A BORSUK-ULAM THEOREM FOR MAPS FROM A SPHERE TO A COMPACT TOPOLOGICAL MANIFOLD

BY

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1. Introduction, notation

It is the purpose of this paper to prove the following Borsuk-Ulam-type-theorem:

THEOREM 1. *Let* $f : S^n \to M^k$ *be a map from the n-sphere to a compact topological k-manifold* M^k; *let* $A(f) = \{x \in S^n; f(x) = f(-x)\}$. *Then*

(a) *if* $n > k$, *then* $\dim(A(f)) \geq n - k$;

(b) *if* $n = k$ *and* $f^* : H^n(M^n; Z_2) \to H^n(S^n; Z_2)$ *is zero, then* $A(f) \neq \emptyset$.

If one restricts to manifolds admitting a differentiable structure the theorem may be found in [1]; the restriction to the case $M^k = R^k$ is known as the Bourgin-Yang-theorem (see [5] and [6]); our line of reasoning is close to that of [1].

As for notation the following should be noted: All coefficient groups are Z_2; therefore, they shall be suppressed from the notation. H_* (H^*) denotes singular homology (cohomology), and $\bar{H}^*$ denotes Alexander-Spanier cohomology in the sense of Section 6.1 of [2] (see also Section 6.4 of [2]). By *dim* we understand the usual topological dimension. Finally *manifold* is taken to mean topological manifold, and the word *closed* (for a manifold) is an abbreviation for "compact and without boundary".

2. Reduction of the problem

Throughout this section and the next one M^k will be a closed, connected manifold of dimension $k \leq n$, and $f : S^n \to M^k$ will be a fixed map, taking the south-pole into x_0. On the manifold $Y = S^n \times M^k \times M^k$ there is an involution T given by the formula $T(x, y, z) = (-x, z, y)$; letting $\Delta(M^k)$ be the diagonal in $M^k \times M^k$ we have in Y two submanifolds $S^n \times (x_0, x_0)$ and $S^n \times \Delta(M^k)$; they are both invariant under T, so they project to give submanifolds

$$(S^n \times (x_0, x_0))/T = P^n \times (x_0, x_0) \quad \text{and} \quad (S^n \times \Delta(M^k))/T = P^n \times \Delta(M^k)$$

of the orbit manifold Y/T. —Also the map $\bar{s} : S^n \to Y$, given by

$$\bar{s}(x) = (x, f(x), f(-x)),$$

induces a map $s : P^n \to Y/T$; letting $A(f) = \{x \in S^n; f(x) = f(-x)\}$ and denoting by $B(f)$ the image of $A(f)$ under the natural map $S^n \to P^n$, we have

Received June 1, 1967.

that

$$B(f) = s^{-1}(P^n \times \Delta(M^k)).$$

Now let $\varphi \in H^k(Y/T)$ be the Poincaré-dual of the orientation class σ of the submanifold $P^n \times \Delta(M^k)$ of Y/T; we then have the following.

LEMMA 2.1. *If* $s^*(\varphi) \neq 0$, *then* $\bar{H}^{n-k}(B(f)) \neq 0$.

Proof. The following proof is just a rearrangement of the proof of [1, (33.2)]. —We first show

(2.1) for every neighbourhood U of $P^n \times \Delta(M^k)$ in Y/T we have

$$\varphi \in \operatorname{Im}\,(H^k(Y/T,\, Y/T - U) \to H^k(Y/T)).$$

To prove this assertion we let V be an open neighbourhood of $P^n \times \Delta(M^k)$ with $V \subseteq U$; we then read off (2.1) from the commutative diagram

$$\begin{array}{ccccccc} & \nearrow H_{n+k}(V) & \xrightarrow{\bar{\gamma}_U} & \bar{H}^k(Y/T, Y/T - V) & \xrightarrow{i} & H^k(Y/T, Y/T - V) & \searrow \\ H_{n+k}(P^n \times \Delta(M^k)) & \downarrow & & \downarrow & & \downarrow & H^k(Y/T, Y/T - U) \\ & \searrow H_{n+k}(Y/T) & \xrightarrow[\bar{\gamma}_U]{} & \bar{H}^k(Y/T) & \xrightarrow{i\,\cong} & H^k(Y/T) & \swarrow \end{array}$$

where $\bar{\gamma}_U$ denotes duality in the sense of [2, (6.2.17)], i is the natural transformation from $\bar{H}$ to H (see [2, p. 289]), and all the unlabelled maps are induced by appropriate inclusions.

