THE LOGARITHMIC EIGENVALUES OF PLANE SETS

BY

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Let S be a bounded set in the complex plane (E) having the same positive two dimensional Lebesgue measure as its closure $(\bar{S})$. Denote the open set complementary to $\bar{S}$ by $\tilde{S}$, and define S^*, the support of S, as follows:

$$S^* = \{z \in S : S \cap \Delta_r(z) \quad \text{has positive measure for each} \quad r > 0\}$$

where $\Delta_r(z)$ is the open disk of radius r and center z.

For such sets a functional μ is defined by

$$\mu(S) = \inf_{f \in L_1^2(S)} \left\{ -\frac{2}{\pi} \int_S \int_S \log |z - \zeta| f(z)\bar{f}(\zeta) \, d\tau_z \, d\tau_\zeta \right\}$$

where $L_1^2(S)$ is the set of all complex-valued functions which are square integrable over S with $\| f \|_2 \leq 1$, and τ is Lebesgue measure in the plane.

Clearly $\mu(S) \leq 0$, and in an earlier paper [1], it was shown that $\mu(S)$ is negative *iff* d, the transfinite diameter of $\overline{S^*}$ exceeds unity in which case the following inequality holds:

$$0 < -\mu < (2A/\pi) \log d$$

where A is the area of S^*.

Since $S \sim S^*$ has measure zero, it follows that $\mu(S) = \mu(S^*)$; hence attention may be restricted to bounded measurable support sets S, i.e. those plane sets for which $S = S^*$. (Observe that $(S^*)^* = S^*$.) Such sets which in addition have closures with transfinite diameter exceeding unity will be called admissible sets.

In the present paper, the dependence of μ on the class of admissible sets will be investigated. It will first be shown that μ is a monotone set functional which is continuous with respect to an appropriate type of convergence.

Next, a variational estimate for μ with respect to an important class of boundary variations is given and this formula is used to attack extremal problems suggested by the inequality: $0 < -\mu < (2A/\pi) \log d$. Specifically, it is proven that among all simply connected admissible domains of given transfinite diameter and sufficiently smooth boundary, the disk is the only one for which the value of the ratio $-\mu/A$ is stationary with respect to the boundary variations considered. Then, by use of specific domains, it is shown that $-\mu/A$ has neither maximum nor minimum under these conditions.

In [1], it was also shown that for admissible sets, μ is the unique negative

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eigenvalue of the logarithmic operator L defined by

$$(2) \qquad (Lf)(z) = -\frac{2}{\pi}\int_S \log|z-\zeta| f(\zeta)\, d\tau_\zeta \quad \text{for} \quad f \in L^2(S)$$

and that within appropriate normalization there is a unique eigenfunction ψ associated with μ. Specifically, ψ may be normalized to have the following properties:

(3)
- (a) ψ has a continuously differentiable extension to the entire complex plane E.
- (b) $\psi > 0$ in E.
- (c) ψ is subharmonic in E [2].
- (d) ψ is harmonic in $\tilde{S}$.
- (e) $\nabla^2\psi = -(4/\mu)\psi$ in S^0, the interior of S.
- (f) $\psi(z) = -\dfrac{2}{\pi\mu}\log|z|\displaystyle\int_S \psi\, d\tau + O\left(\frac{1}{|z|}\right)$ near infinity.
- (g) $(\|\psi\|_2)^2 = \displaystyle\int_S \psi^2(z)\, d\tau_z = 1.$

Monotonicity

That $-\mu$ is a montone set functional follows immediately from its definition. However, using the known properties of the associated normalized eigenfunctions, this simple monotonicity may be considerably strengthened.

THEOREM 1 (Monotonicity). *Let S_1 and S_2 be admissible sets with $S_1 \subset S_2$. Then $-\mu(S_1) \leq -\mu(S_2)$ with equality iff $S_2 \sim S_1$ has zero measure.*

Proof. Let ψ_1, ψ_2 be the associated eigenfunctions. By definition (2),

$$\mu(S_k)\psi_k(z) = -\frac{2}{\pi}\int_{S_k} \log|z-\zeta|\,\psi_k(\zeta)\, d\tau_\zeta, \qquad k = 1, 2, \quad \text{for all} \quad z \in E.$$

After an obvious calculation,

$$\left[\frac{\mu(S_2)}{\mu(S_1)} - 1\right]\int_{S_1} \psi_1\psi_2\, d\tau = \int_{S_2\sim S_1} \psi_1\psi_2\, d\tau.$$

Since each $\psi_k > 0$, and S_1 has positive measure, the integral on the left is positive while that on the right is non-negative and vanishes *iff* $S_2 \sim S_1$ has zero measure. The result follows.

