ALGEBRAIC GROUPS AND FINITE GROUPS

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Our object is to indicate how large classes of finite simple groups, specifically those introduced by Chevalley [5], Suzuki [21], Ree [14], and us [16], can be studied profitably with the aid of the theory of linear algebraic groups. We shall refer to these groups as groups of Chevalley type, the first-mentioned as untwisted, the rest as twisted.

First we recall some facts about linear algebraic groups, all taken from [7]. Assume given an algebraically closed field k. A (linear) algebraic group is a subgroup of some $GL_n(k)$ which is at the same time an algebraic set (i.e. the complete set of solutions of a set of polynomials) in the space determined by the n^2 matric coefficients. The Zariski topology, in which the closed sets are the algebraic sets, is used. An algebraic group is said to be simple if it is connected, has only discrete, i.e. finite, nontrivial normal (algebraic) subgroups, and is not Abelian. The simple algebraic groups have been classified by Chevalley [7] in the Killing-Cartan tradition. Thus there are the classical types A_n, B_n, C_n, D_n and the five exceptional types E_6, E_7, E_8, F_4 and G_2. As examples we may mention the groups PSL_n or SL_n (of type A), SO_n or Spin_n (of type B or D depending on the parity of n), Sp_n (of type C), and the group of automorphisms of the Cayley algebra (of type G_2).

The connection between simple algebraic groups and simple finite groups comes from the fact that many of the latter, all of those mentioned in the first sentence above, arise as fixed-point groups of endomorphisms of the former.

The basic tool for studying the latter with the aid of the theory of algebraic groups is the following extension of a result of Lang [13].

(A) Let G be a connected linear algebraic group and σ an (algebraic) endomorphism of G onto G such that G_σ, the group of fixed points, is finite. Then the map $\varphi : G \longrightarrow G$ defined by $\varphi x = x\sigma x^{-1}$ is surjective.

This is proved in [20]. Here we will sketch a proof in a special case, in which $\sigma x = x^{(q)}$, the result of replacing each entry of x by its q^{th} power, it being assumed that this operation maps G onto itself. Here q is a power of p, the characteristic of k, and is assumed to be greater than 1. From the rules of differentiation, the differential at $y = 1$ of the map $y \longrightarrow y\sigma y^{-1}$ is the same as that of $y \longrightarrow y$, hence is surjective. It follows that the map covers an open set in G. The same holds for the map $z \longrightarrow zg\sigma z^{-1}$ with g an arbitrary element of G. Since G is connected, it is irreducible as an algebraic set [7, Exp. 3]. Thus the preceding open sets intersect: $y\sigma y^{-1} = zg\sigma z^{-1}$ for some y, z. Then $g = \varphi x$ with $x = z^{-1}y$, which proves (A), in this special case.

An immediate consequence of (A) is that if i_g is any inner automorphism, then $i_g\,\sigma$ is conjugate to σ: if $g = x\sigma x^{-1}$, then $i_g\,\sigma = i_x\,\sigma i_x^{-1}$. For the study of

G_σ this means that σ may be altered by an arbitrary inner automorphism and thus be brought to a form in which G_σ can be analyzed exactly. In particular, we get [20]:

(B) If G and σ are as in (A), then every composition factor of G_σ is either cyclic or of Chevalley type.

As a second application of (A) let us consider the conjugacy classes of G_σ. In an ideal situation we would have:

(C) (1) Every class of G fixed by σ contains an element fixed by σ.
(2) Every such class meets G_σ in a single class.

Here (1) is true. Let C be the class. If $y \in C$, then $g\sigma y g^{-1} = y$ for some $g \in G$, whence, writing $g = x\sigma x^{-1}$ by (A), we conclude that σ fixes $x^{-1}yx \in C$. On the other hand (2) is in general false. Assume however that C is such that

(*) G_x is connected for each $x \in C$.

Then (2) holds: if $x, y \in C_\sigma$ and $g \in G$ are such that $gxg^{-1} = y$, then $\sigma g . x . \sigma g^{-1} = y$, whence $g^{-1}\sigma g \in G_x$, $g^{-1}\sigma g = h\sigma h^{-1}$ with $h \in G_x$ by (*) and (A) applied to G_x, and x is conjugate to y under $gh \in G_\sigma$, which yields (2).