Next we prove (c is the generator of $H^1(P^n)$)

(2.2) for every neighbourhood V of $B(f)$ in P^n we have

$$c^k \in \operatorname{Im}\,(H^k(P^n,\, P^n - V) \to H^k(P^n)).$$

Since for every neighbourhood V of $B(f)$ in P^n there is a neighbourhood U of $P^n \times \Delta(M^k)$ in Y/T with $s^{-1}(U) \subseteq V$, it is clearly sufficient to prove (2.2) with $V = s^{-1}(U)$, U a neighbourhood of $P^n \times \Delta(M^k)$ in Y/T; and in this case the assertion follows immediately from the commutative diagram

$$\begin{array}{ccc} H^k(Y/T,\, Y/T - U) & \to & H^k(Y/T) \\ \downarrow s^* & & \downarrow s^* \\ H^k(P^n,\, P^n - s^{-1}(U)) & \to & H^k(P^n) \end{array}$$

using (2.1) and the hypothesis that $s^*(\varphi) = c^k$.

Now, *assume that* $\bar{H}^{n-k}(B(f)) = 0$; then c^{n-k} maps to zero under the composition

$$H^{n-k}(P^n) \underset{\cong}{\xrightarrow{i^{-1}}} \bar{H}^{n-k}(P^n) \to \bar{H}^{n-k}(B(f));$$

therefore, by the definition of $\bar{H}$ there is an open neighbourhood U of $B(f)$

in P^n, such that c^{n-k} maps to zero under $H^{n-k}(P^n) \to H^{n-k}(U)$, i.e. we have

(2.3) $c^{n-k} \in \operatorname{Im}(H^{n-k}(P^n, U) \to H^{n-k}(P^n))$ for some open neighbourhood U of $B(f)$ in P^n.

Using (2.3) and (2.2) with V closed and $V \subseteq U$ we get that

$$c^n = c^k \cdot c^{n-k} \in \operatorname{Im}(H^n(P^n, U \cup (P^n - V)) \to H^n(P^n));$$

since $H^n(P^n, U \cup (P^n - V)) = H^n(P^n, P^n) = 0$ this gives the desired contradiction and Lemma 2.1 is proved.

This lemma reduces the proof of Theorem 1 to a consideration of $s^*(\varphi)$; however, there is a further reduction which is only implicitly contained in [1], but which we shall here need in an explicit form. It is stated in the next two lemmas.

LEMMA 2.2. *If $k < n$, and*

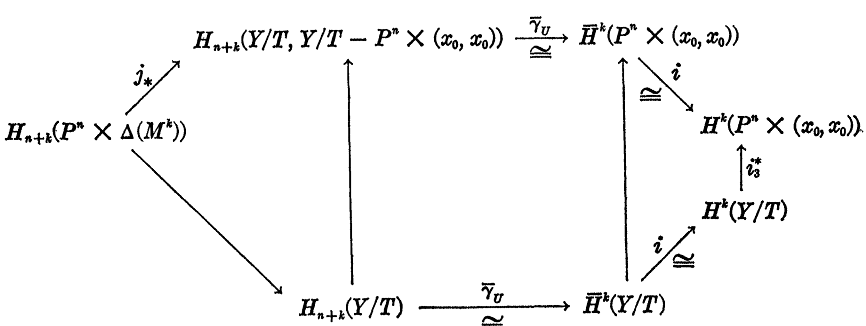
$$j_* : H_{n+k}(P^n \times \Delta(M^k)) \to H_{n+k}(Y/T, Y/T - P^n \times (x_0, x_0))$$

is non-zero, then $\bar{H}^{n-k}(B(f)) \neq 0$.