Characteristic convergence and continuous dependence

The possibility of continuous dependence of μ and ψ upon the class of admissible sets depends, of course, upon the topologies considered. Convergence of the sets in the sense of Fréchet [3] is sufficient but the following weaker type of convergence seems to be more natural:

Let $\{S_n\}_0^\infty$ be a sequence of plane sets and for each $n = 0, 1, 2, \cdots$, let χ_n be the characteristic function of S_n considered as a subset of E. Then S_n

is said to converge characteristically to S_0 *iff* $\chi_n \to \chi_0$ pointwise a.e. $[\tau]$ as $n \to \infty$.

Observe that characteristic convergence is implied by Fréchet convergence, but it is unrelated to convergence in the sense of Carathéodory[4]. Unlike either of these classical convergences, characteristic convergence does not preserve the degree of connectivity as is evident from elementary examples. This makes it useful in the construction of counterexamples. Of more importance is the following application:

THEOREM 2 (Convergence). *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of admissible sets, each contained in a common disk Δ, and let μ_n, ψ_n be respectively the unique negative eigenvalue and associated normalized eigenfunction of the logarithmic operator L on S_n, $n = 1, 2, \cdots$. Suppose S_n converges characteristically to S, (so that S is a measurable set). Then*

(a) $\lim_{n\to\infty} \mu_n$ *exists* $= \mu_0$ *say;*

(b) *if $\mu_0 < 0$, then S^* is admissible, $\mu_0 = \mu(S^*)$ and ψ_n converges almost uniformly in E to ψ_0, the unique normalized eigenfunction associated with μ_0.*

Proof. Let χ_n be the characteristic function of S_n $n = 1, 2, \cdots$, and χ_0 that of S relative to the disk Δ. Consider the logarithmic operators L_n of integral type on the Hilbert space $L^2(\Delta)$ having as their kernels

$$(4) \qquad l_n(z, \zeta) = -\frac{2}{\pi} \chi_n(z)\chi_n(\zeta) \log|z - \zeta|, \qquad z \neq \zeta, n = 0, 1, 2, \cdots$$

From these definitions, it follows that

$$\mu_n = \min_{f \in L_1^2(\Delta)} (L_n f, f)_\Delta = (L_n \psi_n, \psi_n)_\Delta$$

where $(\ ,\ )_\Delta$ is the scalar product on $L^2(\Delta)$. Hence

$$((L_m - L_n)\psi_n, \psi_n)_\Delta \geq \mu_m - \mu_n \geq ((L_m - L_n)\psi_m, \psi_m)_\Delta$$

or

$$|\mu_m - \mu_n| \leq \sup_{f \in L_1^2(\Delta)} |((L_m - L_n)f, f)| \leq \left(\int_\Delta d\tau_z \int_\Delta d\tau_\zeta [l_m(z, \zeta) - l_n(z, \zeta)]^2\right)^{1/2}$$

by the Schwarz inequality and definitions (4).

Since $\chi_n \to \chi_0$ pointwise a.e. $[\tau]$, then

$$(l_m - l_n)^2 \to 0 \quad \text{pointwise a.e.} \quad [\tau \times \tau]$$

as n, $m \to \infty$, and each l_n is dominated by the square integrable function $|\log|z - \zeta||$ in the product space $\Delta \times \Delta$. Therefore by the Lebesgue theorem, it follows that the $\{\mu_n\}_{n=1}^{\infty}$ form a Cauchy sequence and so converge to a unique non-positive limit which is denoted by μ_0.

Let L_Δ denote the logarithmic operator of the disk Δ. Then

$$(5) \qquad \mu_n \psi_n(z) = (L_\Delta \chi_n \psi_n)(z) \qquad \text{all} \quad z \in E.$$

In [1], it was proven that L_Δ is a compact operator from $L^2(\Delta) \to C(K)$

for each compact set $K \subset E$. From (3 g)

$$\int_\Delta (\chi_n \psi_n)^2 \, d\tau = \int_{S_n} \psi_n^2 \, d\tau = 1 \qquad n = 1, 2, \cdots.$$

Hence a subsequence of the functions $\mu_n \psi_n$ may be chosen which converges almost uniformly in E to a limit function which is necessarily continuous, non-positive and superharmonic. If $\mu_0 = 0$, little more can be said; but if $\mu_0 < 0$, the limit function may be written as $\mu_0 \psi_0$. ψ_0 as defined is non-negative and subharmonic in E. Then denoting the subsequence as $\{\psi_{n_k}\}$, it is clear that $\chi_{n_k} \psi_{n_k} \to \chi_0 \psi_0$ pointwise and boundedly on Δ.

