PERMUTATION REPRESENTATIONS

BY
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A *permutation representation* of a group can be defined as a homomorphism of the given group G into a symmetric group, the group of all permutations of a given set Ω. We shall call the elements of Ω points and refer to Ω as a space, but this does not imply that any geometric notions are intended. We can also discuss permutation representations without talking about homomorphisms, by actually writing down the permutation for each g in G:

$$\varphi(g) = \begin{pmatrix} \omega \\ \omega^g \end{pmatrix},$$

ω the general point of Ω. Everything is given once one is given the function ω^g of two variables; it will be called an *action* of G on Ω. This function is subject to the following requirements: for all ω in Ω, and g, h in G, we must have $(\omega^g)^h = \omega^{gh}$, and $\omega^1 = \omega$. These two ways of treating permutation representations are equivalent; I prefer to speak of actions because later we will also be talking about *linear representations*.

There are three major tools that have been developed for the purpose of studying the actions of a group G on a set Ω. The first of these is the well known theory of linear representations over a field, a theory developed by Frobenius around 1900.

The second method is due to Schur, and dates from 1933: this is the method of *Schur rings* [1], [3]. For the purposes of this paper we may define Schur rings in the following slightly simplified manner. Given a group H and a field F, consider the group ring FH. This is the ring of formal linear combinations of elements of H with coefficients in the field F, that is, the set

$$\{\sum a_h h : a_h \in F, h \in H\},$$

with coefficient-wise addition, and multiplication induced from the multiplication in H. A Schur ring is then a subring of FH, which is closed with respect to the additional operation of coefficient-wise multiplication, the operation defined by

$$(\sum a_h h)*(\sum b_h h) = \sum a_h b_h h.$$

Later we shall see that this peculiar type of operation turns up naturally in the theory of permutation representations.

There is one more method, of rather recent origin [4]. This is the study of those relations between points of Ω that remain invariant under the action of G. By studying these *invariant relations*, we hope to get information on the action of G on Ω. The particular case of binary relations can conveniently be represented by graphs. Graph theory has recently contributed considerably to the theory of permutation groups, e.g. in the work of Sims [2].

These three methods seem quite different. It would be desirable to have a common source, something from which they could all be derived. This is what we want to give in this paper. To find a central concept, it will be best to start with the classical method of linear representations; we shall try to find out what the essential points of the theory are, and how they are used to investigate actions.

Let me begin by describing the standard procedure that is used to introduce linear representations into the theory of actions. Suppose we are given an action of G on Ω; we wish to construct a linear representation of G over some field F, that is, a linear action of G on an F-vector space. Let V be the space of formal linear combinations of elements of Ω over F,

$$V = \{v = \sum a_\omega \omega : \omega \in \Omega, a_\omega \in F\}.$$

Addition is defined in the natural manner. The action of G is extended linearly from the basis, defining $v^g = \sum a_\omega \omega^g$. Thus we obtain a linear action of G on V which makes V into a representation module for G, associated with the original action of G on Ω.

However, there is a less formal way to obtain a representation module from the action of G on Ω. Consider the set F_1 of all functions from Ω into the field F. These functions form a natural vector space structure. F_1 is essentially the same space as V, defined above; the correspondence is given by $f \leftrightarrow \sum f(\omega)\omega$. Let G act on F_1 according to the definition $f^g(\omega^g) = f(\omega)$. It is easy to show that this function of two variables gives an action on F_1 equivalent to the action given on V before.

This new point of view has a number of important consequences. Functions are more convenient objects to consider than linear combinations of points; we may compose them, we may consider their inverses, and so on. The functional approach also gives rise to an immediate generalization, to functions of more than one variable. Let F_k be the set of functions from Ω^k to F. This set again has a vector space structure, and we may define an action of G on it as before,

$$f^g(\xi_1^g, \cdots, \xi_k^g) = f(\xi_1, \cdots, \xi_k).$$

Thus we have associated with the given action of G on Ω a whole series of vector spaces on which G acts, $F_0, F_1, F_2, \cdots$ (we have completed the series by adding F_0, the functions of no variables, which is the same as F, with G acting trivially.) Note that F_m has dimension $| \Omega |^m$.

The central concept in the theory of linear representations is that of G-homomorphisms between representation modules. So we seek to describe $\mathrm{Hom}_G(F_l, F_k)$, the F-linear homomorphisms which are compatible with the action of G. Let $n = | \Omega |$, and $m = l + k$. We may regard the elements of F_l as vectors with n^l coordinates; the image by a homomorphism φ must depend linearly on these. Thus φ may be written

$$\varphi(f)(\xi_1, \cdots, \xi_k) = \sum t(\xi_1, \cdots, \xi_k, \eta_1, \cdots, \eta_l)f(\eta_0, \cdots, \eta_l)$$

where the sum extends over all $(\eta_1, \cdots, \eta_l) \in \Omega^l$. Here t is a function of $k + l$ variables with values in F, i.e. $t \in F_m$. One can show easily that φ is compatible with the action of G if and only if t is invariant in the obvious sense:

$$t(\xi_1^g, \cdots, \xi_m^g) = t(\xi_1, \cdots, \xi_m), \qquad \forall g \in G.$$

Hence if we denote the set of invariant functions of m variables by $F_m(G)$, each φ in $\mathrm{Hom}_G(F_l, F_h)$ gives rise to a function $t \in F_m(G)$, and conversely; moreover this correspondence is linear in t. We formulate this as our first theorem.