Proof. Changing f by a homotopy will change s by a homotopy; since we only have to prove that $s^*(\varphi) \neq 0$, we may, therefore, assume that f maps the lower hemisphere E^n_- to x_0; then the restriction of s to P^{n-1} imbeds P^{n-1} in the standard manner in $P^n \times (x_0, x_0)$; we then have the commutative diagram

$$\begin{array}{ccc} P^{n-1} & \xrightarrow{i_1} & P^n \times (x_0, x_0) \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} \\ P^n & \xrightarrow{s} & Y/T \end{array}$$

and it is sufficient to prove that $i_3^*(\varphi) \neq 0$ (since then $i_2^*\, s^*(\varphi) = i_1^*\, i_3^*(\varphi) = i_1^*(c^k \otimes 1) = c^k$, and $s^*(\varphi) \neq 0$); but $i_3^*(\varphi) \neq 0$ follows immediately from the assumptions of the lemma combined with the commutative diagram



LEMMA 2.3. *If $k = n, f^* : H^n(M^n) \to H^n(S^n)$ is zero and*

$$j_* : H_{n+k}(P^n \times \Delta(M^k)) \to H_{n+k}(Y/T, Y/T - P^n \times (x_0, x_0))$$

is non-zero, then $\bar{H}^0(B(f)) \neq 0$.

Proof. As above we may assume that $f : S^n, E_-^n \to M^n, x_0$; then s factors through $Y'/T = (S^n \times (M^n \vee M^n))/T$ as shown in the diagram

$$\begin{array}{ccc} P^n & \xrightarrow{s} & (S^n \times M^n \times M^n)/T = Y/T \\ & {\scriptstyle s_1}\searrow & \nearrow \\ & & (S^n \times (M^n \vee M^n))/T = Y'/T. \end{array}$$

Consider now the diagram

$$\begin{array}{ccccc} P^n \times (x_0, x_0) & & & & \\ & \overset{i_5}{\searrow} & & \overset{i_3}{\searrow} & \\ {\scriptstyle h}\searrow & & Y'/T & \xrightarrow{i_4} & Y/T \\ & & p_1 \downarrow \uparrow s_1 & & \uparrow s \\ & & P^n & = & P^n \end{array}$$

where i_3, i_4, and i_5 are inclusions, h is the obvious homeomorphism, and p_1 is the map induced by the projection $Y' = S^n \times (M^n \vee M^n) \to S^n$. Since $p_1 s_1 = 1$ we have that $s_1^*(p_1^*(c^n)) = c^n$; let $\gamma = p_1^*(c^n)$; then

$$i_5^*(\gamma) = (p_1 i_5)^*(c^n) = h^*(c^n) = c^n \otimes 1 \in H^n(P^n \times (x_0, x_0));$$

also, precisely as in the proof of Lemma 2.2 we have that $i_3^*(\varphi) \neq 0$, i.e. $i_3^*(\varphi) = c^n \otimes 1$; now

$$i_5^*(i_4^*(\varphi) + \gamma) = i_3^*(\varphi) + i_5^*(\gamma) = c^n \otimes 1 + c^n \otimes 1 = 0,$$

so that $i_4^*(\varphi) + \gamma \in \operatorname{Im}(j_1^*)$, where j_1 is the inclusion $Y'/T \to Y'/T$, $P^n \times (x_0, x_0)$. If we can now prove that the composition

$$H^n(Y'/T, P^n \times (x_0, x_0)) \xrightarrow{j_1^*} H^n(Y'/T) \xrightarrow{s_1^*} H^n(P^n)$$

is zero, we then get that $s_1^*(i_4^*(\varphi) + \gamma) = s^*(\varphi) + c^n = 0$, from which $s^*(\varphi) = c^n \neq 0$.