Therefore by the bounded convergence theorem

$$\int_S \psi_0^2 \, d\tau = \int_\Delta \chi_0 \psi_0^2 \, d\tau = \lim_{k\to\infty} \int_\Delta \chi_{n_k} \psi_{n_k}^2 \, d\tau = 1. \tag{6}$$

Similarly, both sides of (5) converge as $k \to \infty$ for each $z \in E$, and so in the limit

$$\mu_0 \psi_0(z) = (L_\Delta \chi_0 \psi_0)(z) = -\frac{2}{\pi} \int_S \log |z - \zeta| \psi_0(\zeta) \, d\tau_\zeta.$$

Therefore ψ_0 is an eigenfunction to the logarithmic operator on the set S associated with the eigenvalue μ_0. Since it is non-negative and normalized (6), it follows that S^* is admissible, and ψ_0 is its unique eigenfunction. The same argument shows that every almost uniformly convergent subsequence of the ψ_n must converge to ψ_0, and thus it may be concluded that the entire sequence $\{\psi_n\}$ converges almost uniformly to ψ_0 in E. Observe that ψ_0 must also be strictly positive.

Extremal problems

In order to attack directly some of the extremal problems associated with the logarithmic eigenvalues of plane sets, it is useful to employ variational methods. The classical apparatus is inapplicable to the present problem, but the basic principle of variation of functionals on which it depends remains valid, and has been successfully applied to analogous problems in conformal mapping and elliptic equations [5], [6]. To facilitate the analysis, we will consider only domains having sufficiently smooth boundaries.

A bounded domain D, whose boundary consists of a finite number of twice continuously differentiable smooth Jordan contours, and having closure of transfinite diameter exceeding unity will be designated as "admissible". Note that an admissible domain is an admissible set so that the terminology is consistent.

Let D be an admissible domain, $\tilde{D}$ the open set complementary to its closure, and ∂D its oriented boundary. Furthermore, let μ and ψ be its unique negative logarithmic eigenvalue and associated positive normalized eigenfunction.

For fixed $\zeta \in D \cup \tilde{D}$ and with $0 < \rho < \frac{1}{4} d(\zeta, \partial D)$, $0 \leq \alpha < 2\pi$, let D_* be the finite domain bounded by the curves:

$$\{w : w = z + \rho^2 e^{i\alpha}/(z - \zeta),\ z \in \partial D\}. \tag{7}$$

Observe that each component of ∂D_* is the conformal image of a unique component of ∂D. Moreover as $\rho \to 0$, the associated domains D_* converge to D in the Fréchet sense, and hence for ρ sufficiently small the transfinite diameter of D_* must also exceed unity [3]. For such ρ, the domain D_* is also admissible and thus possesses a logarithmic eigenvalue μ_* and an associated eigenfunction ψ_*. We wish to relate μ_* to μ as a function of ρ. This is accomplished by a straightforward but lengthy modification of the technique developed in [6], and is given in full detail in [7]. The principal device is Green's theorem in its various forms. There results the desired formula:

$$\begin{aligned} \mu_* - \mu &= -2\pi\rho^2\mu \operatorname{Re} e^{i\alpha}\left\{\frac{1}{2\pi i}\oint_{\partial D}\frac{\psi^2(z)}{z-\zeta}\,d\bar{z} - 2\mu\chi_{\tilde{D}}(\zeta)\left(\frac{\partial\psi}{\partial\zeta}\right)^2\right\} + o(\rho^2) \\ &= \delta\mu + o(\rho^2) \end{aligned} \tag{8}$$

where $\chi_{\tilde{D}}$ is the characteristic function of $\tilde{D}$, and the integration along the oriented boundary is to be interpreted in the conventional way. $\delta\mu$, the leading term of (8), is the familiar first variation of μ with respect to the particular boundary variation considered.

