cdot 2^{-n}, (j + 1)2^{-n}).$$

*Let $r$ and $N$ be positive integers, $N \geqq n$. Then there is a function $g$ with the following properties.*

(a) $g(x) \equiv 0$ *outside of* $I(n, j)$.

(b) $g(x) = \chi_{n,j}(x)$ *except on a set of measure* $2^{-n-r}$.

(c) $g \in V(N, N + r)$.

(d) $\max_x | g(x) | < 2^r$

(e) $\max_x | s_k(x; g) | < 2^r$ *for every* $k \geqq 0$.

*Proof.* Define

$$(5) \qquad E_{N,r} = \bigcup_{\nu=0}^{2^N-1} [\alpha_\nu , \beta_\nu); \quad \alpha_\nu = \nu \cdot 2^{-N}, \qquad \beta_\nu = \alpha_\nu + 2^{-N-r}$$

or equivalently,

$$E_{N,r} = [x : d_{N+1}(x) = d_{N+1}(x) = \cdots = d_{N+r}(x) = 0].$$

Set

$$\begin{aligned} g(x) &= 0, && \text{if} \quad x \notin I(n, j), \\ &= -(2^r - 1), && \text{if} \quad x \in I(n, j) \cap E_{N,r} , \\ &= 1, && \text{otherwise.} \end{aligned}$$

(a) and (d) hold by definition. $g(x)$ differs from $f(x)$ only on $I(n, j) \cap E_{N,r}$ , a set of measure $2^{-n-r}$ as can be seen from (5). Thus, (b) holds. Assertions (c) and (e) are verified directly from the Walsh series for $g$.

$$(6) \qquad g(x) = 2^{-n}\prod_{i=0}^{n-1} (1 + \psi_{2^i}(\bar{x})\psi_{2^i}(x))(1 - \prod_{i=N}^{N+r-1} (1 + \psi_{2^i}(x))).$$

This is so since

$$2^{-n} \prod_{i=0}^{n-1} (1 + \psi_{2^i}(\bar{x})\psi_{2^i}(x)) = \chi_{n,j}(x)$$

and for similar reasons,

$$\prod_{i=N}^{N+r-1} (1 + \psi_{2^i}(x)) = 2^r\varphi_{N,r}(x)$$

where $\varphi_{N,r}$ is the characteristic function of $E_{N,r}$ . The product (6) can be expanded with the use of (4). All terms $\psi_i(x)$, $i < 2^N$, cancel out and

$$g(x) = 2^{-n} \sum_{i=2^N}^{2^{N+r}-1} a_i \psi_i(x)$$

where $a_i = \pm 1$ or 0 and exactly $2^{n+r} - 2^n$ of these coefficients are nonzero. Consequently, $g \in V(N, N + r)$ and

$$\max_x | a_k(x; g) | \leqq 2^{-n} \sum | a_i | = 2^{-n}(2^{n+r} - 2^n) = 2^r - 1.$$

This completes the proof of the lemma.

Because the integer $N$ may be chosen arbitrarily large, the lemma enables us to adjust a sequence of characteristic functions, each time using a new block of Walsh functions.

## 4. Proof of the theorem

It will suffice to prove the theorem for continuous functions. A measurable function finite almost everywhere agrees with a continuous function except on some set of measure less than $\varepsilon/2$. That continuous function may then be modified on a set of measure less than $\varepsilon/2$.

We start with $\varepsilon > 0$, a Walsh subsystem $W$ defined by (1) and (2), and a function $\rho(\delta)$. Let $f$ be any continuous function whose modulus of continuity satisfies $\omega(\delta) \leqq \rho(\delta)$. $f$ can be expressed as the sum of a uniformly convergent series of step functions.

$$f(x) = \sum_{r=0}^{\infty} f_r(x). \tag{7}$$

For each $r$, $f_r$ can be taken to be a dyadic step function, i.e. constant on each interval $I(n_r, j)$ for some $n_r$. The sequence $\{n_r\}$ can be taken to increase so fast that the partial sums of the series (10) converge to $f(x)$ as rapidly as desired. Choose $\{n_r\}$ such that

$$\max_x | f_r(x) | < 2^{-2r}, \qquad r > 0. \tag{8}$$

Once $\rho(\delta)$ is given, $\{n_r\}$ can be selected independent of $f$.

Now arrange all the dyadic intervals $I(n_r, j)$ $r \geqq 0$, $0 \leqq j < 2^{n_r}$, into one sequence. Take first all intervals $I(n_0, j)$, then all intervals $I(n_1, j)$, etc. Define

$$\mu_r = \sum_{\nu=0}^{r} 2^{n_\nu}, \qquad r \geqq 0, \qquad \mu_{-1} = 0.$$

Then according to this enumeration,

$$I_i = I(n_r, k) \quad \text{if} \quad i = \mu_{r-1} + k < \mu_r .$$

Let $\chi_i$ be the characteristic function of $I_i$. Equation (7) may be written as follows:

$$f(x) = \sum_{i=0}^{\infty} a_i \chi_i(x) \tag{9}$$

where $\{a_i\}$ is a suitable sequence of constants. This series converges uniformly. Because of (8)

$$|a_i| < 2^{-2r} \quad \text{if} \quad \mu_{r-1} \leqq i < \mu_r . \tag{10}$$

We now apply the lemma to modify each of the characteristic functions $\chi_i$. Let $m$ be an integer such that

$$\sum_{r=m}^{\infty} 2^{-r} < \varepsilon.$$

Suppose $\mu_{r-1} \leqq i < \mu_r$. There is a function $g_i$ with the following properties.