We may, therefore, concentrate on proving that $s_1^* j_1^* = 0$. —To that end let t be the involution on $M^n \vee M^n$ given by

$$t(y, x_0) = (x_0, y) \quad \text{and} \quad t(x_0, y) = (y, x_0);$$

the projection $S^n \times (M^n \vee M^n) \to M^n \vee M^n$ induces a map

$$b : Y'/T \to (M^n \vee M^n)/t,$$

and the map $\bar{F} : S^n \to M^n \vee M^n$, given by $\bar{F}(x) = (f(x), f(-x))$, induces a map

$$F : P^n \to (M^n \vee M^n)/t;$$

these two maps serve to make the diagram

$$
(2.4)\qquad
\begin{array}{ccc}
H^n(Y'/T, P^n \times (x_0, x_0)) & \xrightarrow{j_1^*} & H^n(Y'/T) \\
\uparrow{\scriptstyle b^*} & & \downarrow{\scriptstyle s_1^*} \\
H^n((M^n \vee M^n)/t, (x_0, x_0)) & \xrightarrow{F^*} & H^n(P^n)
\end{array}
$$

commutative. —Looking at the commutative diagram

$$
\begin{array}{ccc}
 & & H^n(S^n, s_0) \\
 & \nearrow{\scriptstyle 0 = f^*} & \uparrow{\scriptstyle \cong} \\
 & & H^n(S^n, E^n_-) \\
 & \nearrow{\scriptstyle f^*} & \downarrow{\scriptstyle \cong} \\
H^n(M^n, x_0) & \xrightarrow{f^*} & H^n(E^n_+, S^{n-1}) \\
\downarrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} \\
H^n((M^n M^n)/t, (x_0, x_0)) & \xrightarrow{F^*} & H^n(P^n, P^{n-1}) \\
 & \searrow{\scriptstyle F^*} & \downarrow{\scriptstyle \cong} \\
 & & H^n(P^n)
\end{array}
$$

where the isomorphism to the left is that induced by the obvious homeomorphism

$$(M^n \vee M^n)/t \to M^n,$$

and the isomorphisms to the right are all standard isomorphisms, we see that $F^* = 0$. Consider next the commutative diagram

$$
\begin{array}{ccc}
S^n \times M^n, S^n \times x_0 & \xrightarrow{a} & (S^n \times (M^n \vee M^n))/T, P^n \times (x_0, x_0) \\
\downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle b} \\
M^n, x_0 & \xrightarrow{a'} & (M^n \vee M^n)/t, (x_0, x_0)
\end{array}
$$

where $a(x, y) = \text{cls}\,(x, y, x_0)$, $a'(y) = \text{cls}\,(y, x_0)$, and b' is projection.

It is easy to see that a is a relative homeomorphism; also $S^n \times x_0$ is a strong deformation retract of one of its closed neighbourhoods N in $S^n \times M^n$ (e.g. $N = S^n \times D$, D a closed disc around x_0 in M^n); hence (see e.g. [2, (4.8.9)])

$$a_* : H_n(S^n \times M^n, S^n \times x_0) \to H_n((S^n \times (M^n \vee M^n))/T, P^n \times (x_0, x_0))$$

is an isomorphism; since coefficients are Z_2 we also get that

$$a^* : H^n((S^n \times (M^n \vee M^n))/T, P^n \times (x_0, x_0)) \to H^n(S^n \times M^n, S^n \times x_0)$$

is an isomorphism. $(a')^*$ and $(b')^*$ are easily seen to be isomorphisms; and we get that

$$b^* : H^n((M^n \vee M^n)/t, (x_0, x_0)) \to H^n((S^n \times (M^n \vee M^n))/T, P^n \times (x_0, x_0))$$

is an isomorphism. —Putting in "$F^* = 0$" and "b^* iso" in the diagram (2.4) we get $s_1^* j_1^* = 0$ as desired.

Remark. What is actually proved in the first part of this section is the following more general proposition:

Let M^k be a (normal, Hausdorff or something like that) topological space; suppose you have an element $\varphi \in H^k(Y/T)$ such that (2.1) holds, and such that $s^*(\varphi) \neq 0$; then $\bar{H}^{n-k}(B(f)) \neq 0$.

3. Proof of "$j_* \neq 0$"

In this section we keep the notation from Section 2; we start the section with the assumption that

(3.1) $j_* : H_{n+k}(P^n \times \Delta(M^k)) \to H_{n+k}(Y/T, Y/T - P^n \times (x_0, x_0))$ is zero,

and we finish it by a contradiction.