We now wish to consider extremal problems suggested by the inequality

$$0 < -\mu/A \log d < 2/\pi$$

which is known to hold for admissible domains [1]. The first such problem is to extremize the ratio $-\mu/A$ among admissible domains of given transfinite diameter, and to describe those domains (if such exist) for which the extremal values are attained. A necessary condition for an admissible domain to be extremal is that the ratio $-\mu/A$ should have a stationary value with respect to "nearby" admissible domains. In particular, it is necessary that the first variation of $-\mu/A$ should vanish for all those boundary variations of the type (7) which preserve the transfinite diameter. From this discussion, we formulate and prove

THEOREM 3. *The disk is the only simply connected admissible domain of given transfinite diameter for which the ratio $-\mu/A$ assumes a stationary value with respect to the boundary variations of type* (7).

Proof. Let D be the domain in question, d its transfinite diameter, A its area and μ, ψ its logarithmic eigenvalue and associated eigenfunction. Let D_* be a domain related to D through a transformation of the type (7), and d_*, A_*, and μ_* the corresponding entities of D_*

The relation between d_* and d is known [8], and may be conveniently expressed as follows:

$$\log d_* = \log d - \rho^2 \operatorname{Re}\{e^{i\alpha}[f'(\zeta)/f(\zeta)]^2\} + o(\rho^2) \tag{9}$$

where $f(z)$ is the holomorphic function defined in $D \cup \tilde{D}$ which is identically unity in D, and which maps $\tilde{D}$ conformally onto the exterior of the unit disk with the following Laurent expansion near infinity:

$$f(z) = z/d + o(1/z). \tag{10}$$

Observe that since ∂D is a closed Jordan curve, the Carathéodory extension theorem implies that the mapping f may be extended to a homeomorphism of $\tilde{D} \cup \partial D$ onto $\{w : w \in E, | w | \geqq 1\}$ with $| f(z) | \equiv 1$ on ∂D.

In order that the transfinite diameter of D_* agree with that of D within terms of order ρ^2, we must restrict α for each given ζ to those values for which

$$\operatorname{Re}\{e^{i\alpha}[f'(\zeta)/f(\zeta)]^2\} = 0. \tag{11}$$

Straightforward application of the transformation (7) to the well-known formula

$$A_* = \frac{i}{2} \oint_{\partial D_*} w \, d\bar{w}$$

gives:

$$\begin{aligned} A^* - A &= -2\pi\rho^2 \operatorname{Re}\left\{\frac{e^{i\alpha}}{2\pi i} \oint_{\partial D} \frac{d\bar{z}}{z - \zeta}\right\} + O(\rho^4) \\ &= \delta A + O(\rho^4). \end{aligned} \tag{12}$$

The hypothesized stationarity of μ/A implies that

$$0 = (A/\mu)\delta(\mu/A) = \delta\mu/\mu - \delta A/A$$

or substituting from (9) and (12)

$$\operatorname{Re} e^{i\alpha}\left\{\frac{1}{2\pi i} \oint_{\partial D} \frac{(\psi^2(z) - A^{-1})}{z - \zeta} \, d\bar{z} - 2\mu\chi_{\tilde{D}}(\zeta) \left(\frac{\partial\psi}{\partial\zeta}\right)^2\right\} = 0$$

when $\operatorname{Re} e^{i\alpha}[f'(\zeta)/f(\zeta)]^2 = 0$.

Since f is univalent in $\tilde{D}$ and $|f| \geqq 1$, then $(f/f')^2$ is analytic in $\tilde{D}$. The preceding relations imply that the ratio

$$\left[\frac{f(\zeta)}{f'(\zeta)}\right]^2 \left\{\frac{1}{2\pi i} \oint_{\partial D} \frac{(\psi^2(z) - A^{-1})}{z - \zeta} \, d\bar{z} - 2\mu \left(\frac{\partial\psi}{\partial\zeta}\right)^2\right\} \tag{13}$$

is real in $\tilde{D}$ and it is obviously analytic there (since ψ is harmonic in $\tilde{D}$). Hence it is a constant which we denote by k. To evaluate k, we refer to (10) and (3f) which yield the following expansions near infinity:

$$f'(z)/f(z) = 1/z + O(1/z^2), \qquad \partial\psi/\partial z = C/z + O(1/z^2) \tag{14}$$

where C is a well-defined positive constant. Substituting these into (13) and equating coefficients of the ζ^{-2} terms gives

$$k = -\frac{1}{2\pi i} \oint_{\partial D} \psi^2(z) z \, d\bar{z} - \frac{i}{2\pi A} \oint_{\partial D} z \, d\bar{z} - 2\mu C^2 \tag{15}$$