The condition (*), hence also (2), holds in a special case, important for representations in characteristic p, viz. when C is semisimple, i.e. consists of diagonalizable elements, and the group G is simple and simply connected (we will not define this term, but remark that SL_n has to be taken rather than PSL_n, and Spin_n rather than SO_n) (see [7, Exp. 23]), the connection with representations coming from the fact that in this case p must be nonzero and the diagonalizable elements of G_σ are those of order prime to p. It leads to a complete survey of the semisimple classes of the finite simple groups of Chevalley type (see [19]). For arbitrary classes, no such survey has as yet been given, except for the classical groups and one or two other types.

One of the most important structural properties of a simple (or more generally connected) algebraic group G is its decomposition into double cosets relative to a Borel (i.e. maximal connected solvable) subgroup, and this carries over to G_σ (as in (A)), basically by (A), where it becomes the decomposition into double cosets relative to the normalizer B of a p-Sylow subgroup P (p is the characteristic of k). In SL_n, for example, the essence of the decomposition is that a system of representatives of the monomial subgroup modulo the diagonal subgroup is also a system of representatives of the double cosets relative to the superdiagonal subgroup. Tits [22] has abstracted the essence of this (Bruhat) decomposition in his theory of B-N pairs. His methods yield a very simple proof of simplicity, i.e. of:

(D) If G and σ are as in (A) and G is simple and simply connected, then G_σ is simple over its center, with a finite number of exceptions which can be listed.

The simple finite groups obtained this way are of course just the groups of Chevalley type. Henceforth G and σ will be as in (D).

The above decomposition leads to a rather uniform determination of the automorphisms of our groups (cf. [17]), as follows. Let α be an automorphism. By Sylow's theorem the group B above may be taken to be fixed by α, and so may some conjugate of P which intersects B trivially, as may easily be proved. By now the situation is quite rigid and the possibilities can be analyzed. In SL_2, for example, the subdiagonal and superdiagonal subgroups are both fixed. Here one knows that α is the conjugation by a diagonal element of GL_2 composed with an automorphism of the base field, and this is the case to which much of the discussion can be reduced. The Schreier conjecture, that the group of outer automorphisms of a simple finite group is always solvable, is verified in every case.

We may also determine the possible isomorphisms among our groups, by extending a method of Artin [3], as follows. The abstract group G_σ usually determines p, the characteristic of k, as the prime making the largest contribution to its order, thus also the p-Sylow subgroup P, its normalizer B, and the numbers of elements in the double cosets relative to B. It rarely happens that these numbers are the same for two choices of G_σ, so that the possible isomorphisms can be severely limited and then analyzed. In some exceptional cases p does not make the largest contribution to the order of G_σ and further argument is necessary.

The Bruhat decomposition leads to a simple presentation of G_σ in terms of generators and relations, at least if G_σ is not twisted, which can be used to study its Schur multiplier and thus the connection between its projective and linear representations (see [18]). The result is that SL_n, Sp_n, Spin_n, E_8, F_4, G_2, and the simply connected versions of E_6 and E_7 all have trivial Schur multipliers, if a finite number of cases are excluded (see [18]; the true exceptions and their exact nature has not been completely worked out yet). Without exception the Schur multiplier is a p-group, since in our presentation each relation either is confined to a p-group or else expresses the conjugacy of two p-elements; hence it causes no trouble for representations in characteristic p. A number of the twisted cases have been treated by Grover [12], and the Suzuki groups by Alperin and Gorenstein [2]. There remain the groups SU_{2n+1} and the Ree groups.

The above ideas are also important in the representation theory of G_σ as far as it has till now been developed.

In characteristic p the theory is in pretty good shape (see [19]). The irreducible representations have been classified, in terms of "highest weights", certain characters on the group B above, and a tensor product theorem, expressing all of them in terms of a few, p^r (r = rank G = dimension of a maximal diagonalizable subgroup), has been proved. Contributors here have been Brauer and Nesbitt [4], Chevalley [7, Exp. 20], and Wong [24]. The information, however, is incomplete, and it is not even known what the degrees of the representations are, except in a few scattered cases. Each representation can be realized in the decomposition of the reduction mod p of a corresponding representation of an algebraic group of characteristic 0, where essentially all is known, by Weyl's character formula [23, p. 389], but the exact nature of the degeneracy that occurs in this reduction is not known. Curtis [8], using the theory of B-N pairs, has presented a more elementary version of the classification, by a construction in the group algebra itself. His approach, however, does not yield the tensor product theorem mentioned above, or, e.g., the degrees.