Since H_{n+k} has compact support (in the sense of [2, 4.8.11]) we have a closed set $B \subseteq Y/T - P^n \times (x_0, x_0)$ such that $H_{n+k}(P^n \times \Delta(M^k)) \to H_{n+k}(Y/T, B)$ is zero; B is of the form B'/T, where B' is a closed subset of $S^n \times (M^k \times M^k - (x_0, x_0))$; now B' is contained in

$$S^n \times (M^k \times M^k - D \times D)$$

for some disc D around x_0 in M^k; also we may suppose that D is an open disc, contained (*properly*) in some other open disc D' around x_0 in M^k. Then $B \subseteq (S^n \times (M^k \times M^k - D \times D))/T$, and from the above we have

(3.2) $j_* : H_{n+k}(P^n \times \Delta(M^k)) \to H_{n+k}(Y/T, (S^n \times (M^k \times M^k - D \times D))/T)$ is zero.

Consider then $P^n \times \Delta(M^k - D)$; this is a submanifold of $P^n \times \Delta(M^k)$ *with* boundary; therefore, in the commutative diagram

$$\begin{array}{ccl} H_{n+k}(P^n \times \Delta(M^k - D)) & & \\ \downarrow & & \\ H_{n+k}(P^n \times \Delta(M^k)) & \xrightarrow{j_*} & H_{n+k}(Y/T, (S^n \times (M^k \times M^k - D \times D))/T) \\ \downarrow & \nearrow j'_* & \\ H_{n+k}(P^n \times \Delta(M^k), P^n \times \Delta(M^k - D)) & & \end{array}$$

(where the column is part of the exact sequence of the pair) the upper left hand group is zero; from (3.2) we then get that j'_* is not monic.

Now

$$P^n \times \Delta(M^k - D')$$

is closed and contained in the interior of $P^n \times \Delta(M^k - D)$; also

$$(S^n \times (M^k \times M^k - D' \times D'))/T$$

is closed and contained in the interior of $(S^n \times (M^k \times M^k - D \times D))/T$; hence in the diagram

$$\begin{array}{l} H_{n+k}(P^n \times \Delta(D'), P^n \times \Delta(D' - D)) \xrightarrow{j''_*} H_{n+k} \\ \qquad\qquad \downarrow \quad \cdot((S^n \times D' \times D')/T, (S^n \times (D' \times D' - D \times D))/T) \\ H_{n+k}(P^n \times \Delta(M^k), P^n \times \Delta(M^k - D)) \qquad \downarrow \\ \qquad \xrightarrow{j'_*} H_{n+k}((S^n \times M^k \times M^k)/T, (S^n \times (M^k \times M^k - D \times D))/T) \end{array}$$

the vertical maps are excision-isomorphisms, and we get that

(3.3) j''_* is not monic.

Considering next the pair-sequences of the pairs involved in (3.3) and noticing that $H_{n+k}(P^n \times \Delta(D')) = 0$ we get

$$\begin{aligned} (3.4) \quad & j^{(3)}_* : H_{n+k-1}(P^n \times \Delta(D' - D)) \\ & \qquad \to H_{n+k-1}((S^n \times (D' \times D' - D \times D))/T) \text{ is not monic.} \end{aligned}$$

We now assume that D is a disc around 0 of radius 1 in euclidean k-space, and that D' is a disc around 0 of radius (say) 2 in euclidean k-space. There is then a continuous map

$$\bar{R} : S^n \times (D' \times D' - D \times D) \times I \to S^n \times (D' \times D' - D \times D)$$

given by

$$\begin{aligned} \bar{R}(x, y, z, t) &= (x, ((1/\| y \| - 1)t + 1)y, z), && y \in D' - D, z \in \bar{D}, \\ &= (x, y, ((1/\| z \| - 1)t + 1)z), && y \in \bar{D}, z \in D' - D, \\ &= (x, ((1/\| y \| - 1)t + 1)y, ((1/\| z \| - 1)t + 1)z), \\ & && y \in D' - D, z \in D' - D. \end{aligned}$$