Since $A = i/2 \oint_{\partial D} z\, d\bar{z}$, the second term may be evaluated immediately. To evaluate the first, we proceed indirectly. From the normalization of ψ, its differential equations (3d, e) and the complex Green's identity, we have

$$1 = \int_D \psi^2(z)\, d_{\tau_z} = \int_D \left[\frac{\partial}{\partial z}(z\psi^2(z)) - 2z\psi\frac{\partial\psi}{\partial z}\right] d_{\tau_z}$$

$$= \int_D \left[\frac{\partial}{\partial z}(z\psi^2(z)) + \mu\frac{\partial}{\partial \bar{z}}\left(z\left(\frac{\partial\psi}{\partial z}\right)^2\right)\right] d_{\tau_z}$$

$$= \frac{i}{2}\oint_{\partial D}\psi^2(z)z\, d\bar{z} - \mu\frac{i}{2}\oint_{\partial D} z\left(\frac{\partial\psi}{\partial z}\right)^2 dz.$$

Using the analyticity of $\partial\psi/\partial z$ in D and (14), the last term is reduced through residue theory to $\pi\mu C^2$. Collecting these results and substituting into (15) gives $k = -3\mu C^2$, and the stationarity condition becomes

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{(\psi^2(z) - A^{-1})}{z - \zeta}\, d\bar{z} = \chi_{\tilde{D}}(\zeta)\mu\left\{2\left(\frac{\partial\psi}{\partial\zeta}\right)^2 - 3\left(\frac{Cf'(\zeta)}{f(\zeta)}\right)^2\right\} \tag{16}$$

for each $\zeta \in D \cup \tilde{D}$

Since $\psi \in C^1(E)$, this relation together with the proven continuity of f on $\tilde{D} \cup \partial D$ and the known boundary behavior of Cauchy type integrals [9], imply that $f'(\zeta)$ also has a continuous extension to $\tilde{D} \cup \partial D$. On ∂D, $|f(\zeta)| \equiv 1$ so

$$f(\zeta(s)) = e^{i\Phi(s)}$$

where s is the arc length along ∂D as measured from any convenient reference point, and $\Phi(s)$ is the principal branch of arg $(f(\zeta(s)))$.

From these remarks and the chain rule, we conclude that $\Phi \in C^1(\partial D)$ and

$$i\Phi'(s) = \frac{f'(\zeta(s))}{f(\zeta(s))}\zeta'(s). \tag{17}$$

From a theorem of Kellog [10], it follows that f provides a diffeomorphism of ∂D onto the circumference of the unit disk. Hence $|\Phi'(s)| = |f'(\zeta(s)| \neq 0$.

Therefore, we may apply the Plemelj formula [9] to (16) to get

$$\psi^2(\zeta(s)) - A^{-1} = -\mu\left\{\left[2\frac{\partial\psi}{\partial\zeta}\zeta'(s)\right]^2 - 3\left[C\frac{f'(\zeta)}{f(\zeta)}\zeta'(s)\right]^2\right\} \tag{18}$$

The left hand side of this expression is real as is

$$\left[C\frac{f'(\zeta)}{f(\zeta)}\zeta'(s)\right]^2 \qquad \text{(see (17))}.$$

Hence

$$I_m\left[\frac{\partial\psi}{\partial\zeta}\zeta'(s)\right]^2 = 0.$$

The ratio $\left(\frac{f}{f'}\frac{\partial\psi}{\partial\zeta}\right)^2$ is analytic in $\tilde{D}$ and is now seen to be real and continuous on ∂D. Near infinity, it has the expansion $C^2 + O(1/\zeta)$. Thus it is identically equal to C^2 in $\tilde{D}$ and (18) becomes

$$\psi^2(\zeta(s)) - A^{-1} = \mu\left(\frac{\partial\psi}{\partial\zeta}\zeta'(s)\right)^2. \tag{19}$$

Moreover

$$\frac{\partial\psi}{\partial\zeta}\zeta'(s) = \frac{1}{2}\left[d\,\frac{\psi(\zeta(s))}{ds} - i\,\frac{\partial\psi}{\partial n_\zeta}(\zeta(s))\right] \tag{20}$$

where n_ζ is the normal to ∂D at ζ which is directed toward the interior of D.