(a) $g_i(x) \equiv 0$ outside of $I_i$.
(b) $g_i(x) = \chi_i(x)$ except on a set $I_i'$ of measure $2^{-n_r-r-m}$
(c) $g_i \in V(N_i, N_i + r + m)$ where $N_i > N_{i-1} + r + m$ and $g_i$ is a linear combination of Walsh functions in $W$.
(d) $\max_x | g_i(x) | < 2^{r+m}$.
(e) $\max_x | s_k(x; g_i) | < 2^{r+m}$.

By the lemma, there is a function $g_i$ satisfying (a), (b), (d), (e), and belonging to $V(N, N + r + m)$ where $N$ may be taken arbitrarily large. By property (1) of the system $W$, there is an index $\nu_i$ and an integer $N_i$ such that

$$P_{\nu_i} \leqq 2^{N_i} < 2^{N_i+r+m} < q_{\nu_i}, \qquad N_i > N_{i-1} + r + m.$$

Choose $N = N_i$ and (c) will be satisfied.

With (9) in mind we define

$$g(x) = \sum_{i=0}^{\infty} a_i\, g_i(x). \tag{11}$$

The series (11) converges uniformly; for a fixed $x$, $g_i(x) = 0$ for all indices $i$, $\mu_{r-1} \leqq i < \mu_r$, with one exception and for that index

$$| a_i\, g_i(x) | < 2^{-2r} \cdot 2^{r+m} = 2^{-r+m}$$

because of (10) and (d).

We assert that $g$ is the desired modification of $f$. First observe that $g(x) = f(x)$ except on the set

$$E = \bigcup_{i=0}^{\infty} I_i' .$$

Since $| I_i' | = 2^{-n_r-r-m}$ if $\mu_{r-1} \leqq i < \mu_r$,

$$| E | = \sum_{r=0}^{\infty} \sum_{i=\mu_{r-1}}^{\mu_r} | I_i' | = \sum_{r=0}^{\infty} 2^{n_r}(2^{-n_r-r-m})$$
$$= \sum_{r=0}^{\infty} 2^{-r-m} = \sum_{r=m}^{\infty} 2^{-r} < \varepsilon.$$

The set $E$ depends on $\varepsilon$, $W$, and $\rho(\delta)$, but not on $f$.

By (c), each function $g_i$ is a linear combination of a block of Walsh functions in $W$ (the blocks are disjoint). Therefore, the series (11) is easily converted into a Walsh series involving only functions in $W$. That Walsh series is the Walsh-Fourier series of $g$ since a subsequence of its partial sums converges uniformly to $g(x)$.

It remains to show that $s_k(x; g) \to g(x)$ uniformly. Because of uniform convergence in (11),

$$s_k(x; g) = \sum_{i=0}^{\infty} a_i\, s_k(x; g_i).$$

From (c),

$$\begin{aligned} s_k(x; g_i) &= 0, && \text{if } k < 2^{N_i}, \\ &= g_i(x), && \text{if } k \geqq 2^{N_{i+1}}. \end{aligned}$$

Now for some $j$, $2^{N_j} \leqq k < 2^{N_{j+1}}$. Hence

$$s_k(x; g) = \sum_{i=0}^{j-1} a_i\, g_i(x) + a_j\, s_k(x; g_j).$$

For some $r$, $\mu_{r-1} \leqq j < \mu_r$. Then from (10) and (e),

$$| a_j\, s_k(x; g_j) | < 2^{-2r} \cdot 2^{r+m} = 2^{-r+m}.$$

As $k \to \infty, j \to \infty$ and so,

$$\sum_{i=0}^{j-1} a_i\, g_i(x) \to \sum_{i=0}^{\infty} a_i\, g_i(x) = g(x)$$

uniformly. As $j \to \infty, r \to \infty$ and so

$$a_j\, s_k(x; g_j) \to 0$$

uniformly. Therefore $s_k(x; g) \to g(x)$ uniformly and the theorem is proved.

We remark that the theorem and corollary are true if we assume in (1) only that $\{q_\nu - p_\nu\}$ is unbounded. The proof, though simple, is somewhat tedious and will be omitted.

## References


1. N. Bary, *A treatise on trigonometric series*, Vol. I, Pergamon Press, New York, 1964.
2. N. J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc., vol. 65, (1949), pp. 372–414.
3. B. D. Kotlyar, *Walsh series and the theorem of Menshov on adjustment of functions*, Izvestia Akad. Nauk S.S.S.R., vol. 30(1966), pp. 1193–1200, (Russian).
4. D. E. Menshov, *Sur les series de Fourier des fonctions continues*, Mat. Sb., vol. 11(53), (1940), pp. 493–518.
5. J. J. Price, *A density theorem for Walsh functions*, Proc. Amer. Math. Soc., vol. 18 (1967), pp. 209–11.


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