For representations over the complex field, in contrast, the theory is in poor shape. Aside from the groups GL_n, whose characters have been determined by Green [11], only a few cases, of small dimensions, have been treated. Gelfand and Graev [10] have announced the following theorem.

(E) Assume that G_σ is untwisted. Let r be a one-dimensional representation of P (a p-Sylow subgroup, as above) such that
(**) r has a nontrivial restriction to each one-parameter subgroup of P corresponding to a simple root.
Then the induced representation R of G_σ is multiplicity-free.

This is proved in [10] for SL_n, and in a parallel fashion in [25] for all the groups, however with an oversight, since the proof uses the fact that, in the notation of [25], the group U_2 is the derived group of U, which fails in some (a finite number of) cases. A version of (E) is undoubtedly true for the twisted groups as well. It seems to us that the determination of the irreducible components of R would be a major step in the construction of a representation theory for G_σ, and that if the same were done without the condition (**) on r then the result would be decisive. A second theorem announced in [10], and proved there for the groups SL_n, states that the components thus obtained contain a complete set of irreducible representations of G_σ, i.e., by Frobenius reciprocity, the restriction to P of every representation of G_σ contains a one-dimensional representation. This is in general false. As M. Kneser has observed, the representation of the group of type C_2 over a field of 3 elements onto the group of elements of determinant 1 in the Weyl group of type E_6 in its usual representation as a reflection group provides a counter-example. It would be useful to know for which groups the theorem holds, for which it fails.

If we induce from larger subgroups than P we can expect the decomposition into irreducible components to be easier. In this connection we would like to mention two results.

(F) Let $B = PT$ be a semidirect product decomposition of the normalizer B of P, r a one-dimensional representation of B which is trivial on P, and R the induced representation of G_σ. If the restriction of r to T is not fixed by any non-identity element of the Weyl group W (of G_σ relative to T), then R is irreducible.

This follows from a theorem of Mackey (see [9, p. 51]). At the other extreme we have:

(G) If r in (F) is the trivial representation, then the multiplicities of the irreducible components of R are just the degrees of the irreducible representations of W. Or, equivalently, the algebra of complex-valued functions on G_σ which are invariant under left and right translations by the elements of B is isomorphic to the group algebra of W.

This is proved in [1, p. 81].

We will close our discussion by presenting a formula for the orders of the groups G_σ. Let T be a maximal torus of G (a torus is a group isomorphic to a product of GL_1's, e.g. the diagonal subgroup of SL_n), taken to be fixed by σ, which is permissible by (A), and of dimension r (the rank). Then X, the character group of T, is isomorphic to Z^r. The Weyl group $W = N(T)/T$ (of G, not of G_σ) acts on T, hence on X, on the latter as a reflection group (see [7, Exp. 11]). By a result of Chevalley [6], the algebra of invariants under W in the symmetric algebra on X extended to the reals is generated by r homogeneous elements $I_1, I_2, \cdots, I_r$ of uniquely determined degrees $d_1, d_2, \cdots, d_r$. σ acts on T, hence on X, and there may be written $q\sigma_1$ with q positive and σ_1 of finite order. σ_1 normalizes W, hence acts on the invariants, and may be assumed to reproduce the I's with scalar factors $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$. Then $| G_\sigma | = q^d \prod (q^{d_i} - \varepsilon_i)$, with $d = \sum (d_i - 1)$. This can be proved by a method of Solomon (see [15] and [20]). The groups $SL_n(q)$ and $SU_n(q^2)$, for example, can be obtained from $SL_n(k)$ by taking σ in the first case the q^{th} power operation, in the second case its composition with the inverse transpose. Taking T to be the diagonal subgroup, we get σ_1 to be 1 (the identity) in the first case, -1 in the second. Now -1 acts as 1 on a homogeneous invariant of even degree, as -1 on one of odd degree. Thus we can convert the formula for the order of $SL_n(q)$ into that of $SU_n(q^2)$ by simply replacing the factor $q^j - 1$ by $q^j + 1$ whenever j is odd. It is suspected that a similar replacement (basically $q \to -q$) will convert the formulas for the degrees of the irreducible representations, in fact all of the entries in the character table, of the first group into those of the second.

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