Equating real and imaginary parts of (19) gives

$$\psi^2(\zeta(s)) - A^{-1} = \frac{\mu}{4}\left[\left(\frac{d\psi}{ds}(\zeta(s))\right)^2 - \left(\frac{\partial\psi}{\partial n_\zeta}(\zeta(s))\right)^2\right] \tag{21}$$

and

$$\frac{\partial\psi}{\partial s}(\zeta(s))\,\frac{\partial\psi}{\partial n_\zeta}(\zeta(s)) \equiv 0 \tag{22}$$

From Green's identity applied to D and the differential equation for ψ in this region (3e) we have

$$\int_{\partial D}\frac{\partial\psi}{\partial n_\zeta}\,ds_\zeta = -\int_D \nabla^2\psi\,d\tau_\zeta = \frac{4}{\mu}\int_D \psi\,d\tau_\zeta < 0.$$

Therefore $\partial\psi/\partial n_\zeta$ cannot vanish identically on ∂D and so from (22), and the continuity of $(\partial\psi/\partial n_\zeta)(\zeta(s))$, there must be an interval $s_0 < s < s_1$ in which $d\psi/ds \equiv 0$. Hence ψ is constant in this neighborhood as is

$$(\partial\psi/\partial n_\zeta)^2 = -(4/\mu)[\psi^2(\zeta(s)) - A^{-1}] \qquad \text{(from (21))}$$

Thus, either ψ is constant on ∂D, or the interval of constancy terminates, say at s_1. In the latter case, there is an interval $s_1 < s < s_2$ in which $d\psi/ds \neq 0$, and (from (22)) $\partial\psi/\partial n_\zeta = 0$. This implies that $\partial\psi/\partial n$ is discontinuous at s_1, since it is constant and non-zero for $s < s_1$, and so furnishes a contradiction to the assumed smoothness of the boundary. Hence, the first alternative holds and so both ψ and $\partial\psi/\partial n$ are constant on ∂D.

The proof of the theorem is completed through the following lemma which is of interest in itself:

LEMMA. *If D is any admissible domain for which both ψ and $\partial\psi/\partial n$ are constant on the outer boundary component ∂D_e, then ∂D_e is a circle.*

Proof. With ψ, μ, A as above, we have near infinity:

$$\partial\psi/\partial z = C/z + O(1/z^2)$$

where from (3f) and the differential equation (3e)

$$C = -\frac{1}{\pi\mu}\int_D \psi\, d\tau = \frac{1}{4\pi}\int_D \nabla^2\psi\, d\tau$$

or, if we let D_e be the finite domain bounded by ∂D_e, and recall that ψ is harmonic in $D_e \sim \bar{D}$, we have

$$C = \frac{1}{4\pi}\int_{D_e} \nabla^2\psi\, d\tau = -\frac{1}{4\pi}\int_{\partial D_e} \frac{\partial\psi}{\partial n}\, ds \tag{23}$$

by Green's identity applied to D_e.

On the other hand, by residue theory and (14),

$$C^2 = \frac{1}{2\pi i}\oint_{\partial D_e}\left(\frac{\partial\psi}{\partial z}\right)^2 z\, dz = \frac{1}{2\pi i}\oint_{\partial D_e}\left(\frac{\partial\psi}{\partial z} z'\right)^2 z\, d\bar{z}$$

where $z' = z'(s)$ is the unit tangent vector to ∂D_e.

Moreover, on ∂D_e, $$\frac{\partial\psi}{\partial z} z' = \frac{1}{2}\left(\frac{d\psi}{ds} - i\frac{\partial\psi}{\partial n}\right)$$

(as in (20)), and by hypothesis, both ψ and $\partial\psi/\partial n$ are constant there, so

$$\left(\frac{\partial\psi}{\partial z} z'\right)^2 = -\frac{1}{4}\left(\frac{\partial\psi}{\partial n}\right)^2.$$

Thus

$$C^2 = -\frac{1}{8\pi i}\left(\frac{\partial\psi}{\partial n}\right)^2 \oint_{\partial D_e} z\, d\bar{z} = \frac{A_e}{4\pi}\left(\frac{\partial\psi}{\partial n}\right)^2 \tag{24}$$

where A_e is the area of D_e.

Since $C > 0$, we see that $\partial\psi/\partial n$ must be a non-zero constant on ∂D_e. Equating (23) with (24) yields

$$A_e = \frac{1}{4\pi}\left[\int_{\partial D_e} ds\right]^2 = \frac{L_e^2}{4\pi},$$

where L_e is the length of ∂D_e.

Hence the curve ∂D_e satisfies the isoperimetric *equality* with respect to its enclosed area, and this, as is well known [11], implies that ∂D_e is a circle.