Since $\bar{R}$ is equivariant it induces a map

$$R : (S^n \times (D' \times D' - D \times D))/T \times I \to (S^n \times (D' \times D' - D \times D))/T,$$

which is easily seen to give deformation retractions from $S^n \times (D' \times D' - D \times D))/T$ to $(S^n \times (\bar{D}^{\cdot} \times \bar{D} \cup \bar{D} \times \bar{D}^{\cdot}))/T$ ($^-$ is closure, $^{\cdot}$ is boundary) and from $(P^n \times \Delta(D' - D))$ to $P^n \times \Delta(\bar{D}^{\cdot})$.

Therefore, in the diagram

$$\begin{array}{ccc} H_{n+k-1}(P^n \times \Delta(\bar{D}^\cdot)) & \xrightarrow{j_*^{\cdot(4)}} & H_{n+k-1}((S^n \times (\bar{D}^\cdot \times \bar{D} \cup \bar{D} \times \bar{D}^\cdot))/T) \\ \downarrow & & \downarrow \\ H_{n+k-1}(P^n \times \Delta(D' - D)) & \xrightarrow{j_*^{\cdot(3)}} & H_{n+k-1}\ ((S^n \times (D' \times D' - D \times D))/T) \end{array}$$

the vertical maps are isomorphisms, and we get

$$(3.5) \quad j_*^{(4)} : H_{n+k-1}(P^n \times \Delta(\bar{D}^\cdot)) \to H_{n+k-1}((S^n \times (\bar{D}^\cdot \times \bar{D} \cup \bar{D} \times \bar{D}^\cdot))/T$$

is not monic (and, hence, zero).

We have now reformulated our assumption in terms of differentiable manifolds, and we may proceed as follows:

Let N denote the normal bundle of the imbedding

$$P^n \times \Delta(\bar{D}^\cdot) \subseteq (S^n \times (\bar{D}^\cdot \times \bar{D} \cup \bar{D} \times \bar{D}^\cdot))/T,$$

and let $\bar{N}$ be the normal bundle of the imbedding

$$\Delta(\bar{D}^\cdot) \subseteq (\bar{D}^\cdot \times \bar{D} \cup \bar{D} \times \bar{D}^\cdot);$$

then from [1, (32.3)] we get

$$(3.6) \qquad w_k(N) = \textstyle\sum_{\mu=0}^{k} c^\mu \otimes w_{k-\mu}(\bar{N}).$$

On the other hand Thom ([4], see also [1, pp. 84, 85]) has proved that $w_k(N)$ is the image of the orientation class of $P^n \times \Delta(\bar{D}^\cdot)$ under the map

$$H_{n+k-1}(P^n \times \Delta(\bar{D}^\cdot)) \xrightarrow{j_*^{\cdot(4)}} H_{n+k-1}((S^n \times (\bar{D}^\cdot \times \bar{D} \cup \bar{D} \times \bar{D}^\cdot))/T)$$
$$\xrightarrow{\gamma_U} H^k((S^n \times (\bar{D}^\cdot \times \bar{D} \cup \bar{D} \times \bar{D}^\cdot))/T) \xrightarrow{(j^{\cdot(4)})^*} H^k(P^n \times \Delta(\bar{D}^\cdot)),$$

so $w_k(N) = 0$, which clearly contradicts (3.6).

4. Proof of Theorem 1

Step 1. *M^k is closed and connected.* Using Lemma 2.2, Lemma 2.3, and Lemma 3.1 one only has to notice that $\dim(A(f)) \geq \dim(B(f))$.

Step 2. *M^k is compact and connected but with boundary.* Since the boundary of M^k is collared in M^k (see [3, IV]) we have the usual construction of the "double of M^k" W (W consists of two copies of M^k, identified along their common boundary); applying step 1 to W we get the result.

Step 3. *M is compact, but not connected.* Since f maps S^n into a connectedness component of M^k, the theorem follows from the other cases.

Remark. If one knew that a compact subset of an arbitrary manifold is contained in some compact submanifold one could of course drop the assumption of compactness of M^k; the author, however, has no knowledge concerning that point.

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