THEOREM I. $\mathrm{Hom}_G(F_l, F_k) \cong_F F_{l+k}(G)$.

We have found a concept closely connected to the theory of linear representations; these *invariant functions* are our proposed unifying idea. We shall now see how they are connected to the theory of invariant relations and to Schur rings. Let us first try to describe the space $F_m(G)$. By definition

$$f \in F_m(G) \Leftrightarrow f(\xi_1^g, \cdots, \xi_m^g) = f(\xi_1, \cdots, \xi_m), \qquad \forall_g \in G.$$

Thus f is invariant if and only if f is constant on the orbits of G on Ω^m. This immediately tells us how many linearly independent invariant functions there are; since the characteristic functions of the orbits form a basis, we must have

THEOREM II. *The dimension, over F, of $F_m(G)$ is equal to the number of G-orbits in Ω^m.*

Invariance of functions can be expressed in a slightly different way, if we consider level surfaces of the functions in F_m. The invariance condition simply means that when f takes a certain value on a point $(\xi_1, \cdots, \xi_m)$ of Ω^m, it takes the same value on all the G-images of that point, that is:

THEOREM III. *$f \in F_m(G)$ if and only if every level surface of f in Ω^m is invariant under G.*

Any subset of Ω^m is a relation between m points. Thus Theorem III gives the correspondence between invariant functions and invariant relations.

We are led to the connection between Schur's method and invariant functions when we consider the special case of $\mathrm{Hom}_G(F_l, F_k)$ where $k = l = 1$. $\mathrm{Hom}_G(F_1, F_1)$ has a natural ring structure given by composition of functions, $\varphi \circ \varphi'(f) = \varphi(\varphi'(f))$. This ring multiplication induces a similar multiplication in $F_2(G)$, through the correspondence given by Theorem I. Going back to the proof of that result, we find

$$(1) \qquad f, g \in F_2(G) \Rightarrow h(\xi_1, \xi_2) = \sum_\omega f(\xi_1, \omega)g(\omega, \xi_2) \in F_2(G).$$

The new operation thus defined on $F_2(G)$ can be identified with matrix multiplication. With this result in mind, we can now describe how Schur's method is related to invariant functions.

Let G be a group acting on Ω, and suppose there is a subgroup $H \leqq G$ which can be put in a natural 1-1 correspondence with the points of Ω. This can be done whenever H acts regularly on Ω, i.e. when there is exactly one element of H carrying a given point of Ω into any other point. Whenever that is the case, we can switch from Ω to H, as follows: fix a point o in Ω; then for x in H, replace o^x by x. Under this identification, F_2 becomes the set of functions from $H \times H$ to F. Now let $f \in F_2(G)$; then by the identification and invariance of f, $f(\xi_1, \xi_2) = f(o^{h_1}, o^{h_2}) = f(o^{h_1 h_2^{-1}}, o)$; $f(\xi_1, \xi_2) = \hat{f}(h_1 h_2^{-1})$ where $\hat{f}(h)$ is a function of a single variable from H to F. The multiplication on $F_2(G)$ defined by (1) can be carried over to $\hat{F}_2(G)$; doing the calculation, we find

$$(2) \qquad \hat{f}_1, \hat{f}_2 \in \hat{F}_2(G) \Rightarrow \textstyle\sum_{h \in H} \hat{f}_1(xh^{-1})\hat{f}_2(h) \in \hat{F}_2(G).$$

This is an expression of a familiar form, closely related to the group ring FH. For $f \in F_2(G)$, let $f^* = \sum_{x \in H} \hat{f}(x)x \in FH$, and let $S = (F_2(G))^*$. Then it is easy to check that (2) becomes

$$(3) \qquad f_1^*, f_2^* \in S \Rightarrow f_1^* f_2^* \in S, \text{ the product in the group ring } FH.$$

Since the invariant functions form a commutative ring under pointwise multiplication and addition, this shows that S is a Schur ring. Hence the theory of Schur rings can be used to investigate the G-invariant functions of two variables in the special case when G contains a transitive regular subgroup.

This unification of the three major methods offers promise for the future; it involves concepts that are very simple and general in nature, and can be carried over to topological groups. In the theory of multiply transitive groups, invariant functions should prove particularly useful. For example, if one studies a precisely fivefold transitive group, there is no non-trivial invariant relation between five points, but there are non-trivial invariant relations between six points, so $\mathrm{Hom}_G(F_3, F_3)$ is a non-trivial ring.

References

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