This concludes the proof of the theorem, since by hypothesis, the extremal domain D is simply connected and so $\partial D_e \equiv \partial D$. The methods used are also applicable in the case where the stationary domain is multiply connected (and admissible). They show that the outer boundary component is always a circle, but furnish no information as to the presence, number or shape of other components of the boundary. However, further insight into the extremal problem is supplied through study of the ratio $-\mu/A \log d$ for certain classes of admissible domains.

Let D be the open annulus with inner radius r_1 and outer radius r_2 centered at the origin.

Introducing

$$u = (2/\sqrt{-\mu})\,|\,z\,|, \qquad u_k = (2/\sqrt{-\mu})r_k \qquad (k = 1, 2)$$

$$\Psi(u) = K_1(u_1)I_0(u) + I_1(u_1)K_0(u) \tag{25}$$

where I_p and K_p are modified Bessel functions [12], it is simple to verify that for any real number, a, the function

$$\begin{aligned} \psi(z) &= a\Psi(u_1), & 0 \leq |\,z\,| \leq r_1, \\ &= a\Psi(u), & r_1 \leq |\,z\,| \leq r_2, \\ &= a\Psi(u_2) \log |\,z\,|/\log r_2, & |\,z\,| \geqq r_2, \end{aligned}$$

is continuous everywhere, harmonic in $\tilde{D}$, satisfies $\Delta^2\psi + (4/\mu)\psi = 0$ in D, and near infinity has the development

$$\psi(z) = k \log |\,z\,| + O(1/|\,z\,|).$$

Moreover ψ is continuously differentiable everywhere under the following condition

$$u_2 \left[\frac{K_1(u_1)I_1(u_2) - I_1(u_1)K_1(u_2)}{K_1(u_1)I_0(u_2) + I_1(u_1)K_0(u_2)}\right] = \frac{1}{\log r_2} \tag{26}$$

which determines μ for given r_1 and r_2. Green's identities applied to the functions ψ and $\log |\,z - \zeta\,|$ for fixed $\zeta \in D$, yield after a familiar limiting process:

$$\mu\psi(z) = -\frac{2}{\pi}\int_D \log |\,z - \zeta\,|\,\psi(\zeta)\,d\tau_\zeta.$$

Hence μ is the unique negative logarithmic eigenvalue of D which exists *iff* r_2, the transfinite diameter of the annulus, exceeds unity (see (26)), and ψ is the unique positive eigenfunction associated with μ, providing $a > 0$ is chosen to fulfill the normalization

$$1 = \int_D \psi^2\,d\tau = -\frac{a^2\pi\mu}{2}\int_{u_1}^{u_2} u\Psi^2(u)\,du. \tag{27}$$

THEOREM 4. *There is no admissible domain of given transfinite diameter for which the ratio $-\mu/A$ is maximal.*

Proof. In view of the fact that the transfinite diameter of an annulus is equal to its outer radius, it is sufficient to show that there exist admissible annuli of any given outer radius for which the value of the ratio $-\mu/A \log d$ is as near to $2/\pi$ as desired. In fact we will show that for the annuli centered at the origin,

$$\lim_{r_1 \to r_2} -\mu/A \log d = 2/\pi.$$

To establish this useful result, we consider the defining equation (26) which may be rewritten as follows:

$$(28) \qquad u\left[\frac{K_1(\lambda u)I_1(u) - I_1(\lambda u)K_1(u)}{K_1(\lambda u)I_0(u) + I_1(\lambda u)K_0(u)}\right] = \frac{1}{\log r_2}$$

where $\lambda = r_1/r_2$, $u = 2r_2/\sqrt{-\mu}$ (the subscript on u_2 is suppressed). From monotonicity, we know that $-\mu$ is a decreasing function of r_1, and since $-\mu/A$ is bounded and $A \to 0$ as $r_1 \to r_2$, it follows that $\mu \to 0$ and so $u \to \infty$ as $r_1 \to r_2$.

From the known asymptotic approximations for the modified Bessel functions, [12], viz

$$(29) \qquad I_p(u) \sim e^u/\sqrt{2\pi u}; \quad K_p(u) \sim e^{-u}\sqrt{\pi/2u}, \qquad p = 0, 1$$

for large positive u, we obtain the following asymptotic approximation for the left hand side of (28):

$$u\left[\frac{e^{-\lambda u}e^{u} - e^{\lambda u}e^{-u}}{e^{-\lambda u}e^{u} + e^{\lambda u}e^{-u}}\right] = u \tanh u(1 - \lambda).$$

Therefore

$$\lim_{r_1 \to r_2} \tanh u(1 - \lambda) = \lim_{r_1 \to r_2} (u \tanh u(1 - \lambda))/u = 0$$

or

$$(30) \qquad \lim_{r_1 \to r_2} u(1 - \lambda) = 0.$$

Next, consider the ratio

$$\frac{-\mu}{A \log r_2} = \frac{4}{\pi u^2(1 - \lambda^2)\log r_2} = \frac{4}{\pi u(1 - \lambda^2)}\left[\frac{K_1(\lambda u)I_1(u) - I_1(\lambda u)K_1(u)}{K_1(\lambda u)I_0(u) + I_1(\lambda u)K_0(u)}\right]$$

$$= \frac{4}{\pi u(1 + \lambda)} \cdot \left\{\frac{K_1(\lambda u)[uI_1(u) - \lambda u I_1(\lambda u)] - I_1(\lambda u)[uK_1(u) - \lambda u K_1(\lambda u)]}{(u - \lambda u)[K_1(\lambda u)I_0(u) + I_1(\lambda u)K_0(u)]}\right\}$$

However,

$$[uI_1(u)]' = uI_0(u)$$
$$[uK_1(u)]' = -uK_0(u) \qquad [12]$$

so by the law of the mean, and monotonicity of the Bessel functions we obtain

$$\frac{-\mu}{A \log r_2} > \frac{4\lambda}{\pi(1 + \lambda)}\left[\frac{K_1(\lambda u)I_0(\lambda u) + I_1(\lambda u)K_0(u)}{K_1(\lambda u)I_0(u) + I_1(\lambda u)K_0(u)}\right].$$

For large u, the bracketed term in the foregoing inequality is asymptotic to

$$\frac{1}{\sqrt{\lambda}}\left[\frac{e^{-\lambda u}e^{\lambda u} + \sqrt{\lambda}\, e^{\lambda u}e^{-u}}{e^{-\lambda u}e^{u} + e^{\lambda u}e^{-u}}\right] > \frac{1 + e^{u(1-\lambda)}}{1 + e^{2u(1-\lambda)}}$$

and in view of (30),

$$\lim_{r_1 \to r_2} \frac{1 + e^{u(1-\lambda)}}{1 + e^{2u(1-\lambda)}} = 1.$$

Collecting these facts, we see that for r_1 sufficiently near r_2,

$$\frac{2}{\pi} \geq \frac{-\mu}{A \log r_2} > \frac{4}{\pi} \frac{\lambda}{1 + \lambda} f(\lambda),$$

where $$\lim_{r_1 \to r_2} f(\lambda) = 1.$$

Therefore $$\lim_{r_1 \to r_2} -\mu/A \log r_2 = 2/\pi$$

as asserted.
Hence there can be no doubly connected domain of given transfinite diameter for which the ratio $-\mu/A \log d$ is maximal, and since by Theorem 2, $-\mu/A$ may be approximated as closely as desired by domains (or admissible sets) of any desired connectivity, it follows that there is no admissible set of prescribed connectivity and transfinite diameter for which $-\mu/A \log d$ is maximal.

A similar argument establishes

THEOREM 4′. *There is no admissible domain of given transfinite diameter for which the ratio $-\mu/A$ is minimal.*

Proof. Let d be the given transfinite diameter and for h, $w > 0$, consider the admissible domain D (h, w) obtained by smoothing the corners on the set

$$\{z : |z| < 1\} \cup \{z : 0 \leq \operatorname{Re} z < w, 0 \leq |\operatorname{Im} z| < h\}$$

in any prescribed manner. For fixed $h > 0$, the transfinite diameter D (h, w) is an increasing continuous function of w which exceeds $w/4$. (The continuity follows from the identification of the transfinite diameter with the exterior mapping radius [3].) Hence for each integer $n = 1, 2 \cdots$, there exists a $w_n \leq 4d$ for which the domain D_n $(\equiv D(1/n, w_n))$ has transfinite diameter d. The sequence D_n converges characteristically to the unit disk which is not an admissible set. Therefore by Theorem 2, μ_n, the eigenvalue of D_n, approaches zero. Since A_n, the area of D_n, exceeds unity it follows that $-\mu_n/A_n \to 0$; i.e. there exist admissible domains of given transfinite diameter for which the ratio $-\mu/A$ is as small as desired.

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